Generalizations and Relationships of the Descartes Circle Theorem

Mara Holloway

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1 Background and Introduction

Dating back to the mathematics studies in ancient Greece, circles and their properties have been studied by hundreds of mathematicians. While it is interesting enough to study a single circle, more possibilities arise when considering a collection of circles in relation to one another. In the mid-1600s, Rene Descartes first introduced a relation connecting the four radii of what would come to be known as Descartes Configurations. For centuries, many mathematicians made discoveries that eventually came together to prove the Descartes Circle Theorem and many generalizations of the relation in it.

Given an arbitrary set of four circles, there is no reason to suspect that the circles will interact in a special way. However, if we begin to impose a few particular conditions, several interesting relationships arise. Descartes configurations present an arrangement of circles with specific properties that lead to surprising relationships. Investigating generalizations of the Descartes Circle Theorem allows for exploration into these relationships. These investigations will be guided primarily by the paper *Beyond the Descartes Circle Theorem* by J. Lagarias, C. Mallows, and A. Wilks [1].

Definition 1. [1] A *Descartes configuration* is an arrangement of four mutually tangent circles in the plane, in which no three circles share a tangent. If the radii of these circles are r_1, r_2, r_3, r_4, then the curvatures are $b_j = \frac{1}{r_j}$.

The four general cases for Descartes configurations are shown in Figure 1. The

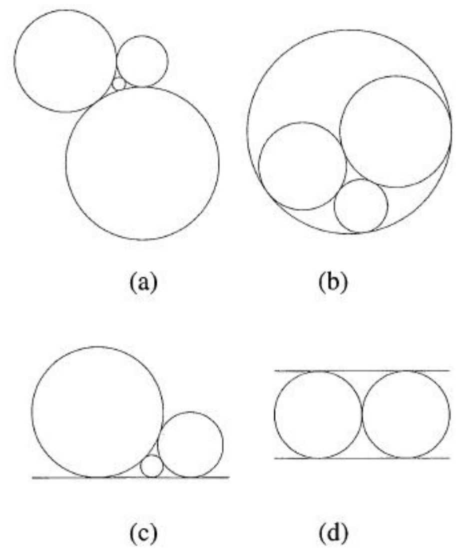

Figure 1: Possible Descartes Configurations [1]

Descartes Circle Theorem relates the curvatures of the circles in a Descartes configuration in an interesting way.

Theorem 1. [1] ***(Descartes Circle Theorem).*** *In a Descartes configuration of circles, the curvatures satisfy*

$$\sum_{j=1}^{4} b_j^2 = \frac{1}{2}\left(\sum_{j=1}^{4} b_j\right)^2. \tag{1}$$

Introducing further specifications involving the centers and orientations of the circles, the surprising relationships continue to develop. First, we can incorporate the centers of the circles to state Theorem 2. The next restriction to impose is orientation. The notion of orientation will be necessary to generalize Theorem 1 to higher dimensions, so I will begin by expanding upon this concept.

Definition 2. [1] An *oriented circle* is a circle together with an assigned direction of unit normal vector, which can point inward or outward.

Definition 3. [1] An *oriented Descartes configuration* is a Descartes configuration in which the orientations of the circles are compatible in the following

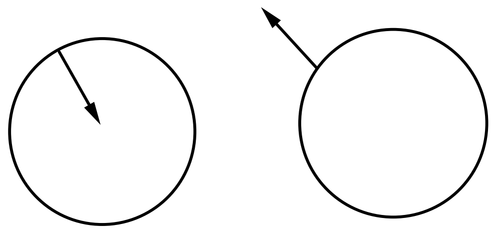

Figure 2: Inward vs. Outward Orientation

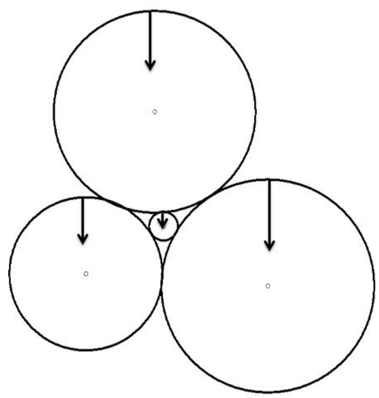

Figure 3: Oriented Descartes Configuration

sense: either (i) the interiors of all four circles are disjoint, or (ii) the interiors are disjoint when all orientations are reversed.

Figure 2 illustrates the two distinct types of orientation: inward and outward. Figure 3 gives one example of an oriented Descartes configuration. Notice that if the orientations were to be changed for up to three of the circles in this specific example, it would no longer fit the definition for an oriented Descartes configuration.

We will now begin to consider Descartes configurations under specific conditions. First, we will incorporate the idea of centers, then we will explore configurations in higher dimensions of real space.

2 Euclidean Generalizations of the Descartes Circle Theorem

The first instance in which we explore a generalization of the Descartes Circle Theorem is to investigate the complex version. Then we will begin representing the relationships between these circles using linear algebra, a technique that will be used throughout the remainder of these discussions.

Theorem 2. [1] ***(Complex Descartes Theorem).*** *Any Descartes configuration of four mutually tangent circles with curvatures b_j and centers $z_j = x_j + iy_j$ satisfies*

$$\sum_{j=1}^{4}(b_j z_j)^2 = \frac{1}{2}\left(\sum_{j=1}^{4} b_j z_j\right)^2. \tag{2}$$

The relations resulting from Equation 2 can be expressed using linear algebra. But first, we must establish how the matrices used are developed from the available information.

Let $n \in \mathbb{N}$ and define $\mathbf{Q}_n$ by

$$\mathbf{Q}_n := I_{n+2} - \frac{1}{n}\mathbf{1}_{n+2}\mathbf{1}_{n+2}^T,$$

where $\mathbf{1}_k$ denotes a column vector of k ones.

Therefore,

$$\mathbf{Q}_2 = I_4 - \frac{1}{2}\mathbf{1}_4\mathbf{1}_4^T = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

We call $\mathbf{Q}_2$ the coefficient matrix of the *Descartes quadratic form*

$$Q_2(x_1, x_2, x_3, x_4) := \mathbf{x}^T\mathbf{Q}_2\mathbf{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - \frac{1}{2}(x_1 + x_2 + x_3 + x_4)^2.$$

This relates directly to Equations 1 and 2. If $b = (b_1, b_2, b_3, b_4)$ then $\mathbf{Q}_2(\vec{b}) = 0$ if and only if Equation 1 is satisfied. Similarly, if $b = (b_1z_1, b_2z_2, b_3z_3, b_4z_4)$

then $\mathbf{Q}_2(\vec{b}) = 0$ if and only if Equation 2 is satisfied. Using the curvatures and centers of the circles in a Descartes configuration along with $\mathbf{Q}_2$, we can develop the Extended Descartes Theorem.

