Appendix 1: Ordinary and Partial Derivatives

A1.1 Ordinary Derivatives

Suppose we study the temperature of a pool of water as time changes, and suppose that the relation between temperature *x* and time *t* is given by the function

$$x = f(t) = t^2 - 1, \qquad \text{(A.1.1)}$$

the plot of which is a parabola (Fig. 1).

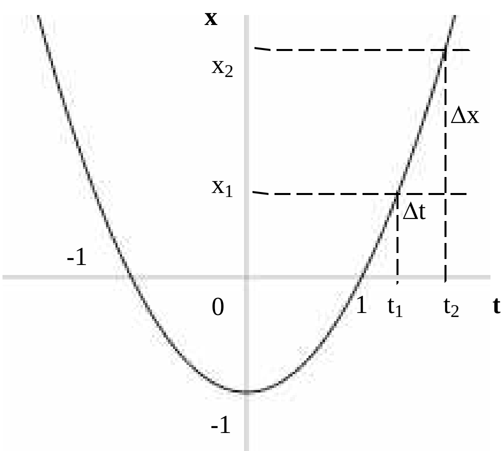

Figure 1

By simple substitution in (A.1.1) or by just looking at the plot, we can see that at time $t = -1$ the temperature is $x = 0$, at time $t = 0$ it is $x = -1$, at time $t = 1$ it is $x = 0$, at time $t = t_1$ it is $x = x_1$, and so on. Suppose that we want to know how quickly the temperature increases between t_1 and t_2, that is, the rate of change of temperature with respect to time between t_1 and t_2. If we denote the average rate of change by $\bar{v}$, the temperature change $x_2 - x_1$ by Δx, and the temporal change $t_2 - t_1$ by Δt, we obtain

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{\Delta t} = \frac{f(t_2) - f(t_1)}{\Delta t}. \qquad \text{(A.1.2)}$$

But $t_2 = t_1 + \Delta t$, and therefore,

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t}. \qquad \text{(A.1.3)}$$

By definition, the *instantaneous* rate of change v is $\bar{v}$ when Δt becomes infinitesimally small. When an interval is infinitesimally small, it is customary to use dx or dt instead of Δx or Δt, and from now on we shall follow this notation. So, (A.1.3) becomes

$$v = \frac{dx}{dt} = \frac{f(t_1 + dt) - f(t_1)}{dt}. \qquad \text{(A.1.4)}$$

In other words, to obtain the instantaneous rate of change, we take the function $f(t)$, substitute $t + dt$ in place of t, subtract $f(t)$ from what we have obtained, and finally divide by dt. Let us see what happens when we apply this procedure to $x = f(t) = t^2 - 1$. Clearly, (A.1.4) becomes

$$v = \frac{dx}{dt} = \frac{\left[(t + dt)^2 - 1\right] - \left(t^2 - 1\right)}{dt} = \frac{t^2 + dt^2 + 2tdt - 1 - t^2 + 1}{dt} = \frac{dt^2 + 2tdt}{dt}. \qquad \text{(A.1.5)}$$

Now, let us consider the quantity dt^2. It is the square of an infinitesimally small quantity, and therefore infinitely infinitely small, as it were. In fact, it is so small that we can disregard it altogether.[1] Hence, (A.1.5) can be written as

$$v = \frac{2tdt}{dt} = 2t. \qquad \text{(A.1.6)}$$

So, the instantaneous rate of change of temperature is a function of time: $v = 2t$. Hence, suppose that we measure time in seconds and temperatures in degree Celsius. Then, at time $t = 2s$, the temperature changes at a rate of $4C^{\circ}/s$ (4 degrees per second); at time $t = 2.5s$, at rate of $5C^{\circ}/s$, and so on. Given any function $x = f(t)$, $\frac{dx}{dt}$ is the (first)

[1] This sounds preposterous, and, in a way, it is. However, infinitesimal talk can be made precise and rigorous.

derivative with respect to t of the function $x = f(t)$. Some other notations are also common: $f(t)'; f'; D_t f(x)$.

