§0. Introduction.

Let $A = A(D^n)$ be a polydisc algebra, that is, the algebra of all continuous functions on $\overline{D}^n \subset \mathbb{C}^n$, which are analytic on $D^n$. We study $A$ from the point of view of the deformation theory of uniform algebras [KJ4]. We show that if the Banach-Mazur distance between $A$ and a Banach algebra $B$ is sufficiently small then $B$ inherits a lot of properties of $A$; in particular, the spectrum of $B$ has a structure of $n$ dimensional complex analytic manifold.

We begin by recalling some results from [KJ4] and [RR7]. Let $A$ be a uniform algebra. A deformation of $A$ is a new normed algebra obtained by putting on the vector space $A$ a new associative multiplication $\times$, which for some small positive $\varepsilon$ satisfies

\begin{equation}
\| f \times g - fg \| \leq \varepsilon \| f \| \| g \| \quad \text{for all } f, g \text{ in } A.
\end{equation}
If \( \varepsilon < 1 \) and this new algebra is renormed with its spectral norm, we then obtain a new uniform algebra, \( A_\times \) [KJ4]. If \( Id \) is the identity map from \( A \) into \( A_\times \), then

\[
\| Id \| \| Id^{-1} \| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \rightarrow 1 \text{ as } \varepsilon \rightarrow 0,
\]

so the Banach-Mazur distance between \( A \) and \( A_\times \) tends to 1 with \( \varepsilon \rightarrow 0 \).

By theorem 3 of [KJ4] the converse is also true. If the Banach-Mazur distance between \( A \) and a Banach algebra \( B \) is smaller than \( 1 + \varepsilon' \), then \( B = A_\times \) for some new multiplication \( \times \) on \( A \), which satisfies (1) with \( \varepsilon \rightarrow 0 \) as \( \varepsilon' \rightarrow 0 \).

Small deformations of uniform algebras were studied in [BJ1-2, KJ1-5, RR1-7] and other papers. The main question concerns stability of various properties of uniform algebras. We say that a property \( \mathcal{P} \) is stable if for any uniform algebra \( A \) having the property \( \mathcal{P} \) there is an \( \varepsilon > 0 \) such that any \( \varepsilon \)-deformation \( A_\times \) of \( A \) has also this property. The following properties are stable.

1. \( A \) is Dirichlet [KJ4].
2. Choquet boundary of \( A \) is compact [KJ4].
3. \( A = C(S), \quad S \) a compact Hausdorff space [KJ4].
4. \( A = A(D) \) [RR6].
5. \( A = H^\infty(D), \) [KJ5].
6. \( A \) is an algebra of analytic functions of a finite bordered, possibly singular, Riemann surface [RR7].

The stability of the last property and related questions were studied by R. Rochberg in a series of papers [RR1—7] ([RR7] gives the most comprehensive exposition).

Our knowledge about deformations of algebras of analytic functions of one variable is still incomplete (especially for non-separable algebras), but already quite extensive. By contrast we know almost nothing about deformations of
algebras of analytic functions of many variables. In this note we prove that a small deformation of a polydisc algebra $A(D^n)$ produces a uniform algebra, whose structure is quite similar to that of $A(D^n)$. The result should be seen only as a very small first step toward a comprehensive description of deformations of algebras of analytic functions of many variables.

§1. Notation.

We use standard Banach space terminology. For a compact space $X$, $C(X)$ denotes Banach algebra of all continuous, complex valued functions on $X$, with sup norm. A uniform algebra is a unital subalgebra of $C(X)$, which separates points of $X$. Further, $\mathfrak{M}(A)$, $\partial A$ and $ChA$ denote the maximal ideal space (= spectrum), Shilov boundary, and the Choquet boundary of $A$, respectively. We frequently identify $A$, via the Gelfand transform, with a subalgebra of $C(\mathfrak{M}(A))$ or of $C(\partial A)$. Hence for $F \in \mathfrak{M}(A)$ and $f \in A$ we may write $F(f)$ as well as $f(F)$. $A^{-1}$ is the set of all invertible elements of $A$.

For a compact subset $K$ of $X$ we put $A \upharpoonright K = \{ f \upharpoonright K : f \in A \subset C(X) \}$. $A \upharpoonright K$ is a subalgebra of $C(K)$. By a linear extension from $A \upharpoonright K$ into $A$ we mean a linear, continuous map $\Psi : A \upharpoonright K \to A$ such that

$$\Psi(g) \upharpoonright K = g \text{ for all } g \in A \upharpoonright K, \text{ and } \Psi(1) = 1.$$ 

$K$ is called a peak set for $A$ if there is a sequence of functions $f_n$ in $A$ such that

$$f_n = 1 \text{ on } K, \quad \| f_n \| = 1 \text{ for all } n \in \mathbb{N},$$

and $f_n \to 0$ uniformly on any compact subset of $X \setminus K$.