Theorem 3. [1] ***(Extended Descartes Theorem).*** *Given a configuration of four oriented circles with nonzero curvatures* (b_1, b_2, b_3, b_4) *and centers* $\{(x_i, y_i) : 1 \leq i \leq 4\}$, *let* $\boldsymbol{M}$ *be the* 4×3 *matrix*

$$\boldsymbol{M} := \begin{bmatrix} b_1 & b_1x_1 & b_1y_1 \\ b_2 & b_2x_2 & b_2y_2 \\ b_3 & b_3x_3 & b_3y_3 \\ b_4 & b_4x_4 & b_4y_4 \end{bmatrix}.$$

Then this configuration is an oriented Descartes configuration if and only if

$$\boldsymbol{M}^T \boldsymbol{Q}_2 \boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Theorem 3 will be of great importance since it contains an if and only if statement about oriented Descartes configurations. For now, however, we return to the idea of orientation in order to generalize the results of Theorem 1 to apply to $\mathbb{R}^n$. In these higher dimensions, the four circles we were considering are now $(n+2)$ mutually tangent $(n-1)$- spheres in $\mathbb{R}^n$, no $(n+1)$ of which share a tangent hyperplane. In order to make these generalizations, we also introduce the concepts of signed curvature and curvature-centered coordinates.

Definition 4. Given an oriented circle, its *signed curvature*, b, gives the curvature and direction of orientation. If a circle is considered to be oriented outward, then $b > 0$; if a circle is considered to be oriented inward, then $b < 0$.

Definition 5. [1] Given an oriented sphere S in $\mathbb{R}^n$, its *curvature-centered coordinates* consist of the $(n+1)$-vector

$$\mathbf{m}(S) = (b, bx_1, bx_2, ..., bx_n),$$

in which b is the signed curvature of S and $\mathbf{x}(S) = (x_1, x_2, ..., x_n)$ is its center.

Extending Theorem 3 to n dimensions and incorporating the idea of curvature-centered coordinates leads to the Euclidean Generalized Descartes Theorem.

Theorem 4. [1] ***(Euclidean Generalized Descartes Theorem).*** *Given a configuration of $n+2$ oriented spheres $S_1, S_2, .., S_{n+2}$ in $\mathbb{R}^n$, let $\boldsymbol{M}$ be the $(n+2) \times (n+1)$ matrix whose jth row entries are the curvature-center coordinates $\boldsymbol{m}(S_j)$ of the jth sphere. If this configuration is an oriented Descartes configuration, then*

$$\boldsymbol{M}^T \boldsymbol{Q}_n \boldsymbol{M} = \begin{bmatrix} 0 & 0 \\ 0 & 2I_n \end{bmatrix}.$$

Conversely, any real solution $\boldsymbol{M}$ to the above equation is the matrix containing the curvature-center coordinates of a unique oriented Descartes configuration.

In order to make further generalizations from Theorem 4 we must be able to invert the original oriented Descartes configuration in the unit sphere. We establish a notion of augmented curvature center coordinates.

Definition 6. [1] *Inversion* in the unit sphere maps the point $\mathbf{x}$ to $\mathbf{x}/|\mathbf{x}|^2$ where

$$|\mathbf{x}|^2 = \sum_{j=1}^{n} x_j^2.$$

Next we aim to understand how this mapping applies to the center and radius of an inverted circle in the unit sphere. In order to do this we use the following lemma and proposition.

Lemma 5. [4] *Let O be a point not lying on circle S. If two lines through O intersect S in pairs of points (P_1, P_2) and (Q_1, Q_2), respectively, then $\overline{OP_1} \cdot \overline{OP_2} = \overline{OQ_1} \cdot \overline{OQ_2}$. This common product is called the power of O with respect to S when O is outside S, and minus this number is called the power of O when O is inside S.*

Proposition 1. [4] *Let U be a circle of radius u and center O, S a circle of radius r and center x. Assume that O lies outside S; let p be the power of O with respect to S. Let $k = u^2/p$. Then the image $\bar{S}$ of S under inversion in U is the circle of radius kr whose center is the image x^* of x under the dilation from O of ratio k.*

Since we are interested in inversion in the unit sphere, let U be the unit sphere with radius $u = 1$ and center $O = (0, 0)$. Let S be a circle of radius r and center $\mathbf{x}$. Based on Lemma 5, the power of O is $p = |\mathbf{x}|^2 - r^2$ and $k = u^2/p = 1/|\mathbf{x}|^2 - r^2$. So the image $\bar{S}$ has radius $kr = r/|\mathbf{x}|^2 - r^2$ and center $k\mathbf{x} = \mathbf{x}/\left|\mathbf{x}\right|^2 - r^2$. Therefore, an oriented sphere S with center $\mathbf{x}$ and oriented radius r inverts to the sphere $\bar{S}$ with center $\bar{\mathbf{x}} = \mathbf{x}/(\left|\mathbf{x}\right|^2 - r^2)$ and oriented radius $\bar{r} = r/(\left|\mathbf{x}\right|^2 - r^2)$. In all cases,

$$\frac{\bar{\mathbf{x}}}{\bar{r}} = \frac{\mathbf{x}/(\left|\mathbf{x}\right|^2 - r^2)}{r/(\left|\mathbf{x}\right|^2 - r^2)} = \frac{\mathbf{x}}{r} \tag{3}$$

and

$$\bar{b} = \frac{1}{\bar{r}} = \frac{(\left|\mathbf{x}\right|^2 - r^2)}{r} = \frac{\left|\mathbf{x}\right|^2}{r} - r.$$

If we add a column to the $(n+2) \times (n+1)$ matrix $\mathbf{M}$ (from Theorem 4) using information about the inversion of the oriented Descartes configuration, it can be expanded into an $(n+2) \times (n+2)$ matrix $\mathbf{W}$.

