Nonlinear generalizations of the Banach–Stone theorem

by

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Abstract. Let \( X, Y \) be locally compact sets. Assume there is a map \( T \) from \( C_0(X) \) onto \( C_0(Y) \) with \((1-\varepsilon)||f-g|| \leq ||Tf-Tg|| \leq (1+\varepsilon)||f-g||\), for \( f, g \in C_0(X) \), where \( \varepsilon \) is sufficiently small. Then \( C_0(X) \) and \( C_0(Y) \) are linearly isometric. Similar results hold for function algebras, extremely regular function spaces, and certain spaces of vector-valued functions.

§ 0. Introduction. In this paper we study the following problem:

Does there exist a universal constant \( \varepsilon > 0 \) such that if \( T \) is a map from a Banach space \( A \) onto a Banach space \( B \) which satisfies

\[ (1-\varepsilon)||f-g|| \leq ||Tf-Tg|| \leq (1+\varepsilon)||f-g||, \quad \text{for } f, g \in A, \]

then

1. \( A \) and \( B \) are linearly homeomorphic,
2. if \( A \) and \( B \) are subspaces of \( C_0(X) \) and \( C_0(Y) \), respectively, then \( X \) and \( Y \) are homeomorphic?

We prove that the answer to the first problem is positive if both \( A \) and \( B \) are \( C_0(X) \) spaces. The answer to the second problem is evidently negative, in general, since any Banach space can be represented in a number of ways as a subspace of a space \( C(X) \), with different \( X \). However, we prove here that the answer to this question is positive if \( A, B \) are extremely regular function spaces.

The source of our problem is the classical Banach–Stone theorem which states that the Banach spaces \( C(X) \) and \( C(Y) \) are linearly isometric if and only if \( X \) and \( Y \) are homeomorphic. In the sixties Amir [2] and Camberra [6, 7] proved that this result is stable: if there is a linear homeomorphism \( T \) from \( C(X) \) onto \( C(Y) \) with \( ||T||\cdot||T^{-1}|| < 2 \) then \( C(X) \) and \( C(Y) \) are actually linearly isometric. During the next years, linear isomorphisms with a small bound were studied in a number of papers; see, e.g., [3–4, 8–9, 11–16, 20].

The assumption about small bound is essential. For example, the Banach space \( C[0, 1] \) is linearly homeomorphic to \( C(X) \), for any compact, metric, uncountable space \( X \). In the nonlinear case we even have the more
far-reaching, well-known result of Kadec [18] that any two separable infinite-dimensional Banach spaces are homeomorphic. The situation changes significantly if we consider uniform homeomorphisms. For example, if a Banach space $E$ is uniformly homeomorphic to a Hilbert space $H$, then $E$ and $H$ are linearly homeomorphic [19]. Hence the answer to our first problem is positive if $A$ is a Hilbert space. On the other hand, in 1977 Aharoni and Lindenstrauss [1] gave an example of two Banach spaces $C_0(X)$ and $C_0(Y)$ which are Lipschitz equivalent but not linearly homeomorphic. This means that the assumption that $\varepsilon$ is small is essential also in the first problem.

§ 1. The result. We use the standard Banach space terminology. For a closed subspace $A$ of $C_0(X)$ we denote by $\mathrm{Ch} A$ the set of all points $x_0 \in X$ such that for any $\varepsilon > 0$ and any neighborhood $U$ of $x_0$ there is an $f \in A$ with $\|f\| = f(x_0) = 1$ and $|f(x)| < \varepsilon$ for $x \in X - U$. $A$ is called an almost extremely regular subspace of $C_0(X)$ if $\mathrm{Ch} A$ is dense in $X$ and is called extremely regular if $\mathrm{Ch} A = X$. The main examples of almost extremely regular subspaces are function algebras. By a function algebra we mean any closed subalgebra of $C_0(X)$ which separates points of $X$. By $\partial A$ we denote the Shilov boundary of $A$ and $\mathrm{Ch} A$ defined above coincides with the Choquet boundary of $A$. A function algebra $A$ is an almost extremely regular subspace of $C(\partial A)$ and it is extremely regular if $\mathrm{Ch} A = \partial A$.

A map $T$ between Banach spaces $A$ and $B$ is called $\varepsilon$-bi-Lipschitz if the condition (*) is satisfied.

All our results hold both in the real and in the complex cases.