If $K$ consists of a single point, this point is called a peak point. It is well-known that, for a separable uniform algebra $A$, $ChA$ is equal to the set of all peak points for $A$. It is also well-known that the algebra $A \upharpoonright K$ is complete if $K$ is a peak set for $A$. 
For \( f_0 \in C(X) \), \( f_0 \cdot A = \{ f_0 \cdot f : f \in A \subset C(X) \} \). If \( \Omega \) is an open and bounded subset of \( \mathbb{C}^n \) then \( A(\Omega) = \{ f \in C(\Omega) : f \big|_{\Omega} \text{ is holomorphic} \} \).

The unit disc in \( \mathbb{C} \) we denote by \( D \). If \( f : U \to \mathbb{C} \), where \( U \subseteq \mathbb{C} \), then \( 1f : U \times \mathbb{C} \to \mathbb{C} \) is defined by \( 1f(z,w) = f(z) \) \( (2f(z,w) = f(w) \), respectively \).

For \( z \in D \), \( L_z \) is the corresponding Blaschke factor, that is
\[
L_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in \overline{D}.
\]
Hence \( 1L_z \) is the function defined on \( \overline{D} \times \mathbb{C} \) by
\[
1L_z(w_1, w_2) = \frac{z - w_1}{1 - \bar{z}w_1}, \quad w_1 \in \overline{D}, \quad w_2 \in \mathbb{C}.
\]

The Banach-Mazur distance \( d_{B-M} \) between Banach spaces \( A \) and \( B \) is defined by
\[
d_{B-M}(A,B) = \inf \{ \| T \| \| T^{-1} \| : T : A \to B \text{ is an isomorphism} \}.
\]

§2. The Results.

**Theorem 1.** Let \( B \) be a complex function algebra such that \( d_{B-M}(A(D^n),B) < 1 + \epsilon \) with \( \epsilon < \epsilon_0 \). Then \( \mathcal{M}(B) \cong \overline{D}^n \), so \( B \) can be seen as a subalgebra of \( C(\overline{D}^n) \). There is a linear isomorphism \( T : A(D^n) \to B \) such that

\[
(1) \quad \| Tf - f \| \leq \epsilon' \| f \| \quad \text{for } f \in A(D^n).
\]

Moreover, \( D^n \) can be given a structure \( \tau \) of an \( n \)-dimensional complex manifold such that all functions from \( B \) are \( \tau \)-holomorphic.

Here \( \epsilon_0 > 0 \) is an absolute constant and \( \epsilon' \to 0 \) as \( \epsilon \to 0 \).

Before we prove the theorem we discuss some general results concerning small
perturbations of functions algebras.

**Proposition 1.** Let $A$ be a complex function algebra and $B$ a complex Banach algebra. If there is a linear isomorphism $T: A \rightarrow B$ with $\| T \| \cdot \| T^{-1} \| < 1 + \varepsilon$ and $T(1) = 1$ then $B$ is a uniform algebra and

$$\| g \| \geq \sigma_B(g) \geq (1 - \varepsilon') \| g \| \quad g \in B,$$

where $\varepsilon' \rightarrow 0$ as $\varepsilon \rightarrow 0$.

**Proof.** We define a second multiplication $\times$ on the Banach space $A$ by

$$f \times g = T^{-1}(Tf \cdot Tg) \quad f, g \in A.$$  

We have

$$\| f \times g \| \leq (1 + \varepsilon^3) \| f \| \cdot \| g \| \quad f, g \in A.$$  

Both multiplications on $A$ have the same unit. Hence by Theorem 3.1 (v) of [KJ4] there is a function algebra $B_1$ and a linear isomorphism $T_1: A \rightarrow B_1$ with $\| T_1 \| \cdot \| T_1^{-1} \| \leq 1 + \varepsilon_1$ ($\varepsilon_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$) and such that

$$T^{-1}(Tf \cdot Tg) = f \times g = T_1^{-1}(Tf \cdot Tg) \quad f, g \in A.$$  

So $T_1 \circ T^{-1}: B \rightarrow B_1$ is an algebra isomorphism and

$$\sigma_B(g) = \sigma_{B_1}(T_1 \circ T^{-1}(g)) \geq \| T_1 \circ T^{-1}(g) \| \geq (1 + \varepsilon)^{-1} (1 + \varepsilon_1)^{-1} \| g \|$$

for any $g \in B$.