Definition 7. [1] Given an oriented sphere S in $\mathbb{R}^n$, its *augmented curvature-center coordinates* are the $(n+2)$-vector

$$\mathbf{w}(S) := (\bar{b}, b, bx_1, ..., bx_n) = (\bar{b}, \mathbf{m}),$$

in which $\bar{b}$ is the oriented curvature of the sphere $\bar{S}$ obtained by inversion of S in the unit sphere, and the entries of $\mathbf{m}$ are the curvature-center coordinates of S.

Augmented curvature-center coordinates are unique to distinct spheres. Because the augmented curvature-center coordinates are found by using both the center and the curvature (which is found using the radius), the only way two spheres could have the same augmented curvature center coordinates would be for them to have the same center and same radius. In this case, the spheres would not be distinct.

Definition 8. [1] Given a collection of $(S_1, S_2, ..., S_{n+2})$ of $n+2$ oriented spheres in $\mathbb{R}^n$, the *augmented matrix* $\mathbf{W}$ associated with it is the $(n+2)\times(n+2)$ matrix whose jth row has entries given by the augmented curvature-center coordinates $\mathbf{w}(S_j)$ of the jth sphere.

Augmented matrix coordinates provide a simple result with inversion in the unit sphere. The radii of the original and inverted spheres can be related to each other via curvatures. The action of inverting an inverted sphere results in obtaining the original sphere. Therefore, $\bar{\bar{b}} = b$. Also, by Equation 3 we can see that $\bar{x}\bar{b} = xb$. Thus we form $\mathbf{W}'$, the augmented matrix associated with an inverted sphere, by merely interchanging the first two columns of $\mathbf{W}$. This can be represented mathematically in the following way

$$\mathbf{W}' = \mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

We will modify Theorem 4 to replace curvature center coordinates with their augmented counterparts. This result gives the Augmented Euclidean Descartes Theorem.

Theorem 6. [1] ***(Augmented Euclidean Descartes Theorem)*** *The augmented matrix* $\mathbf{W}$ *of an oriented Descartes configuration of* $n+2$ *spheres* $\{S_i : 1 \leq i \leq n+2\}$ *in* $\mathbb{R}^n$ *satisfies*

$$\mathbf{W}^T \mathbf{Q}_n \mathbf{W} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_n \end{bmatrix}. \tag{4}$$

Conversely, any real solution $\mathbf{W}$ *to Equation 4 is the augmented matrix of a unique oriented Descartes configuration.*

Theorem 6 will be proven later, however we must first delve into spherical generalizations of the Descartes Circle Theorem. Once the spherical results of the Descartes Circle Theorem are proven, we will be able to prove the accompanying Euclidean generalizations.

3 Spherical Generalizations of the Descartes Circle Theorem

Now we turn our attention to spherical geometry in order to expand upon these investigations. First, we clarify the spherical analogs to the already-established Euclidean forms. The standard model for spherical geometry is the unit n-sphere S^n in $\mathbb{R}^{n+1}$ as

$$S^n := \{\mathbf{y} : y_0^2 + y_1^2 + ... + y_n^2 = 1\}.$$

The distance between two points of S^n is the angle α between the radii that join the origin of $\mathbb{R}^{n+1}$ to these points on S^n, such that $0 \leq \alpha \leq \pi$. We know this to be the distance because of the relationship between central angles and arc length. Arc length is calculated by taking a fraction of the circumference proportional to the central angle intercepting the given arc. Since the radius is 1 in the unit sphere, the circumference of the circle is equal to 2π, the amount of radians in an entire circle. Therefore, the central angle, α determines and is equal to the arc length. A *sphere* C is the locus of points that are equidistant from a point in S^n called its *center*. There are two choices for the center of a given sphere which form a pair of antipodal points of S^n. The choice of center amounts to orienting the sphere. The interior of a sphere is a spherical cap, formed by the intersection of the sphere S^n and a plane in $\mathbb{R}^{n+1}$. Thus, a *spherical cap* is an oriented sphere. The two spherical caps formed by the intersection of a single plane with a given sphere are called *complementary*. The *interior* of an oriented sphere contains all points of S^n on the same side of the plane as the center of the sphere.

Similarly to its analog in Euclidean geometry, a *spherical Descartes configuration* consists of $n+2$ mutually tangent spherical caps on the surface of the unit n-sphere such that either (i) the interiors of all spherical caps are mutually disjoint, or (ii) the interiors of all complementary spherical caps are mutually disjoint. The idea of spherical Descartes configurations along with an understanding of spherical curvature-center coordinates will lead us to the Spherical Generalized Descartes Theorem.

Definition 9. [1]If C is a spherical cap with center $\mathbf{y} = (y_0, y_1, ..., y_n)$ and angular radius α, the *spherical curvature-center coordinates* $\mathbf{w}_+(C)$ are defined by

$$\mathbf{w}_+(C) := (w_{-1}, w_0, w_1, ...w_n) = (\cot\alpha, \frac{y_0}{\sin\alpha}, \frac{y_1}{\sin\alpha}, ..., \frac{y_n}{\sin\alpha}). \tag{5}$$

(Note: This indexing accounts for the $(n+2)$ entries and will be consistent with the indexing for curvature center coordinates, discussed later.)

We obtain the $(n+2)\times(n+2)$ *spherical curvature-center coordinate matrix* $\mathbf{W}_+$ by making each row the spherical curvature-center coordinates of the corresponding $n+2$ caps.

Now that we have established a basic understanding of spherical curvature-center coordinates, we use this knowledge to present the Spherical Generalized Descartes Theorem. This theorem is an integral part of the proof of Theorem 6.

Theorem 7. [1] ***(Spherical Generalized Descartes Theorem).*** *Consider a configuration of n+2 oriented spherical caps, C_j, that is a spherical*

Descartes configuration. The $(n+2) \times (n+2)$ matrix $\boldsymbol{W}_+$ whose jth row is the spherical curvature-center coordinates of C_j satisfies

$$\boldsymbol{W}_+^T \boldsymbol{Q}_n \boldsymbol{W}_+ = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2I_n \end{bmatrix}. \tag{6}$$

Conversely, any real matrix $\boldsymbol{W}_+$ that satisfies Equation 6 is the spherical curvature-center coordinate matrix of some spherical Descartes configuration.