Theorem 1. Let $X, Y$ be locally compact Hausdorff spaces, let $A$ be an almost extremely regular subspace of $C_0(X)$ and $B$ an extremely regular subspace of $C_0(Y)$. Assume there is an $\varepsilon$-bi-Lipschitz map $T$ from $A$ onto $B$ with $T0 = 0$ and with $\varepsilon \leq \varepsilon_0$. Then there is a homeomorphism $\varphi$ from $X$ onto $Y$ and

\[
(\ast \ast) \quad \|((Tf) \circ \varphi) - |f|\| \leq c(\varepsilon)\|f\|, \quad \forall f \in A,
\]

where $\varepsilon_0$ is an absolute constant and $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We postpone the proof of this result to the next section. Now we get two corollaries.

Theorem 2. Let $X, Y, T, A, B, \varphi$ be as in Theorem 1 and assume $X$ is paracompact. Then there is a scalar-valued continuous function $u$ defined on $X$, of modulus one, such that:

(a) if the spaces $A$ and $B$ are real, then

\[
(1) \quad \|((Tf) \circ \varphi - u \cdot f)\| \leq c(\varepsilon)\|f\|, \quad \forall f \in A;
\]
Banach–Stone theorem

(b) if \( A, B \) are complex, then \( X = X_1 \cup X_2 \) and for any \( f \in A \) with \( \|f\| \leq 1 \) we have

\[
\begin{align*}
|Tf(\varphi(x)) - u(x)f(x)| & \leq c'(\varepsilon), \quad \forall x \in X_1, \\
|Tf(\varphi(x)) - u(x)f(x)| & \leq c'(\varepsilon), \quad \forall x \in X_2,
\end{align*}
\]

where \( c'(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

Note that (1) and (2)–(3) are formulated in different ways. In (1) and (**) we have 

\[ \ldots \leq c(\varepsilon)\|f\|, \quad \forall f \in A, \] 

while in (2) and (3) we have 

\[ \ldots \leq c(\varepsilon), \quad \forall f \in A \text{ with } \|f\| \leq 1 \]. 

The first type of statement is stronger, in general, since \( T \) is nonlinear.

Assume we can prove Theorem 1 with the weaker statement

\[
\|(Tf) \circ \varphi| - |f|| \leq c(\varepsilon), \quad \forall f \in A, \|f\| \leq 1,
\]

in place of (**) if \( T \) is an \( \varepsilon \)-bi-Lipschitz map, then for any \( t > 0 \) the map

\[
A \ni f \rightarrow (1/t) T(tf) \in B
\]

is also \( \varepsilon \)-bi-Lipschitz. Hence for any \( t > 0 \) there is a homeomorphism \( \varphi_t : X \rightarrow Y \) such that

\[
\|(Tf) \circ \varphi_t| - |f|| \leq tc(\varepsilon), \quad \forall f \in A, \|f\| \leq t.
\]

Hence

\[
\|(Tf) \circ \varphi_t| - |(Tf) \circ \varphi_{t'}|| \leq (t' + t')c(\varepsilon), \quad \forall f \in A, \|f\| \leq \min(t', t').
\]

Using (4) and the regularity of \( B \) it is easy to note that the map \( R^+ \ni t \rightarrow \varphi_t \) is locally constant, so it is constant.

The above means that in Theorem 1 our two types of equation are equivalent. The same situation is in the first part of Theorem 2 because two unimodular real-valued functions are identical or far from each other (with the distance equal to 2). A problem arises in the complex case. From (2) we deduce that for any \( t > 0 \) there is a unimodular function \( u \) such that

\[
|Tf(\varphi(x)) - u(x)f(x)| \leq tc(\varepsilon), \quad \forall f \in A, \|f\| \leq t, \quad \forall x \in X,
\]

but now \( u \) may depend on \( t \). This may happen even if \( X \) and \( Y \) are just one-point sets \([17]\). To deduce Theorem 2 from Theorem 1 and then to prove Theorem 1, we need the following proposition, which is an immediate consequence of Proposition 2 of \([10]\) (put, in Proposition 2, \( f := T, \ x := 2f, \ y := 2g \) and also \( x := 2f, \ y := 0 \)).