**Proposition 2.** Let $A$ be a function algebra, let $f_0 \in A \setminus A^{-1}$ be such that $|f_0| \equiv 1$ on $\partial A$ and such that $f_0A = \{ f \in A: f |_K \equiv 0 \}$, where $K = \{ x \in M(A): f_0(x) = 0 \}$. Assume also that there is a linear, norm one extension $\Psi: A |_K \rightarrow A$. If $T$ is a linear isomorphism from $A$ onto a function algebra $B \subseteq C(\partial A)$ such that

$$(2) \quad | Tf(x) - f(x) | \leq \varepsilon \| f \| \quad f \in A, \ x \in \partial A,$$
with \( \varepsilon \leq \varepsilon_0 \) (absolute constant), then

(i) \( g_0 = Tf_0 \in B \setminus B^{-1} \),

(ii) \( g_0B = \{ g \in B : g \|_L = 0 \} \), where \( L = \{ x \in \mathcal{M}(B) : g_0(x) = 0 \} \),

and

(iii) \( \Phi : A \mid_K \to B \mid_L \) defined by \( \Phi(f) = T(\Psi(f)) \mid_L \) is a surjective linear isomorphism with \( \| \Phi \| \| \Phi^{-1} \| \leq 1 + \varepsilon' \), where \( \varepsilon' \to 0 \) as \( \varepsilon \to 0 \).

**Proof.** The part (i) follows from Proposition 15.3 of [KJ4], we show (ii) and (iii).

Let \( f_1, f_2 \in A \) be such that \( f_0f_1 = f_0f_2 \). Since \( f_0 \neq 0 \) on \( \partial A \), it follows that \( f_1 - f_2 \equiv 0 \) on \( \partial A \), so \( f_1 = f_2 \). Hence we can define a linear, surjective map \( S : f_0A \to g_0B \) by \( S(f_0 \circ f) = g_0 \circ Tf \). By (2) we have

\[
(3) \quad |Sf(x) - f(x)| \leq 2\varepsilon \|f\| \quad f \in f_0A.
\]

Hence \( g_0B \) is a closed ideal of \( B \) contained in the closed ideal \( \{ g \in B : g \|_L = 0 \} \).

If these two ideals were not equal then \( B / g_0B \) would not be a uniform algebra, since the spectral radius of any element of \( \{ g \in B : g \|_L = 0 \} / g_0B \) would be equal to zero (note that any linear-multiplicative functional that annihilates \( g_0B \) annihilates also \( \{ g \in B : g \|_L = 0 \} \)). We will show that \( B / g_0B \) is a uniform algebra which will prove (ii).

We define a map \( \widetilde{S} : A \to B \) by
\[
\tilde{S}(f) = T(\Psi(f|_K)) + S(f - \Psi(f|_K)), \quad f \in A.
\]

By (2) and (3), for any \( f \in A \) we have
\[
\| \tilde{S}(f) - f \|_{\partial A} = \| \tilde{S}(f) - \Psi(f|_K) - (f - \Psi(f|_K)) \|_{\partial A} \leq \varepsilon \| \Psi(f|_K) \| + 2\varepsilon \| f - \Psi(f|_K) \| \leq 5\varepsilon \| f \|.
\]

Hence, and by (2), \( \tilde{S} \) is a linear surjective isomorphism of \( A \) onto \( B \) with
\[
\| \tilde{S} \| \| \tilde{S}^{-1} \| \leq (1+5\varepsilon)/(1-5\varepsilon) = 1+\varepsilon_1.
\]

On the subspace \( f_0 A = \{ f \in A : f|_K \equiv 0 \} \) of \( A \), \( \tilde{S} \) coincides with \( S \) and maps \( f_0 A \) onto \( g_0 B \).

Hence the map \( \Phi : A/f_0 A \to B/g_0 B \) defined by \( \Phi(f+f_0 A) = \tilde{S}(f) + g_0 B \) is a linear surjective isomorphism of the Banach algebra \( A/f_0 A \) onto \( B/g_0 B \) such that \( \| \Phi \| \| \Phi^{-1} \| \leq 1+\varepsilon_1 \) and \( \Phi(1) = 1 \). The map on \( A|_K \) defined by \( f \mapsto \Psi(f) + f_0 A \in A/f_0 A \) is an isometry and algebra isomorphism of a function algebra \( A|_K \) onto \( A/f_0 A \). Hence, by Proposition 1, \( B/g_0 B \) is a uniform algebra and the spectral radius and the norm almost coincide in \( B/g_0 B \).