In order to prove Theorem 7, we introduce three lemmas. Let $\mathbf{J}_n$ be the $(n+2) \times (n+2)$ matrix

$$\mathbf{J}_n = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

Lemma 8. [1]

(i) For any $(n+2)$-vector $\boldsymbol{w}_+$, there is a spherical cap C with $\boldsymbol{w}_+(C)$=$\boldsymbol{w}_+$ if and only if

$$\boldsymbol{w}_+ \boldsymbol{J}_n \boldsymbol{w}_+^T = 1. \tag{7}$$

(ii)The spherical caps C and C' are externally tangent if and only if

$$\boldsymbol{w}_+(C) \boldsymbol{J}_n \boldsymbol{w}_+(C')^T = -1. \tag{8}$$

Proof. (i) Let C be a spherical cap with $\mathbf{w}_+(C)$=$\mathbf{w}_+ = (w_{-1}, w_0, ...w_n)$. Multiplying $\mathbf{w}_+$ by $\mathbf{J}_n$ will result in $\mathbf{w}_+$ with the first entry negated. All other entries will remain the same. We can then think of the entries of this row vector as being $(-w_{-1}, w_0, ...w_n)$. Multiplying this result by the column vector $\mathbf{w}_+^T$ results in a (1×1) matrix with the entry $(-w_{-1}^2 + w_0^2 + ... + w_n^2)$. Based on the definition given in Equation 5, these entries can be rewritten using trigonometric functions and the curvature centered coordinates from $\mathbf{y}$. Therefore, if $\mathbf{w}_+$ comes from a spherical cap with center $\mathbf{y}$ and angular radius α, then

$$\mathbf{w}_+ \mathbf{J}_n \mathbf{w}_+^T = \frac{-(\cos\alpha)^2 + \sum_{i=0}^{n} y_i^2}{(\sin\alpha)^2}. \tag{9}$$

Because we are working on a unit n-sphere, the sum of the squares of the curvature centered coordinates must equal 1. Thus, we can rewrite the result from Equation 9 in the following way

$$\frac{-(\cos\alpha)^2+\sum_{i=0}^{n} y_i^2}{(\sin\alpha)^2} = \frac{1-(\cos\alpha)^2}{(\sin\alpha)^2} = 1.$$

If we take Equation 7 to be true, then we could use the same mathematical concepts in the reverse order to obtain Equation 9. Define α by setting $\cot\alpha := (\mathbf{w}_+)_{-1}$ (note: $\alpha \neq 0$), and define a vector $\mathbf{y} = (y_0, y_1, ..., y_n)$ by $y_i := (\mathbf{w}_+)_i \cdot \sin\alpha$. Using these definitions,

$$\begin{aligned}
\mathbf{w}_+\mathbf{J}_n\mathbf{w}_+^T &= 1\\
-w_{-1}^2 + w_0^2 + w_1^2 + ... + w_n^2 &= 1\\
-(\cot\alpha)^2 + \frac{\sum_{i=0}^{n} y_i^2}{\sin\alpha^2} &= 1\\
\frac{-\cos\alpha^2 + \sum_{i=0}^{n} y_i^2}{\sin\alpha^2} &= 1\\
-\cos\alpha^2 + \sum_{i=0}^{n} y_i^2 &= \sin\alpha^2\\
\sum_{i=0}^{n} y_i^2 &= \sin\alpha^2 + \cos\alpha^2\\
\sum_{i=0}^{n} y_i^2 &= 1.
\end{aligned}$$

Therefore, the sum of the squares of the curvature centered coordinates equals one. Thus, $\mathbf{y}$ lies on the unit sphere, and there is a spherical cap that gives the vector $\mathbf{w}_+$.

(ii) Let C and C' be externally tangent spherical caps with centers $\mathbf{y}$ and $\mathbf{y}'$, and angular radii, α and α'. Because the spheres are externally tangent, the angle between their centers is $\alpha + \alpha'$.

The computation of $\mathbf{y}(\mathbf{y}')^T$ to find the distance between the centers amounts to taking the dot product of the two vectors. Therefore,

$$\mathbf{y}(\mathbf{y}')^T = |\mathbf{y}||(\mathbf{y}')^T|\cos\theta$$

where θ is the angle between the vectors. The magnitudes of both vectors are 1 since we are working on a unit sphere. Therefore, since $\theta = \alpha + \alpha'$,

$$\mathbf{y}(\mathbf{y}')^T = \cos(\alpha + \alpha'). \tag{10}$$

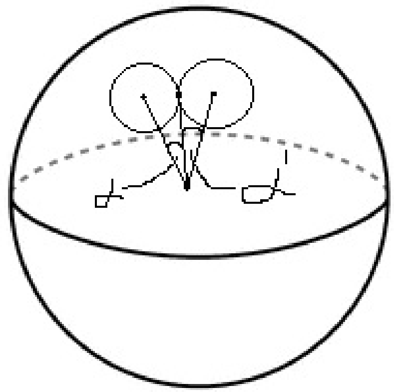

Figure 4: Sphere with externally tangent spherical caps

Now,

$$\begin{aligned}
\mathbf{w}_+\mathbf{J}_n\mathbf{w}_+^T &= -w_{-1}w'_{-1} + w_0w'_0 + ... + w_nw'_n \\
&= -\cot\alpha\cot\alpha' + \frac{y_0y'_0}{\sin\alpha\sin\alpha'} + ... + \frac{y_ny'_n}{\sin\alpha\sin\alpha'} \\
&= \frac{1}{\sin\alpha\sin\alpha'}(-\cos\alpha\cos\alpha' + \mathbf{y}(\mathbf{y}')^T).
\end{aligned}$$

Using Equation 10 and the trigonometric identity

$$\cos(\alpha+\alpha') = \cos\alpha\cos\alpha' - \sin\alpha\sin\alpha'$$

we obtain

$$\frac{1}{\sin\alpha\sin\alpha'}(-\cos\alpha\cos\alpha' + \mathbf{y}(\mathbf{y}')^T) =$$
$$\frac{1}{\sin\alpha\sin\alpha'}(-\cos\alpha\cos\alpha' + \cos\alpha\cos\alpha' - \sin\alpha\sin\alpha') = -1.$$

