Appendix 3: Information and Quantum Teleportation

The existence of entanglement may produce the impression that it is possible to transmit information instantaneously through great distances. For, as we saw in the EPR paper setup, a measurement return in one particle is instantaneously correlated to one in the other. However, such an impression is wrong. One can easily verify that if the system S made up of particles *a* and *b* is in state $\frac{1}{\sqrt{2}}\left(\left|\uparrow_z^a\downarrow_z^b\right\rangle - \left|\downarrow_z^a\uparrow_z^b\right\rangle\right)$, then a measurement of S_z on *b* will return $+1$ half the times and -1 half the times independently of whether we measure S_z on *a* first or do not do any measurement on *a* at all. The same is true if we exchange the particles. So, no information can be instantaneously transmitted from one subsystem of S to the other because any observation of S_z on one subsystem will return random results no matter what observation is performed on the other. Empirically, the correlation appears only if we look at the returns of *both* subsystems. Notice that this failure is due to one's inability to control the measurement return of S_z on the first observed particle. In short, entanglement *alone* cannot be used to exchange information.

However, entanglement plays a role in quantum teleportation, a procedure whereby by exploiting the correlation between entangled particles and the transmission of a bit of information in a 'normal' way (for example, by phone), the quantum state of a particle is exactly reproduced at an arbitrary distance in another particle (Home, D., (1997): 254-57). The simplest example of teleportation involves three spin half particles, 1, 2, and 3. Particles 2 and 3 are put in an EPR singlet configuration, so that

$$\left|\Psi\right\rangle_{23} = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_2 \otimes \left|\downarrow\right\rangle_3 - \left|\downarrow\right\rangle_2 \otimes \left|\uparrow\right\rangle_3\right). \qquad \text{(A3.1)}$$

Particle 2 is in Alice's laboratory and particle 3 in Bob's, possibly at a great distance from Alice's. Alice is also in possession of particle 1 in state

$$\left|\Psi\right\rangle_1 = a\left|\uparrow\right\rangle_1 + b\left|\downarrow\right\rangle_1 \qquad \text{(A3.2)}$$

that may be unknown to her.[1] The goal is to reproduce $\left|\Psi\right\rangle_1$ in particle 3.

Since 1 is not entangled with 2 or 3, the system 1+2+3 is the factorizable state

$$\begin{aligned}\left|\Psi\right\rangle_{123} = \left|\Psi\right\rangle_1 \otimes \left|\Psi\right\rangle_{23} = \frac{a}{\sqrt{2}}\left(\left|\uparrow\right\rangle_1 \otimes \left|\uparrow\right\rangle_2 \otimes \left|\uparrow\right\rangle_3 - \left|\uparrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2 \otimes \left|\uparrow\right\rangle_3\right) + \\ \frac{b}{\sqrt{2}}\left(\left|\downarrow\right\rangle_1 \otimes \left|\uparrow\right\rangle_2 \otimes \left|\downarrow\right\rangle_3 - \left|\downarrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2 \otimes \left|\uparrow\right\rangle_3\right)\end{aligned} \qquad \text{(A3.3)}$$

Now consider the vectors

$$\begin{aligned}\left|\Psi^{\pm}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2 \pm \left|\downarrow\right\rangle_1 \otimes \left|\uparrow\right\rangle_2\right) \\ \left|\Phi^{\pm}\right\rangle_{12} &= \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_1 \otimes \left|\uparrow\right\rangle_2 \pm \left|\downarrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2\right)\end{aligned} \qquad \text{(A3.4)}$$

which constitute an orthonormal basis in the tensor product space of the vector space for 1 and that for 2. Using (A3.4) we can rewrite the product states of 1 and 2 in (A3.3) and obtain

$$\left|\Psi\right\rangle_{123} = \frac{1}{2}\left[\begin{aligned}&\left(-a_2\left|\uparrow\right\rangle_3 - b\left|\downarrow\right\rangle_3\right)\otimes\left|\Psi^-\right\rangle_{12} + \left(-a_2\left|\uparrow\right\rangle_3 + b\left|\downarrow\right\rangle_3\right)\otimes\left|\Psi^+\right\rangle_{12} + \\ &\left(a_2\left|\downarrow\right\rangle_3 + b\left|\uparrow\right\rangle_3\right)\otimes\left|\Phi^-\right\rangle_{12} + \left(a_2\left|\uparrow\right\rangle_3 - b\left|\downarrow\right\rangle_3\right)\otimes\left|\Phi^+\right\rangle_{12}\end{aligned}\right]. \qquad \text{(A3.5)}$$

Now Alice performs a measurement on $\left|\Psi\right\rangle_{123}$ by using a device *D* such that the state function becomes

[1] For simplicity we assume that particle 1 is not entangled with a fourth particle, but this assumption is not essential to the procedure.