On the other hand the maximal ideal space of \( B/g_0 B \) can be identified with \( L \) and the spectral norm is given by the supremum over the maximal ideal space so \( B/g_0 B \) and \( B|_L \) are almost isometric. This gives (iii).

**Proposition 3.** Let \( B \) be a complex function algebra such that
\[
d_{B-M}(A(D^n), B) < 1+\varepsilon \quad \text{with} \quad \varepsilon < \varepsilon_0.
\]
If \( K \subseteq \partial A(D^n) = \partial B \) is a peak set for the algebra \( A(D^n) \) then \( K \) is also a peak set for the algebra \( B \) and
\[
d_{B-M}(A|_K, B|_K) \leq 1+\varepsilon_1 \to 1 \quad \text{as} \quad \varepsilon \to 0.
\]
The above proposition follows immediately from Theorem 16.7 of [KJ4].

§3. Proof of Theorem 1.

To simplify the notation we prove the result for \( n = 2 \). It will be quite transparent how to write the general case. Put \( X = \partial D \times \partial D = \partial A(D^2) \). By Theorem 3.1 of [KJ4] \( \partial B \cong X \) so \( B \) can be seen as a subalgebra of \( C(X) \). There is also a linear isomorphism \( T: A(D^2) \to B \) such that

\[
\left| Tf(z,w) - f(z,w) \right| \leq \varepsilon' \| f \| \quad f \in A(D^2), \ (z,w) \in X.
\]

To prove the result we have to show that \( \mathfrak{M}(B) \cong \overline{D}^2 \), that (4) holds for all \( (z,w) \in \overline{D}^2 \) and that \( D^2 \) can be given a structure \( \tau \) of 2-dimensional complex manifold such that all functions from \( B \) are \( \tau \)-analytic.

We construct a homeomorphism \( \varphi \) from \( \overline{D}^2 \) onto \( \mathfrak{M}(B) \) in several steps. \( \overline{D}^2 \) is a union of three disjoint sets: \( X, \partial D \times D \cup D \times \partial D = \partial(D^2) \setminus X, \) and \( D^2 \).

Step 1. Definition of \( \varphi \) on \( X \).

We put \( \varphi(x) = x \) for \( x \in X \).

Step 2. Definition of \( \varphi \) on \( \partial(D^2) \setminus X \).

We need the following lemma which is a combination of lemmas 3.1, 3.2 and 3.3 of [RR6].

Lemma 1. Let \( S \) be a linear isomorphism from \( A(D) \) onto a uniform algebra \( B_1 \subseteq C(\partial D) \) such that
\[ |Sf(w) - f(w)| \leq \varepsilon \| f \|, \quad f \in A(D), \ w \in \partial D. \]

For any \( z \in D \) we put \( \psi(z) = \{ g \cdot S(L_z) \in B_1 : g \in B_1 \} \) and for \( z \in \partial D \) we put \( \psi(z) = z \in \M(B_1) \). Then for any \( z \in \overline{D} \) we have

(i) \( \psi(z) \in \M(B_1) \),

(ii) \( \psi \) is a homeomorphism of \( \overline{D} \) onto \( \M(B_1) \), and

(iii) \[ |Sf(\psi(z)) - f(z)| \leq 2\varepsilon \| f \| , \quad f \in A(D), \ z \in \overline{D}. \]

To define \( \varphi \) on \( \overline{D} \times \partial D \) let \( w_0 \in D \) and put \( K_{w_0} = \overline{D} \times \{ w_0 \} \). \( K_{w_0} \) is a peak set for \( A(D^2) \) and \( A(D^2)|_{K_{w_0}} \) is isometrically isomorphic with the disc algebra. The maximal ideal space of \( A(D^2)|_{K_{w_0}} \) can be identified with \( K_{w_0} \). By Proposition 3, Theorem 3.1 of [KJ4], and Lemma 1 we get a homeomorphism \( \varphi(\cdot, w_0) \) from \( K_{w_0} \) onto \( \M(B|_{K_{w_0}}) \subseteq \M(B) \). For \( z \in D \), \( \varphi(z, w_0) \) is the maximal ideal of \( B \) consisting of all functions \( g \in B \subseteq C(\M(B)) \) such that \( g|_{\varphi(K_{w_0})} \) is divisable by \( T^{(1)}L_z|_{\varphi(K_{w_0})} \).

Recall that according with the notation described in the previous section \( ^1L_z: \overline{D}^2 \rightarrow \C \) is defined by \( ^1L_z(\alpha, \beta) = L_z(\alpha) \), where \( L_z \) is the Blaschke factor.

In the same way we define \( \varphi \) on \( \partial D \times \overline{D} \).