Therefore, Equation 8 holds. Next, assume $\mathbf{w}_+(C)\mathbf{J}_n\mathbf{w}_+(C')^T = -1$ for two spherical caps, C and C'. Using the same mathematical logic as in the previous part of the proof we know that

$$\begin{aligned}
\mathbf{w}_+\mathbf{J}_n\mathbf{w}_+^T &= -w_{-1}w'_{-1} + w_0w'_0 + ... + w_nw'_n \\
&= \frac{1}{\sin\alpha\sin\alpha'}(-\cos\alpha\cos\alpha' + \mathbf{y}(\mathbf{y}')^T) = -1.
\end{aligned}$$

Solving for $\mathbf{y}(\mathbf{y}')^T$ and using the cosine addition formula gives $\mathbf{y}(\mathbf{y}')^T = \cos(\alpha + \alpha')$. Thus, C and C' are externally tangent when $\mathbf{w}_+(C)\mathbf{J}_n\mathbf{w}_+(C')^T = -1$. $\square$

Lemma 9. [1] *If we assume that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are non-singular* $n \times n$ *matrices and* $\mathbf{W}\mathbf{A}\mathbf{W}^T = \mathbf{B}$*, then* $\mathbf{W}^T\mathbf{B}^{-1}\mathbf{W} = \mathbf{A}^{-1}$.

Proof. If the determinant of matrix $\mathbf{W}$ equals zero, then the determinant of matrix $\mathbf{B}$ would have to be zero, which is a contradiction because $\mathbf{B}$ is non-singular. Therefore, the determinant of $\mathbf{W}$ is not zero and $\mathbf{W}$ is also non-singular. If we invert both sides of $\mathbf{WAW}^T$=$\mathbf{B}$ then we have $(\mathbf{W}^T)^{-1}\mathbf{A}^{-1}\mathbf{W}^{-1} = \mathbf{B}^{-1}$. Multiplying on the left on both sides of the equation by $\mathbf{W}^T$ and on the right on both sides of the equation by $\mathbf{W}$ gives $\mathbf{W}^T\mathbf{B}^{-1}\mathbf{W}$=$\mathbf{A}^{-1}$. $\square$

Lemma 10. *For all* $n \in \mathbb{N}$,

$$2\mathbf{I}_{n+2} - \mathbf{1}_{n+2}\mathbf{1}_{n+2}^T = 2\mathbf{Q}_n^{-1}.$$

Proof. This relationship will be shown in a general manner for the case $n = 2$, however the same approach can be applied for all values of n. We begin by expanding $\mathbf{Q}_n$ in order to find the inverse of the matrix. Expanding $\mathbf{Q}_n$ gives

$$\mathbf{Q}_n = \begin{bmatrix} 1-1/n & -1/n & -1/n & -1/n \\ -1/n & 1-1/n & -1/n & -1/n \\ -1/n & -1/n & 1-1/n & -1/n \\ -1/n & -1/n & -1/n & 1-1/n \end{bmatrix}$$
$$= \frac{1}{n}\begin{bmatrix} n-1 & -1 & -1 & -1 \\ -1 & n-1 & -1 & -1 \\ -1 & -1 & n-1 & -1 \\ -1 & -1 & -1 & n-1 \end{bmatrix}.$$

We will find the inverse of the matrix by using elemtary row operations on an augmented matrix. The result will then be multiplied by n, the inverse of $1/n$.

$$\left[\begin{array}{cccc|cccc} n-1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & n-1 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & n-1 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & n-1 & 0 & 0 & 0 & 1 \end{array}\right] =$$
$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{n-3}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} \\ 0 & 1 & 0 & 0 & \frac{1}{n(n-4)} & \frac{n-3}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} \\ 0 & 0 & 1 & 0 & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{n-3}{n(n-4)} & \frac{1}{n(n-4)} \\ 0 & 0 & 0 & 1 & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{n-3}{n(n-4)} \end{array}\right].$$

So we have

$$\mathbf{Q}_n^{-1} = n \begin{bmatrix} \frac{n-3}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} \\ \frac{1}{n(n-4)} & \frac{n-3}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} \\ \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{n-3}{n(n-4)} & \frac{1}{n(n-4)} \\ \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{1}{n(n-4)} & \frac{n-3}{n(n-4)} \end{bmatrix} = \begin{bmatrix} \frac{n-3}{n-4} & \frac{1}{n-4} & \frac{1}{n-4} & \frac{1}{n-4} \\ \frac{1}{n-4} & \frac{n-3}{n-4} & \frac{1}{n-4} & \frac{1}{n-4} \\ \frac{1}{n-4} & \frac{1}{n-4} & \frac{n-3}{n-4} & \frac{1}{n-4} \\ \frac{1}{n-4} & \frac{1}{n-4} & \frac{1}{n-4} & \frac{n-3}{n-4} \end{bmatrix}.$$

Because we were doing this for the case of $n = 2$, we can rewrite the constants 3 and 4 in terms of n. Thus,

$$\mathbf{Q}_n^{-1} = \begin{bmatrix} \frac{n-(n+1)}{n-(n+2)} & \frac{1}{n-(n+2)} & \frac{1}{n-(n+2)} & \frac{1}{n-(n+2)} \\ \frac{1}{n-(n+2)} & \frac{n-(n+1)}{n-(n+2)} & \frac{1}{n-(n+2)} & \frac{1}{n-(n+2)} \\ \frac{1}{n-(n+2)} & \frac{1}{n-(n+2)} & \frac{n-(n+1)}{n-(n+2)} & \frac{1}{n-(n+2)} \\ \frac{1}{n-(n+2)} & \frac{1}{n-(n+2)} & \frac{1}{n-(n+2)} & \frac{n-(n+1)}{n-(n+2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

□

We now have the necessary information to prove the Spherical Generalized Descartes Theorem.