EXAMPLE A1.1

Since the derivative can be generally understood as the rate of change of the function f with respect to the independent variable t, we could interpret $x = f(t) = t^2 - 1$ as follows. Suppose x is the amount of dollars needed to buy a standard number of items at time t, where t is measured in days. Then, the function represents inflation, and the derivative the rate of inflation, that is, the velocity, as it were, with which inflation grows. Notice that when $t < 0$ the rate of inflation is negative: the standard number of items costs less and less and we experience deflation. For example, when $t = -3$ (3 days before zero day), x decreases by 6 dollars for each day. However, the rate of deflation is slowing down, since the following day (at $t = -2$) x decreases by 4 dollars for each day. When $t = 0$ the rate of deflation is zero: on that day, x increases by zero dollars for each day. After $t = 0$, inflation picks up.

The function f' is the derivative of f. The derivative of f' is the *second* derivative of f, and is denoted by f''. Other common notations are $\frac{d^2x}{dt^2}; \frac{d^2f}{dt^2}$.

Consider $f(t) = t^2 - 4$; its derivative is $f'(t) = 2t$, another function of t. The derivative of $f'(t)$ is

$$\frac{2(t + dt) - 2t}{dt} = \frac{2dt}{dt} = 2, \tag{A.1.7}$$

which is the second derivative of $x = t^2 - 4$. In other words,

$$\frac{d^2}{dt^2}\left(t^2 - 4\right) = 2. \tag{A.1.8}$$

Suppose that in the previous example $x = t^2 - 4$ expresses the particle's motion (that is, *x* represents position and *t* time). Then, its first derivative gives the particle's instantaneous velocity (the instantaneous rate of change of position), and its second derivative gives the particle's instantaneous acceleration (the instantaneous rate of change of velocity).

A1.2 Derivatives of Some Common Functions

In the previous section, we have been able to calculate some simple derivatives, but the process proved rather cumbersome. To simplify things, here a set of rules, stated without proof, allowing us to calculate the derivatives of most common functions.

i. If *c* is a constant, then $\frac{d}{dx}c = 0$. For example, $\frac{d}{dx}5 = 0$.

ii. $\frac{d}{dx}x^n = nx^{n-1}$. For example, $\frac{d}{dx}x^3 = 3x^2$; $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$;

$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = -4x^{-5} = -\frac{4}{x^5}$ (Notice that –4 went down to –5).

iii. $\frac{d}{dx}\sin x = \cos x$.

iv. $\frac{d}{dx}\cos x = -\sin x$.[2]

v. If *c* is a constant, then $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$. For example,

$\frac{d}{dx}4x^3 = 4\frac{d}{dx}x^3 = 12x^2$.

[2] For an introduction to the sine and cosine functions, see appendix 2.

vi. Sum Rule: the derivative of the sum of two functions is the sum of the derivatives: $(f+g)' = f' + g'$. For example, $\frac{d}{dx}\left(3x^2+2x\right) = 6x+2$; $\frac{d}{dx}\left(\sin x - x^{-3}\right) = \cos x + 3x^{-4}$.

vii. Product Rule: the derivative of a product is the first factor times the derivative of the second plus the derivative of the first factor times the second: $(fg)' = fg' + f'g$. For example, $\frac{d}{dx}(3x\sin x) = 3x\cdot\frac{d}{dx}\sin x + \frac{d}{dx}3x\cdot\sin x = 3x\cos x + 3\sin x$.

viii. Chain Rule: a composition is a function of a function, such as $\sin x^2$, where a function ($\sin u$) is applied to another function ($u = x^2$). Informally, a composition has the form *f*(thing), where the 'thing' (in this case x^2) is another function. Then, $D_x f(thing) = f'(thing)\cdot D_x(thing)$. For example, $\frac{d}{dx}\sin x^2 = \cos x^2\cdot\frac{d}{dx}(x^2) = 2x\cos x^2$; $\frac{d}{dx}\cos 5x = -\sin 5x\cdot\frac{d}{dx}(5x) = -5\cos 5x$.

A1.3 Partial Differentiation

Consider a flat surface lying on the *xy*-plane with hot and cold spots on it. Suppose now that the temperature on the surface is a function of position. This can be expressed as $T = f(x,y)$, where *f* is a function of *two* variables, *x* and *y*. For example, $20 = f(2,5)$ means that at the point of coordinates 2, 5 the temperature is 20 degrees. Or, suppose that the temperature is not only a function of position, but of time as well: the hot spots flare up periodically, let us say. Then, the function expressing such state of affairs is $T = f(x,y,t)$, a function of three variables. Of course, in place of temperature, we might have other quantities such as the magnitude of acceleration, or pressure. In these cases as well we may become interested in rates of change. For example, given

$T = f(x, y)$, we might be interested in how quickly temperature changes (its rate of change) as we move along, say, the *x*-direction while *y* does not change. This is exactly what partial derivatives tell us.