**Step 3. Definition of \( \varphi \) on \( D^2 \).**

Let \((z_0, w_0) \in D^2 \). We put
\[ \varphi(z_0, w_0) = \left\{ T(1L_{z_0})g_1 + T(2L_{w_0})g_2 : g_1, g_2 \in B \right\}. \]

We have to show that \( \varphi(z_0, w_0) \) is a proper maximal ideal of \( B \). We first show that \( \varphi(z_0, w_0) \) is a proper ideal. To this end assume that there are \( g_1, g_2 \in B \) such that

\[ T(1L_{z_0})g_1 + T(2L_{w_0})g_2 \equiv 1. \]

Put \( L = \left\{ x \in \mathfrak{M}(B) : T(1L_{z_0})(x) = 0 \right\} \). By (5) we have

\[ T(2L_{w_0}) \cdot g_2 \equiv 1 \quad \text{on} \quad L. \]

Put \( f_0 = 1L_{z_0} \) and \( K = \left\{ x \in \overline{D}^2 : 1L_{z_0}(x) = 0 \right\} = \{z_0\} \times \overline{D} \). By Proposition 2, there is an almost isometry \( \Phi \) from \( A(D) \cong A(D^2) \vert K \) onto \( B \vert L \) given by \( \Phi(f) = T(2f) \vert L \). By [RR6] \( \Phi(L_{w_0}) = T(2L_{w_0}) \vert L \) is a noninvertible element of \( B \vert L \), which contradicts (6) and proves that \( \varphi(z_0, w_0) \) is a proper ideal.

We show that \( \varphi(z_0, w_0) \) is a maximal ideal. Set

\[ S_1 : A(D^2) \rightarrow A(D), \quad S_1(f) = f(\cdot, w_0); \quad S_2 : A(D^2) \rightarrow A(D^2), \quad S_2(f) = \frac{(f - S_1(f))}{2L_{w_0}}. \]

Any \( f \in A \) can be decomposed as follows

\[ f = f(z_0, w_0) + 1L_{z_0} \cdot \frac{S_1(f) - f(z_0, w_0)}{L_{z_0}} + 2L_{w_0} \cdot S_2(f). \]

So we can define a linear map \( \Psi_{z_0, w_0} : A(D^2) \rightarrow B \) by

\[ \Psi_{z_0, w_0}(f) = f(z_0, w_0) + T(1L_{z_0}) \cdot T\left( \frac{S_1(f) - f(z_0, w_0)}{L_{z_0}} \right) + T(2L_{w_0}) \cdot T(S_2(f)). \]

By (4) \( \Psi \) is close to the identity map so it is a surjective isomorphism. Hence
\[ B_{z_0,w_0} = \Psi_{z_0,w_0}(\{ f \in A(D^2) : f(0,0) = 0 \} ) \] is a closed subspace of \( B \) of dimension one. We also have \( B_{z_0,w_0} \subseteq \varphi(z_0,w_0) \varsubsetneq B \). Hence \( B_{z_0,w_0} = \varphi(z_0,w_0) \) is a proper maximal ideal of \( B \).

**Step 4.** \( \varphi \) is continuous on \( D^2 \).

A maximal ideal of a function algebra \( B \) can be identified with a linear-multiplicative functional on \( B \) or with an element from the domain of a function \( g \in B \). From the definitions of \( \varphi \) and \( \Psi_{z_0,w_0} \) we get

\[ g(\varphi(z_0,w_0)) = (\Psi_{z_0,w_0})^{-1}(g)(z_0,w_0). \] (7)

Hence, for any \( g \in B \), with \( \| g \| \leq 1 \) we have

\[ |g(\varphi(z_0,w_0)) - g(\varphi(z_1,w_1))| \leq d_H((z_0,w_0), (z_1,w_1)) \| (\Psi_{z_0,w_0})^{-1} - (\Psi_{z_1,w_1})^{-1} \| \]

where \( d_H(\cdot, \cdot) \) is the hyperbolic distance on \( D^2 \). Hence \( \varphi |_{D^2} : D^2 \to \mathcal{M}(B) \) is a continuous map if \( \mathcal{M}(B) \) is equipped with the norm topology, it is more so continuous if we take \( \mathcal{M}(B) \) with its original weak * topology.

**Step 5.** \( \varphi \) is continuous.

The following is an obvious topological observation.

*Let \( X \) be a compact metric space, \( Y \) a topological space, \( G \) a dense subspace of \( X \) and \( \phi \) a function from \( X \) into \( Y \). If \( \phi | G \) is continuous and if for any \( x_0 \in X \setminus G \) and any sequence \( x_n \) in \( G \) convergent to \( x_0 \) and such that \( \phi(x_n) \) is convergent we have \( \lim_{n \to \infty} \phi(x_n) = \phi(x_0) \), then \( \phi \) is*
continuous.