Proof. The matrix $\mathbf{W}_+\mathbf{J}_n\mathbf{W}_+^T$ provides information about the relationships between n spherical caps. By Lemma 8, if the entries along the diagonal of a matrix are 1 the information used to obtain those values describes a spherical cap. If the other entries in the matrix are -1 then the caps whose coordinates were used to calculate that value are externally tangent. Therefore, if the caps C_j form a spherical Descartes configuration, Lemma 8 ensures that

$$\mathbf{W}_+\mathbf{J}_n\mathbf{W}_+^T = 2\mathbf{I}_{n+2} - \mathbf{1}_{n+2}\mathbf{1}_{n+2}^T = 2\mathbf{Q}_n^{-1}.$$

By applying Lemma 9 to this result (with $\mathbf{A} = \mathbf{J}_n = \mathbf{J}_n^{-1}$ and $\mathbf{W} = \mathbf{W}_+$) we obtain

$$\mathbf{W}_+^T\mathbf{Q}_n\mathbf{W}_+ = 2\mathbf{J}_n^{-1} = 2\mathbf{J}_n.$$

□

In order to translate the proof of Theorem 7 back into Euclidean geometry, the idea of stereographic projection must be employed. As a reference for stereographic projection we use [2]. This explanation will be presented in $\mathbb{R}^3$, however it can be generalized to higher dimensions. Let the Euclidean plane be represented by $y_0 = 0$ in three-space and consider the unit sphere, $y_0^2 + y_1^2 + y_2^2 = 1$, then for any point $Q = (0, Q_0, Q_1)$ in the Euclidean plane, draw a line through Q and the South Pole of the sphere $P = (-1, 0, 0)$. This line intersects the sphere at point Q'. In this way, the entire Euclidean plane can be mapped to the sphere, with the exclusion of one point (the North Pole, also called the point of projection). Mappings by stereographic projection hold the following special properties. We state without proof this well known theorem.

Theorem 11. [3] *Stereographic projection maps lines and circles in the plane to circles on the sphere, and, conversely, circles on the sphere map stereographically to lines and circles in the plane.*

Definition 10. [3] A mapping of a subset of the plane into the plane is said to be *conformal* at a point P if it preserves the angle between any two curves at P.

Because the mapping of stereographic projection is conformal and bijective, we can use the same idea to map the sphere to the plane.

For the sake of this project, we will instead consider the point of projection, P, to be located at the "South Pole", or $P = (-1, 0, ..., 0)$ on the unit sphere in $\mathbb{R}^{n+1}$. The points on this sphere will be mapped to the plane $y_0 = 0$ by stereographic projection. If we are mapping $(y_0, y_1, ..y_n)$ to $(x_1, x_2, ..., x_n)$, we can determine the equation of the line through the point of projection and the initial point. If we have the equation of the line through the point of projection and a point on the sphere, we can find where the line intersects the plane $y_0 = 0$. We can move between P and the initial point, Q, by the expression $(1-t)(y_0, y_1, ..., y_n) = t(-1, 0, ..., 0)$ where we obtain point P when $t = 1$ and we obtain point Q when $t = 0$. Expanding this to give the expressions for individual coordinates along the line gives a generalization that can be used for other points, (i.e. $(x_1, x_2, ..., x_n)$). This expansion is $((1-t)y_0 - t, (1-t)y_1, ..., (1-t)y_n)$. Because we are wanting to see where the line intersects the plane $y_0 = 0$, we need to find the value for t that will make our 1st coordinate zero. Therefore,

$$
\begin{aligned}
(1-t)y_0 - t &= 0 \\
y_0 - ty_0 - t &= 0 \\
y_0 - t(y_0+1) &= 0 \\
t &= \frac{y_0}{y_0+1}.
\end{aligned}
$$

We can find the remaining coordinates by substituting this value in for t. For example, the second coordinate, x_2 can be found as follows,

$$x_2 = (1-t)y_1 = \left(1 - \frac{y_0}{y_0+1}\right) y_1 = \frac{y_1}{y_0+1}.$$

Generalizing this to all subsequent coordinates, the mapping of $(y_0, ..., y_n) \to (x_1, ..., x_n)$ is given by

$$x_i = \left(\frac{y_i}{1+y_0}\right), 1 \leq i \leq n.$$

A spherical cap C with center $(p_0, ..., p_n)$ and angular radius α is the intersection of the unit sphere with the plane

$$\sum_{i=1}^{n} p_i y_i = \cos\alpha \tag{11}$$

where y_i is a variable point on the plane. If we take $\mathbf{y}$ to be on the edge of the spherical cap C, then

$$
\begin{aligned}
\sum_{i=1}^{n} p_i y_i &= \mathbf{y} \cdot \mathbf{p} \\
&= |\mathbf{y}||\mathbf{p}| \cos\alpha \\
&= 1 \cdot 1 \cos\alpha.
\end{aligned}
$$

If $\mathbf{y}$ lies elsewhere on the plane, we can write $\mathbf{y}$ in terms of $\mathbf{z}$, a point that does lie on the edge of C. Let $\mathbf{y} = (\mathbf{y} - \mathbf{z}) + \mathbf{z}$. Note that $(\mathbf{y} - \mathbf{z})$ is perpendicular to $\mathbf{p}$. So we have

$$\begin{aligned}\sum_{i=1}^{n} p_i y_i &= \mathbf{y} \cdot \mathbf{p} \\ &= ((\mathbf{y} - \mathbf{z}) + \mathbf{z}) \cdot \mathbf{p} \\ &= \mathbf{z} \cdot \mathbf{p} \\ &= \cos \alpha.\end{aligned}$$

Finding the mapping of the stereographic projection of cap C in the plane $y_0 = 0$ is accomplished using a similar technique, however this time we are looking for the inverse of the projection. To do this we find the vector resulting from

$$t(-1, 0, 0) + (1 - t)(0, x_1, x_2), \tag{12}$$

which is

$$(-t, (1 - t)x_1, (1 - t)x_2).$$

The magnitude of this vector must equal one since we are still working on a unit sphere. Therefore, we set the sum of the squares of the components equal to one in order to solve for t. The initial step is represented mathematically by

$$(-t)^2 + ((1 - t)x_1)^2 + ((1 - t)x_2)^3 = 1.$$

Therefore,

$$t = 1, t = \frac{-1 + {x_1}^2 + {x_2}^2}{1 + {x_1}^2 + {x_2}^2}.$$

Because $t = 1$ is a trivial case, we will focus on the value $t = \frac{-1+{x_1}^2+{x_2}^2}{1+{x_1}^2+{x_2}^2}$. Substituting this value in for t in expression 12 and simplifying gives the vector

$$\left(-1 + \frac{2}{1 + {x_1}^2 + {x_2}^2}, \frac{2x_1}{1 + {x_1}^2 + {x_2}^2}, \frac{2x_2}{1 + {x_1}^2 + {x_2}^2}\right).$$