$$
\left|\Psi\right\rangle_{123D} = \frac{1}{2}\left[\begin{array}{l} \left(-a_2\left|\uparrow\right\rangle_3 - b\left|\downarrow\right\rangle_3\right)\otimes\left|\Psi^-\right\rangle_{12}\otimes\left|D_1\right\rangle + \\ \left(-a_2\left|\uparrow\right\rangle_3 + b\left|\downarrow\right\rangle_3\right)\otimes\left|\Psi^+\right\rangle_{12}\otimes\left|D_2\right\rangle + \\ \left(a_2\left|\downarrow\right\rangle_3 + b\left|\uparrow\right\rangle_3\right)\otimes\left|\Phi^-\right\rangle_{12}\otimes\left|D_3\right\rangle + \\ \left(a_2\left|\uparrow\right\rangle_3 - b\left|\downarrow\right\rangle_3\right)\otimes\left|\Phi^+\right\rangle_{12}\otimes\left|D_4\right\rangle \end{array}\right] \tag{A3.6}
$$

with consequent collapse into one of the four equally probable orthogonal summand states. Hence, because of Alice's operations, particle 3, Bob's particle, is left in one of the following four states

$$
-\begin{pmatrix} a \\ b \end{pmatrix}, \tag{A3.7}
$$

$$
\begin{pmatrix} -a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{A3.8}
$$

$$
\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{A3.9}
$$

$$
\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{A3.10}
$$

each with probability 1/4.

Now Alice must tell Bob which results she got from her measurement, that is, in which state D, a macroscopic classical object, is. If D is in state $\left|D_1\right\rangle$, then particle 3 is in the state described by (A3.7), that is, in the same state as particle 1 was, namely, $\begin{pmatrix} a \\ b \end{pmatrix}$, apart from an irrelevant phase factor, and therefore teleportation has occurred. If D is in state $\left|D_2\right\rangle$, then particle 3 is in the state described by (A3.8), which is just the result of the application of the rotation matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} a \\ b \end{pmatrix}$, the state of particle 1. All Bob needs to do is apply the reverse the rotation to $\begin{pmatrix} -a \\ b \end{pmatrix}$, thus achieving teleportation, since after the reverse rotation particle 3 will be in state $\begin{pmatrix} a \\ b \end{pmatrix}$. The other two cases are treated analogously. Notice that even before teleportation is achieved, in fact as soon as Alice performs her measurement, particle 1 becomes entangled with 2 and therefore loses its original state.

Quantum teleportation looks like magic or like a cheap trick ("teleportation", after all, does sound bombastic), depending on how one looks at it. The phone call between Alice and Bob, a classical type of non-superluminal communication, is a necessary ingredient of the teleportation scheme. As we saw, (A3.7)-(A3.10) are all equally probable, which is not only a consequence of quantum theory, but also a requirement of relativity theory for, were there a special correlation between any of (A3.7-10) and the state of particle 1 before teleportation, this correlation could be exploited to send a superluminal signal (a signal faster than light). Hence, teleportation is not instantaneous, and one might wonder whether it would not be easier for Alice to do it the old fashioned way, namely, to tell Bob what state particle 1 is in, so that Bob could duplicate the state in his laboratory, achieving the same result without the bombast. However, we should note two related points. First, with teleportation neither Alice nor anyone else need know the state of particle 1, while doing it the old fashioned way requires that Alice, or someone on her side, knows the state of 1. This leads us to the second point, namely, that the information Alice conveys to Bob can be very scant, since she is just reporting the result of her measurement. Most of the information, as it were, has been instantaneously conveyed from Alice's location to Bob's by the entanglement between particles 2 and 3.

It is as if one were to receive a very detailed drawing of the four sides of a very ornate building (a lot of information) and then be told by phone which sheet represents the front, which the back, and so on (little, if crucial, information). The difference, of course, is that in quantum teleportation, most of the information is already contained, as if by magic, in particle 3.

Exercise A3.1

Show that the set of vectors given by (A3.4) is complete for the vector space of system 1+2.

Answers to the Exercises

Exercise A3.1

A basis for the vector space of 1+2 is $\{|\uparrow\rangle_1|\downarrow\rangle_2, |\downarrow\rangle_1|\uparrow\rangle_2, |\uparrow\rangle_1|\uparrow\rangle_2, |\downarrow\rangle_1|\downarrow\rangle_2\}$, which means that every vector in it is expressible as

$$a|\uparrow\rangle_1|\downarrow\rangle_2 + b|\downarrow\rangle_1|\uparrow\rangle_2 + c|\uparrow\rangle_1|\uparrow\rangle_2 + d|\downarrow\rangle_1|\downarrow\rangle_2 = \alpha\left|\Psi^{\pm}\right\rangle_{12} + \beta\left|\Phi^{\pm}\right\rangle_{12} .$$