To end the proof of the continuity of $\varphi$ let $(z_0, w_0) \in \overline{D^2} \setminus D^2$ and let $(z_n, w_n)$ be a sequence in $D^2$ convergent to the point $(z_0, w_0)$ and such that $\varphi(z_n, w_n)$ is convergent to $x_0 \in \mathcal{M}(B)$. From (7), (4) and the definition of $\Psi_{z_0, w_0}$ by a direct computation we get

(8) \[ |Tf(\varphi(z_0, w_0)) - f(z_0, w_0)| \leq 10\varepsilon' \| f \|, \quad f \in A(D^2), \ (z_0, w_0) \in D^2.\]

Hence

(9) \[ |Tf(z_0) - f(z_0, w_0)| \leq 10\varepsilon' \| f \| \quad f \in A(D^2).\]

Assume first that $(z_0, w_0) \in X = \partial A(D^2)$. From (10) and (4) we have

(10) \[ |Tf(z_0) - Tf(z_0, w_0)| \leq \varepsilon' \| f \|, \quad f \in A(D^2).\]

Since $(z_0, w_0)$ is a peak point for $A(D^2)$ it is also a peak point for $B$ ([KJ], Theorem 16.7) so the distance, in the norm topology, between the functional $(z_0, w_0) \in \mathcal{M}(B)$ and any other functional $x_0$ from $\mathcal{M}(B)$ is equal to two. Hence, by (10) we get $(z_0, w_0) = x_0$. Assume now that $(z_0, w_0) \in \partial D^2 \setminus X$, say $z_0 \in D$ and $w_0 \in \partial D$. Put as before $K_{w_0} = \overline{D} \times \{w_0\}$. We know that $K_{w_0}$ is a peak set for $A(D^2)$ and that $\varphi(K_{w_0}) \subseteq \mathcal{M}(B)$ is a peak set for $B$. Assume that $x_0 \in \varphi(K_{w_0})$. Let $\mu$ be a probabilistic measure on $X = \partial B$ which represents the functional $x_0$, this is such that $\int_X g \, d\mu = g(x_0)$ for $g \in B$. Since $\varphi(K_{w_0})$ is a peak set not containing $x_0$, $\mu$ is concentrated outside of $K_{w_0} \cap X = \partial D \times \{w_0\}$, so $\mu(K_{w_0}) = 0$. Let $f_n$ be a sequence of norm one elements of $A(D^2)$ such that $f_n \equiv 1$ on $K_{w_0}$ and $f_n \to 0$ uniformly on any open subset of $\overline{D^2} \setminus K_{w_0}$. By (4) we get

\[ |Tf_n(z_0)| = | \int_X f_n \, d\mu | \to \varepsilon' \quad \text{as} \quad n \to \infty.\]

On the other hand, from (10), for any $n \in \mathbb{N}$ we get
\[ |Tf_n(x_n) - 1| = |Tf_n(z_0, w_0)| \leq 10\varepsilon'. \]

Hence \( x_0 \in \varphi(K_{w_0}), \ x_0 = \varphi(\alpha, w_0) \). We need to show that \( \alpha = z_0 \). By the definitions of \( \Psi_{z,w} \) and \( \varphi \) on \( D^2 \), for any \( n \in \mathbb{N} \) we have

\[ T(1L_{z_n})(\varphi(z_n, w_n)) = 0. \]

The sequence \( T(1L_{z_n}) \) is convergent, in norm, to \( 1L_{z_0} \) and \( \varphi(z_n, w_n) \) is weak * convergent to \( \varphi(\alpha, w_0) \), hence

\[ T(1L_{z_0})(\varphi(\alpha, w_0)) = 0. \]

By the definition of \( \varphi \) on \( K_{w_0} = \overline{D} \times \{w_0\} \), \( \varphi(z_0, w_0) \) is the only point of \( \varphi(K_{w_0}) \) where \( T(1L_{z_0}) \) is equal to zero. Hence \( \varphi(z_0, w_0) = \varphi(\alpha, w_0) \). The map \( \varphi \) is a homeomorphism of \( K_{w_0} \) onto \( \varphi(K_{w_0}) \) so \( z_0 = \alpha \).

**Step 6.** \( \varphi \) is one to one.

By the definition of \( \varphi \) and Theorem 16.7 of [KJ4] \( \varphi \) is one to one on \( \partial D^2 \) and \( \varphi(\partial D^2) \cap \varphi(D^2) = \emptyset \). We have to show that \( \varphi \) is one to one on \( D^2 \).