**Proposition.** Let \( T \) be an \( \varepsilon \)-bi-Lipschitz map from a Banach space \( E \) onto a Banach space \( F \), with \( \varepsilon < 1/3 \) and \( T0 = 0 \). Then for any \( f, g \in E \) with \( \|f\|, \|g\| \leq 2 \) we have

\[
\|(Tf + Tg) - T(f + g)\| \leq \varepsilon'
\]
and

\[ \|f + g\| - \varepsilon' \leq \|Tf + Tg\| \leq \|f + g\| + \varepsilon', \]

where \( \varepsilon' = 100\varepsilon^{1/10}. \)

Proof of Theorem 2. We define a carrier \( F: X \to 2^B \) by

\[ F(x) = \{ g \in B : \|g\| = 1 = g(\varphi(x)) \} \]

Since \( B \) is extremely regular \( F(x) \) is nonempty for any \( x \in X \). It is also easy to check that \( F \) is a norm lower semicontinuous, convex, complete carrier, so by the Michael Selection Theorem [5] there is a norm continuous function \( F: X \to B \) such that \( \|F(x)\| = 1 = F(x)(\varphi(x)) \) for all \( x \in X \). We define \( v \) on \( X \) by

\[ v(x) = T^{-1}(F(x))(x), \quad \text{for } x \in X. \]

The map \( v \) is continuous and by (**) \n
\[ \|v(x) - 1\| \leq c(\varepsilon)(1 + 2\varepsilon), \quad \forall x \in X. \] \hspace{1cm} (5)

Put \( \tilde{A} = A/v := \{ f/v : f \in A \} \) and define \( \tilde{T}: \tilde{A} \to B \) by \( \tilde{T}(f/v) = T(f) \). The space \( \tilde{A} \) is an almost extremely regular subspace of \( C_0(X) \) and \( \tilde{T} \) is an \( (\varepsilon + 2c(\varepsilon)) \)-bi-Lipschitz map. By the definition of \( \tilde{T} \) and (5) it is sufficient to prove Theorem 2 for \( \tilde{T} \) and \( \tilde{A} \) in place of \( T \) and \( A \). We have also \( \tilde{T}^{-1}(F(x))(x) \equiv 1 \). Hence to simplify notation we can just assume that \( v \equiv 1 \), \( X = Y \), \( \varphi = \text{id} \) and that

\[ \|Tf - |f|\| \leq \varepsilon, \quad \forall f \in A, \|f\| \leq 3. \]

Fix \( x \in X \). For any \( f \in A \) with \( \|f\| \leq 1 \) we have

\[ \|T(f + T^{-1}(F(x))) - |f + T^{-1}(F(x))|\| \leq \varepsilon. \] \hspace{1cm} (6)

By the Proposition and since \( \varphi(x) = x \) we have also

\[ \|1 + Tf(x) - |T(f + T^{-1}(F(x)))(x)|\| \leq \|(F(x) + Tf) - T(f + T^{-1}(F(x)))\| \leq \varepsilon'. \]

Hence evaluating (6) at the point \( x \) we get

\[ \|1 + Tf(x) - |1 + f(x)|\| \leq \varepsilon + \varepsilon'. \]

Hence, since \( |f(x)| \leq 1 \) and \( |Tf(x)| \leq 1 + \varepsilon \), by a direct computation we get, in the real case,

\[ |Tf(x) - f(x)| \leq 2\varepsilon + \varepsilon', \]

and in the complex case:

(a) \[ |Tf(x) - f(x)| \leq 2\varepsilon + \varepsilon', \] or
(b) \(|Tf(x) - \overline{f}(x)| \leq 2\varepsilon + \varepsilon'.\)

Now to end the proof we have to show that in the complex case, for any \(x \in X\), we have always \(Tf(x) \approx \overline{f}(x)\) or always \(Tf(x) \approx f(x)\) independently of \(f\). Assume that

\[ T(if(x))(x) \approx if(x)(x) = i, \]

and let \(f \in A\) with \(\|f\| \leq 1\). Assume \(Tf(x) \approx \overline{f(x)}\). Hence, by the previous result and the Proposition, we get

\[ f(x) + i = f(x) + if(x)(x) \approx T(f + if(x))(x) \approx Tf(x) + i \]

or

\[ f(x) + i = f(x) + if(x)(x) \approx \overline{T(f + if(x))(x)} \]

\[ = \overline{Tf(x)} - i \approx f(x) - i. \]

The latter is impossible, so (7) implies \(Tf(x) \approx f(x)\) for all \(f \in A\). If \(T(if(x))(x) = -i\), then by exactly the same argument we get \(Tf(x) \approx \overline{f(x)}\) for all \(f \in A\).