As we know, a function $f(x)$ of one variable has one first order derivative $\frac{d}{dx} f(x)$. A function $f(x,y)$ of two variables has two first order *partial* derivatives, one with respect to *x* and one with respect to *y*. The partial derivative $\frac{\partial}{\partial x} f(x,y)$ is the ordinary derivative of *f* with respect to *x*, with *y* treated as if it were a constant. Similarly, the partial derivative $\frac{\partial}{\partial y} f(x,y)$ is the ordinary derivative of *f* with respect to *y*, with *x* treated as if it were a constant. The value of the partial derivative $\frac{\partial}{\partial x} f(x,y)$ at points $x = a$ and $y = b$ is denoted by $\left.\frac{\partial}{\partial x} f(x,y)\right|_{x=a,y=b}$. As in the case of ordinary derivatives, there are partial derivatives of higher order; their calculation is obvious.

EXAMPLE A1.3.1

Suppose that temperature is a function of position according to

$$T = x^3 y^2. \tag{A1.3.1}$$

To find $\frac{\partial}{\partial x}(x^3 y^2)$, we treat y^2 as if it were a constant, and then differentiate as usual to obtain

$$\frac{\partial}{\partial x}(x^3 y^2) = 3x^2 y^2. \tag{A1.3.2}$$

Analogously, we obtain

$$\frac{\partial}{\partial y}(x^3 y^2) = 2yx^3. \tag{A1.3.3}$$

Moreover, by plugging $x=2$ and $y=1$ into (A1.3.2) we obtain

$$\frac{\partial}{\partial x}(x^3y^2)\Big|_{x=2,y=1}=12, \qquad \text{(A1.3.4)}$$

which means that at $x=2$ and $y=1$, if *y* is kept constant and *x* increases, then for every unit of increase in *x*, the function $f(x^3y^2)$ increases by 12 units (the temperature *T* increases by 12 degrees). Analogously,

$$\frac{\partial}{\partial y}(x^3y^2)\Big|_{x=2,y=1}=16 \quad , \qquad \text{(A1.3.5)}$$

which means that that at $x=2$ and $y=1$, if *x* is kept constant and *x* increases, then for every unit of increase in *y*, *T* increases by 16 degrees. Moreover,

$$\frac{\partial^2}{\partial x^2}(x^3y^2)=\frac{\partial}{\partial x}(3x^2y^2)=6xy^2. \qquad \text{(A1.3.5)}$$

Exercises

Exercise A1.1

1. Suppose that $x = f(t) = t^2 - 4$ expresses the temperature of an object at time *t*. What does f' express? What is the value of f' when $t = 4$, and what does it signify?
2. Suppose that $x = f(t) = t^2 - 4$ expresses the pressure in a pipe at point *t*. along the pipeline. What does f' express? What does it signify at $t = 0$?
3. Suppose that $x = f(t) = t^2 - 4$ expresses the enrollment at Heisenberg High at year *t*. What does f' express? What does it signify at $t = 2$?

Exercise A1.2

Calculate: a. $\frac{d}{dx}(4x^3)$; b. $\frac{d}{dx}2\cos x$; c. $\frac{d}{dx}\frac{2}{x^3}$; d. $\frac{d}{dy}\left(\sqrt{y}\right)$; e. $\frac{d}{dx}x^{\frac{3}{4}}$; f. $\frac{d}{dx}3$; g. $\frac{d}{dx}5x^{\frac{6}{5}}$; h. $\frac{d}{dx}\sqrt{x^5}$; i. $\frac{d}{dx}x$; j. $\frac{d}{dx}\frac{1}{x}$.

Exercise A1.3

Calculate the derivatives of the following functions: a. $3x - \sin x$; b. $\sin x \cos x$; c. $3x^2 - \sin x^3$; d. $\cos(2x - x^3)$; e. $\cos^3 5x$ (apply the chain rule twice); f. $\sin\frac{1}{x}$; g. $4x^{-2}\cos\sqrt{x^5}$; h. $\cos^4 x$; i. $\cos x^4$.

Exercise A1.4

1. Suppose that the equation of motion of a particle is $x = t^2 + 3t - 2$. Determine the instantaneous velocity and acceleration at time $t = 1$.
2. Suppose that the equation of motion of a particle is $x = 2t^2 + 4t - 3$. Determine the instantaneous velocity and acceleration at time $t = 4$.