Based on Equation 11, if this vector is dotted with the vector (p_0, p_1, p_2), the result should be $\cos \alpha$. Rearranging this equation to be equal to zero, the resulting quadratic equation can then be put into standard form for a circle, making the center and radius easily identifiable. Therefore,

$$\left(-1 + \frac{2}{1 + {x_1}^2 + {x_2}^2}, \frac{2x_1}{1 + {x_1}^2 + {x_2}^2}, \frac{2x_2}{1 + {x_1}^2 + {x_2}^2}\right) \cdot (p_0, p_1, p_2) - \cos \alpha = 0$$

results in

$$\left(x_1 - \frac{p_1}{p_0 + \cos\alpha}\right)^2 + \left(x_2 - \frac{p_2}{p_0 + \cos\alpha}\right)^2 = \frac{1 - \cos\alpha^2}{(p_0 + \cos\alpha)^2}.$$

Generalizing this result to higher dimensions, the intersection of the cap C in the plane $y_0 = 0$ is the sphere S with center $(x_1, ..., x_n)$ and radius r, where

$$x_i = \frac{p_i}{p_0 + \cos\alpha}, 1 \leq i \leq n, \tag{13}$$

and

$$r = \frac{\sin\alpha}{p_0 + \cos\alpha}. \tag{14}$$

Using this information, we are now able to prove Theorem 6. Working backwards, we will then also be able to prove other intermediate theorems and ultimately the Descartes Circle Theorem.

Proof of the Augmented Euclidean Descartes Theorem. By definition, the spherical curvature-center coordinates of a given spherical cap C are given by the vector

$$\mathbf{w}_+(C) = (\cot\alpha, \frac{p_0}{\sin\alpha}, \frac{p_1}{\sin\alpha}, ..., \frac{p_n}{\sin\alpha}).$$

This definition can be related to the augmented Euclidean curvature-center coordinate vector $\mathbf{w}(S)$, which is associated with the projected sphere S in the plane $y_0 = 0$. By Equations 13 and 14 we have

$$\frac{x_i}{r} = \frac{\frac{p_i}{p_0+\cos\alpha}}{\frac{\sin\alpha}{p_0+\cos\alpha}} = \frac{p_i}{\sin\alpha}$$

and

$$b = \frac{1}{r} = \frac{p_0 + \cos\alpha}{\sin\alpha} = \cot\alpha + \frac{p_0}{\sin\alpha}.$$

This results in

$$
\begin{aligned}
\bar{b} &= \frac{(|\mathbf{x}|^2 - r^2)}{r} \\
&= \frac{\frac{p_1^2+p_2^2+\ldots+p_n^2+p_0^2-p_0^2}{(p_0+\cos\alpha)^2} - \frac{\sin^2\alpha}{(p_0+\cos\alpha)^2}}{\frac{\sin\alpha}{p_0+\cos\alpha}} \\
&= \frac{1-p_0^2-\sin^2\alpha}{(p_0+\cos\alpha)^2}\cdot\frac{p_0+\cos\alpha}{\sin\alpha} \\
&= \frac{\cos^2\alpha - p_0^2}{(p_0+\cos\alpha)\cdot\sin\alpha} \\
&= \frac{(\cos\alpha+p_0)(\cos\alpha-p_0)}{(p_0+\cos\alpha)\cdot\sin\alpha} \\
&= \cot\alpha - \frac{p_0}{\sin\alpha}.
\end{aligned}
$$

Thus, by Definition 7,

$$
\mathbf{w}(S) = \left(\cot\alpha - \frac{p_0}{\sin\alpha}, \cot\alpha + \frac{p_0}{\sin\alpha}, \frac{p_1}{\sin\alpha}, ..., \frac{p_n}{\sin\alpha}\right) = \mathbf{w}_+(C)\mathbf{G}, \tag{15}
$$

where

$$
\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}.
$$

By Theorem 11, a configuration of $n+2$ spherical caps $C_1, ..., C_{n+2}$ on the unit sphere will project stereographically into a configuration of Euclidean spheres $S_1, ..., S_{n+2}$ in the plane $y_0 = 0$. Conversely, all configurations of Euclidean spheres map to configurations of spherical caps. The mapping takes spherical Descartes configurations to Euclidean Descartes Configurations. The rows of $\mathbf{w}_+(C_j)$ are assembled into the matrix $\mathbf{W}_+$ and the corresponding rows of $\mathbf{w}(S_j)$ are entered into the matrix $\mathbf{W}$. Then, by extension of Equation 15,

$$
\mathbf{W} = \mathbf{W}_+\mathbf{G}.
$$

Applying this result to Theorem 7 gives

$$
\mathbf{W}^T\mathbf{Q}_n\mathbf{W} = \mathbf{G}^T\mathbf{W}_+^T\mathbf{Q}_n\mathbf{W}_+\mathbf{G} = \mathbf{G}^T\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2I_n \end{bmatrix}\mathbf{G} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_n \end{bmatrix}.
$$

This result proves Theorem 6, the Augmented Euclidean Descartes Theorem. □

Therefore, if $\mathbf{W}$ is an augmented matrix of an oriented Descartes configuration of $n+2$ spheres then Equation 4 is satisfied.

Proof of the Euclidean Generalized Descartes Theorem. Originally, matrix $\mathbf{W}$ was constructed by augmenting matrix $\mathbf{M}$ by adding another column. If we remove that column and focus on matrix $\mathbf{M}$ again, we can see that the calculation of $\mathbf{M}^T\mathbf{Q_n}\mathbf{M}$ amounts to deleting the first row and first column of the matrix $\mathbf{W}^T\mathbf{Q}_n\mathbf{W}$. Now we have

$$\mathbf{M}^T\mathbf{Q}_n\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 2I_n \end{bmatrix}.$$

This result proves Theorem 4, the Euclidean Generalized Descartes Theorem. □

We can now apply this theorem to the case where $n = 2$ to prove Theorem 3.

Proof of the Extended Descartes Theorem. Based on the previous proof, if matrix $\mathbf{M}$ contains the curvature center coordinates of $n+2$ spheres, then these spheres form a Descartes configuration. Let $n = 2$. Then

$$\mathbf{M}^T\mathbf{Q}_2\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

Therefore, the four oriented circles whose curvature-center coordinates are contained in $\mathbf{M}$ make up a Descartes configuration. We can further restrict our focus to only consider the first column of matrix $\mathbf{M}$, which contains the curvatures of the $n+2$ oriented circles.