Assume \( (z_i, w_i) \in D^2, i = 1,2 \) are such that \( \varphi(z_1, w_1) = \varphi(z_2, w_2) \).

Expanding \( 1L_{z_2} \) in powers of \( 1L_{z_1} \) we get

\[ 1L_{z_2} = L_{z_2}(z_1) \cdot (1 - |L_{z_2}(z_1)|^2) \cdot 1L_{z_1} + \overline{L_{z_2}(z_1)} \cdot (1L_{z_1})^2 \cdot f, \]

where \( f \in A(D^2), \ |f| \leq 2 \).

Applying the operator \( T \) to the above equation, then evaluating at the point

\[ \varphi(z_1, w_1) = \varphi(z_2, w_2) \in \mathcal{M}(B) \]

yields

\[ 0 = L_{z_2}(z_1) + \overline{L_{z_2}(z_1)} \cdot T((1L_{z_1})^2 \cdot f)(\varphi(z_1, w_1)). \]

By (9) we have \( |T((1L_{z_1})^2 \cdot f)| \leq 10\varepsilon' |f| \) so finally we get

\[ |L_{z_2}(z_1)| \leq |L_{z_2}(z_1)| 10\varepsilon'. \]

If \( 10\varepsilon' < 1 \), this proves that \( z_1 = z_2 \); the same argument shows that \( w_1 = \)
Step 7. For any \((z,w) \in D^2\) there is a neighborhood \(U\) of \(\varphi(z_0,w_0)\) in \(\mathcal{W}(B)\) and a homeomorphism \(\tau\) of \(U\) onto \(D^2\) such that \(g \circ \tau^{-1}\) is analytic for any \(g \in B\).

Note that after we prove that \(\varphi\) is surjective this will give us the desired analytic structure on \(D^2 \cong \varphi(D^2)\).

It follows from the definition of \(\varphi(z_0,w_0)\), for \((z_0,w_0) \in D^2\) that for any \(g \in B\) with \(g(\varphi(z_0,w_0)) = 0\) there are \(g_1, g_2 \in B\) such that \(\|g_i\| \leq 3\|g\|,\ i = 1,2\) and

\[
g = T(1)z_0 \cdot g_1 + T(2)w_0 \cdot g_2.
\]

By the Gleason Embedding Theorem ([TG], p.154) there is a neighborhood \(U\) of \(\varphi(z_0,w_0)\) in \(\mathcal{W}(B)\) and a homeomorphism \(\tau\) of \(U\) onto an analytic variety \(V \subseteq \mathbb{C}^2\). Since \(\varphi\) defines a continuous embedding of \(D^2\) into \(\mathcal{W}(B)\), the topological dimension of \(U\), and so of \(V\) is at least 4, hence \(V\) is an open subset of \(\mathbb{C}^2\). The homeomorphism \(\tau\) given by the Gleason Theorem is such that \(g \circ \tau^{-1}\) is analytic for any \(g \in B\).

Step 8. Let \(F \in \mathcal{W}(B)\) and \((z_0,w_0) \in D^2\). Assume that

\[
\| F - \varphi(z_0,w_0) \| < \frac{1}{27}, \quad \text{where } F \text{ and } \varphi(z_0,w_0) \text{ are considered to be functionals on } B. \]

Then \(F \in \varphi(D^2)\).

The above statement will follow from Step 7 and from the proof of the Gleason Embedding Theorem ([TG], p.154-155). Since \(\|g_i\| \leq 3\|g\|,\ i = 1,2\) in (11), we get

\[
U = \left\{ x \in \mathcal{W}(B) : \ |T(1)z_0(x)| < \frac{1}{24}, \quad |T(2)w_0(x)| < \frac{1}{24}\right\},
\]

\[
V = D^2(\frac{1}{24}) \quad \text{and}
\]