Exercise A1.5

1. Find $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial^2}{\partial y}$ for: a. $z = y^3 - 3x^2y^2$; b. $z = x^2 + 4x^2y$; c. $z = xe^{-y}$; d. $z = \frac{x^2}{y}$.

2. Suppose that the temperature at point (x, y) is xy^2. What is the rate of change of temperature as one moves through point $(2,1)$ in the *x*-direction? In the *y*-direction?

Answers to the Exercises

Exercise A1.1

1. The first derivative expresses the rate of change of temperature; $f'(4)=8$ tells us that at point 4, the temperature increases 8 degrees for every meter of positive displacement.

2. The first derivative expresses the rate of change of pressure; $f'(0)=0$ tells us that at point 0 the pressure increases by 0 units for every one unit of positive displacement.

3. The first derivative expresses the rate of change of enrollment; $f'(2)=4$ tells us that on year 2, the enrollment increases by 4 students per year.

Exercise A1.2

a: $12x^2$; b: $-2\sin x$; c: $\frac{d}{dx}\left(\frac{2}{x^3}\right)=\frac{d}{dx}2x^{-3}=-6x^{-4}=-\frac{6}{x^4}$;

d: $\frac{d}{dy}\sqrt{y}=\frac{d}{dy}y^{\frac{1}{2}}=\frac{1}{2}y^{-\frac{1}{2}}=\frac{1}{2\sqrt{y}}$; e: $\frac{3}{4}x^{-\frac{1}{4}}=\frac{3}{4\sqrt[4]{x}}$; f: 0; g: $6x^{\frac{1}{5}}$;

h: $\frac{d}{dx}\left(\sqrt{x^5}\right)=\frac{d}{dx}x^{\frac{5}{2}}=\frac{5}{2}x^{\frac{3}{2}}$; i: 1; 2j: $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}x^{-1}=-x^{-2}=-\frac{1}{x^2}$.

Exercise A1.3

a: $3-\cos x$; b: $\sin x\cdot D_x\cos x+D_x\sin x\cdot\cos x=-\sin^2 x+\cos^2 x$; c: $6x-3x^2\cos x^3$;

d: $(3x^2-2)\sin(2x-x^3)$; e: $3\cos^2 5x\cdot D_x\cos 5x=-15\cos^2 5x\cdot\sin 5x$;

f: $-\cos\frac{1}{x}\cdot D_x\frac{1}{x}=\frac{1}{x^2}\cos\frac{1}{x}$; g: $-8x^{-3}\cos\sqrt{x^5}-4x^2\sin\sqrt{x^5}\cdot\frac{5}{2}x^{\frac{3}{2}}=$

$=-8x^{-3}\cos\sqrt{x^5}-10x^{-2}x^{\frac{3}{2}}\sin\sqrt{x^5}=-8x^{-3}\cos\sqrt{x^5}-10x^{-\frac{1}{2}}\sin\sqrt{x^5}$; h: $-4\cos^3 x\cdot\sin x$;

i: $-4x^3\sin x^4$.

Exercise A1.4

1. $v_{t=1}=5$; $a_{t=1}=2$.

2. $v_{t=4} = 20$; $a = 4$.

Exercise A1.5

1a: $6xy^2; 6y^2; 3y^2 - 6x^2y; 6y - 6x^2$; 1b: $2x + 8xy; 2 + 8y; 4x^2; 0$; 1c: $e^{-y}; 0; -xe^{-y}; xe^{-y}$;

1d: $2xy^{-1}; 2y^{-1}; -x^2y^{-2}; 2x^2y^{-3}$.

2. If we move in the *x*-direction, then *y* remains constant, and the partial derivative needed is $\frac{\partial}{\partial x}(xy^2)\bigg|_{x=2;y=1} = y^2\big|_{x=2;y=1} = 1$. So, at point (2.1), as we move 1 meter in the *x*-direction the temperature increases 1 degree. If we move in the *y*-direction, then *x* remains constant, and the partial derivative needed is $\frac{\partial}{\partial y}(xy^2)\bigg|_{x=2;y=1} = 2xy\big|_{x=2;y=1} = 4$. So, at point (2.1), as we move 1 meter in the *y*-direction the temperature increases 4 degree.