Proof of the Descartes Circle Theorem. Based on Theorem 3, $\mathbf{b}^T\mathbf{Q}_2\mathbf{b} = 0$, as noted in the $(1,1)$ entry of $\mathbf{M}^T\mathbf{Q}_2\mathbf{M}$. Therefore,

$$b_1^2 - 2b_2b_1 - 2b_3b_1 - 2b_4b_1 + b_2^2 + b_3^2 + b_4^2 - 2b_2b_3 - 2b_2b_4 - 2b_3b_4 = 0.$$

Thus,

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = 2b_2b_1 + 2b_3b_1 + 2b_4b_1 + 2b_2b_3 + 2b_2b_4 + 2b_3b_4. \tag{16}$$

Adding $b_1^2 + b_2^2 + b_3^2 + b_4^2$ to both sides of Equation 16 results in

$$2\sum_{j=1}^{4} b_j^2 = \left(\sum_{j=1}^{4} b_j\right)^2.$$

Therefore,

$$\sum_{j=1}^{4} b_j^2 = \frac{1}{2}\left(\sum_{j=1}^{4} b_j\right)^2.$$

□

Using the results from Theorem 3 we can also prove Theorem 2, the Complex Descartes Theorem.

Proof of the Complex Descartes Theorem. The Complex Descartes Theorem is true if and only if

$$\sum_{j=1}^{4}(b_j z_j)^2 - \frac{1}{2}\left(\sum_{j=1}^{4} b_j z_j\right)^2 = 0 \tag{17}$$

where $z_j = x_j + iy_j$. For a complex number of the form $a + bi$ to equal 0, it must be the case that both $a = 0$ and $b = 0$. For this reason, the real and imaginary parts will be addressed separately. First we focus on the real component of Equation 17. This gives

$$\begin{aligned}
&\operatorname{Re}\left(\sum_{j=1}^{4}(b_j z_j)^2 - \frac{1}{2}\left(\sum_{j=1}^{4} b_j z_j\right)^2\right) \\
&\quad = b_1^2x_1^2 - 2b_1b_2x_2x_1 - 2b_1b_3x_3x_1 - 2b_1b_4x_4x_1 + b_2^2x_2^2 + b_3^2x_3^2 + b_4^2x_4^2 \\
&\qquad\qquad - 2b_2b_3x_2x_3 - 2b_2b_4x_2x_4 - 2b_3b_4x_3x_4 \\
&\quad - (b_1^2y_1^2 - 2b_1b_2y_2y_1 - 2b_1b_3y_3y_1 - 2b_1b_4y_4y_1 + b_2^2y_2^2 + b_3^2y_3^2 + b_4^2y_4^2 \\
&\qquad\qquad - 2b_2b_3y_2y_3 - 2b_2b_4y_2y_4 - 2b_3b_4y_3y_4) \\
&\qquad\qquad\qquad = \mathbf{M}^T\mathbf{Q}_2\mathbf{M}_{(2,2)} - \mathbf{M}^T\mathbf{Q}_2\mathbf{M}_{(3,3)}.
\end{aligned}$$

Based on Theorem 3, both the $(2,2)$ and $(3,3)$ entries of $\mathbf{M}^T\mathbf{Q}_2\mathbf{M}$ equal 2. Therefore,

$$\operatorname{Re}\left(\sum_{j=1}^{4}(b_j z_j)^2 - \frac{1}{2}\left(\sum_{j=1}^{4} b_j z_j\right)^2\right) = 0.$$

Next, we address the imaginary component. Based on Theorem 3, $\mathbf{M}^T\mathbf{Q}_2\mathbf{M}_{(2,3)} = 0$. Therefore,

$$b_1^2x_1y_1-b_1b_2x_2y_1-b_1b_3x_3y_1-b_1b_4x_4y_1-b_1b_2x_1y_2+b_2^2x_2y_2-b_2b_3x_3y_2-b_2b_4x_4y_2$$
$$-b_1b_3x_1y_3-b_2b_3x_2y_3+b_3^2x_3y_3-b_3b_4x_3y_3-b_1b_4x_1y_4-b_2b_4x_2y_4-b_3b_4x_3y_4+b_4^2x_4y_4 = 0.$$

This implies that

$$\begin{aligned} &ib_1^2x_1y_1 - ib_2b_1x_2y_1 - ib_3b_1x_3y_1 - ib_4b_1x_4y_1 - ib_2b_1x_1y_2 - ib_3b_1x_1y_3 \\ &\quad - ib_4b_1x_1y_4 + ib_2^2x_2y_2 - ib_2b_3x_3y_2 - ib_2b_4x_4y_2 - ib_2b_3x_2y_3 + ib_3^2x_3y_3 \\ &\qquad - ib_3b_4x_4y_3 - ib_2b_4x_2y_4 - ib_3b_4x_3y_4 + ib_4^2x_4y_4 = 0. \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{Im}\left(\sum_{j=1}^{4}(b_jz_j)^2 - \frac{1}{2}\left(\sum_{j=1}^{4} b_jz_j\right)^2\right) \\ &\quad = ib_1^2x_1y_1 - ib_2b_1x_2y_1 - ib_3b_1x_3y_1 - ib_4b_1x_4y_1 - ib_2b_1x_1y_2 - ib_3b_1x_1y_3 \\ &\quad - ib_4b_1x_1y_4 + ib_2^2x_2y_2 - ib_2b_3x_3y_2 - ib_2b_4x_4y_2 - ib_2b_3x_2y_3 + ib_3^2x_3y_3 \\ &\qquad - ib_3b_4x_4y_3 - ib_2b_4x_2y_4 - ib_3b_4x_3y_4 + ib_4^2x_4y_4 = 0. \end{aligned}$$

Therefore, both the real and imaginary parts of Equation 17 are equal to 0. So it is true that

$$\sum_{j=1}^{4}(b_jz_j)^2 - \frac{1}{2}\left(\sum_{j=1}^{4} b_jz_j\right)^2 = 0$$

□

The Descartes Circle Theorem is simple to state, yet challenging to prove. By generalizing the theorem to incorporate more information and applying it to higher dimensions, we were able to employ the tools of linear algebra and spherical geometry to prove these generalized theorems. Ultimately, these tools allowed us to also prove the Descartes Circle Theorem. These proofs provide one interesting example of how years of mathematics from various fields can be used together to prove what appears to be simple theorem.

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