(12) \(\tau: U \to V\) is given by \(\tau(x) = (T(1)z_0(x), T(2)w_0(x))\).
Put \( G = \{(z,w) \in D^2 : \max \{|L_{z_0}(z)|, |L_{w_0}(w)|\} < \frac{1}{25}\} \). \( G \) is an open subset of \( \mathbb{C}^2 \), diffeomorphic with \( D^2 \). By (8), if \( \varepsilon' \) is small enough, \( \varphi(G) \subseteq U \). Hence \( \tau \circ \varphi : G \to V \) is a continuous, injective map from \( G \) into \( D^2 \). We define \( \kappa : D^2(\frac{1}{25}) \to \mathbb{C}^2 \) by
\[
\kappa(z,w) = \tau \circ \varphi\left(L_{z_0}^{-1}(z), L_{w_0}^{-1}(w)\right).
\]
The map \( \kappa \) is continuous, one to one and by (12) and (9) we have
\[
d_E(\kappa(z,w) - (z,w)) \leq 10\sqrt{2} \varepsilon', \quad (z,w) \in V,
\]
where \( d_E(\cdot, \cdot) \) is the Euclidean metric on \( \mathbb{C}^2 \). Hence we get
\[
D^2(\frac{1}{25}) \subseteq \kappa\left(D^2(\frac{1}{25})\right) \subseteq D^2(\frac{1}{24}).
\]
This shows that
\[
\varphi(D^2) \supseteq \left\{x \in \mathbb{M}(B) : |T^{(1)}L_{z_0}(x)| < \frac{1}{25}, |T^{(2)}L_{w_0}(x)| < \frac{1}{25}\right\},
\]
and since the norms of \( T^{(1)}L_{z_0} \) and \( T^{(2)}L_{w_0} \) are not smaller than \( 1 - \varepsilon' \) this ends the proof of this Step.

\textbf{Step 9.} \( \varphi \) maps \( \overline{D^2} \) onto \( \mathbb{M}(B) \).

We need the following Theorem of B. Johnson [BJ2].

\textbf{Theorem.} Let \( F \) be linear functional on \( A(D^2) \) such that
\[
|F(f \cdot g) - F(f) \cdot F(g)| \leq \varepsilon \|f\| \|g\| \quad f, g \in A(D^2),
\]
where \( \varepsilon < \varepsilon_0 \) (positive absolute constant). Then there is \( (z,w) \in \overline{D^2} \) such that
\[ | F(f) - f(z, w) | \leq \varepsilon' \| f \|, \quad f \in A(D^2), \]

where \( \varepsilon' \to 0 \) as \( \varepsilon \to 0 \).

Let \( F \in \mathcal{M}(B) \). Put \( \widetilde{F} = F \circ T \). By (4) \( \widetilde{F} \) is almost multiplicative on \( A(D^2) \) so by Theorem of Johnson there is \( (z_0, w_0) \in \overline{D}^2 \) such that

\[ | \widetilde{F}(f) - f(z_0, w_0) | \leq \varepsilon_1 \| f \|, \quad f \in A(D^2). \]

By (8) we have

(13) \[ | T_{\varphi}(\varphi(z_0, w_0) - g(\varphi(z_0, w_0)) | \leq \varepsilon_2 \| g \|, \quad g \in B. \]

If \( (z_0, w_0) \in D^2 \) then \( F \in \varphi(D^2) \) by the previous step so assume \( (z_0, w_0) \in \partial(D^2) \). Without loss of generality we may assume that \( w_0 \in \partial D \).

As before \( K_{w_0} = \overline{D} \times \{ w_0 \} \) is a peak set for \( A(D^2) \) and \( A(D^2) |_{K_{w_0}} \) can be identified with the disc algebra. By the definition of \( \varphi \) on \( K_{w_0} \) and since \( A(D) \) is stable [RR6], \( \varphi(K_{w_0}) \) is the maximal ideal space of \( B |_{\varphi(K_w)} \). By Theorem 16.1 of [KJ4] \( \varphi(K_{w_0}) \) is a peak set for \( B \) so

\[ \| F_1 - \varphi(z, w) \| = 2 \]

for any \( (z, w) \in K_{w_0} \) and functional \( F_1 \in \mathcal{M}(B) \setminus \varphi(K_{w_0}) \). By (13) this gives \( F \in \varphi(K_{w_0}) \).

§4. Open Problems.

**Problem 1.** Let \( B \) be as in Theorem 1. Is \( (D^n, \tau) \) holomorphically equivalent to an open subset \( \Omega \) of \( \mathbb{C}^n \)? Or, equivalently, is \( B \) isometric with \( A(\Omega) \)?

**Problem 2.** Let \( B \) be as in Theorem 1. Is there a continuous (real analytic)
family of deformations from $A(D^n)$ to $B$. That is, does it exist a continuous (real analytic) map $\alpha \mapsto T_\alpha$ from the unit segment into $\mathcal{L}(A(D^n), C(D^n))$, such that $\text{Im}(T_\alpha)$ is a uniform algebra for $\alpha \in [0,1]$?

**Problem 3.** Extend the result of Theorem 1 to other domains in $\mathbb{C}^n$.

**Problem 4.** Can Theorem 1 be extended to cover non-separable algebra $H^\infty(\Omega)$ of analytic, bounded functions on $\Omega$?

References


[RR1] R. Rochberg, “Almost isometries of Banach Spaces and moduli of