**Corollary.** Let \(A, B\) be uniform algebras such that \(\partial B = \text{Ch} B\). Assume that there is an \(\varepsilon\)-bi-Lipschitz map from \(A\) onto \(B\), with \(\varepsilon \leq \varepsilon_0\) (absolute constant). Then \(\partial A\) and \(\partial B\) are homeomorphic. Furthermore, if \(B = C(\partial B)\) then \(A = C(\partial A)\).

§ 2. **Proof of Theorem 1.** To prove the theorem we first need some notation. For \(x_0\) in \(X\) a net \((f_a)_{a \in A} \subset C_0(X)\) is called **peaking** at \(x_0\) if:

(i) for any \(a \in A\), \(\|f_a\| = 1 = f_a(x_0)\), and

(ii) \(f_a \to 0\) uniformly off any neighborhood of \(x_0\).

We denote by \(P_1^A(x_0)\) the set of all nets \((f_a)_{a \in A}\) in \(A\) such that \((\|f_a\|)_{a \in A}\) peaks at \(x_0\) and by \(P_2^A(x_0)\) the subset of \(P_1^A(x_0)\) consisting of nets which peak at \(x_0\).

Fix \(M \geq 0\). For any \(x_0 \in X\) and \(i = 1, 2\) we define

\[ S_{x_0}^i = \{ y \in Y : \exists (f_a)_{a \in A} \in P_1^A(x_0) \exists (y_a)_{a \in A} \in Y, \]

\[ y_a \to y \text{ and } \forall a \in A \ |Tf_a(y_a)| \geq M_i \}. \]

Evidently \(S_{x_0}^2 \subset S_{x_0}^1\).

We divide the proof into a number of simple steps and at various points of the proof we use inequalities involving \(\varepsilon\) which are valid only if \(\varepsilon\) is sufficiently small; in these circumstances we will merely assume that \(\varepsilon\) is near zero. These assumptions are the source of \(\varepsilon_0\).

The strategy of the proof is the following: Steps 1 and 2 show that, for a suitable \(M\) and any \(x \in \text{Ch} A\), the sets \(S_{x}^1, S_{x}^2\) coincide and contain exactly
one point. This gives a function $\varphi: \text{Ch}\ A \to Y$ defined by $\{\varphi(x)\} = S^1_x = S^2_x$. Steps 3 and 4 show that

$$|Tf(\varphi(x))| \leq |f(x)| + o(\varepsilon_0),$$

and Steps 5,6 prove that

$$|Tf(\varphi(x))| \geq |f(x)| - o(\varepsilon),$$

which together give (**) In the last four steps we show that $\varphi$ is continuous and injective and can be extended to a continuous and still injective map from $X$ onto $Y$. The main tools of the proof are the Proposition, which says that $T$ is "almost" additive, and the peaking functions, which, as we show, are mapped to "almost" peaking functions.

**Step 1.** If $M \leq 1 - \varepsilon - \varepsilon'$, then for any $x_0 \in \text{Ch} A$ we have $S^2_{x_0} \neq \emptyset$.

**Proof.** If $Y$ is compact, this is an immediate consequence of (G) with $\varepsilon = 0 = T_{x_0}$. To consider the general case, let $(f_{x_{0 \in A}}) \in \mathcal{P}^1_A(x_0)$. Fix $x_0 \in A$. For any $x \in A$ we have $\|f_{x} + f_{x_0}\| = 2$, so by (G), $\|Tf_{x} + Tf_{x_0}\| \geq 2 - \varepsilon'$. On the other hand, $\|Tf_{x_0}\| \leq 1 + \varepsilon$, so we get

$$\sup \{y \in Y: |Tf_{x_0}(y)| \geq 1 - \varepsilon\} \geq 1 - \varepsilon - \varepsilon'.$$

The set $\{y \in Y: |Tf_{x_0}(y)| \geq 1 - \varepsilon\}$ is compact, so its intersection with $S^2_{x_0}$ is not empty.

**Step 2.** If $M \geq \sqrt{2}/2 + 2\varepsilon'$, then for any $x_0 \in \text{Ch} A$ the set $S^1_{x_0}$ has at most one point.

**Proof.** Assume $y^1, y^2$ are two distinct points of $S^1_{x_0}$ and let $(f_{x_{0 \in A}}) \in \mathcal{P}^1_A(x_0), (y^i_{x_{0 \in A}}) \subset Y, i = 1, 2$, be the corresponding nets given by the definition of $S^1_{x_0}$. Without loss of generality we can assume that $Tf^i_{x_{0 \in A}}(y^i_{x_{0 \in A}}) \rightarrow x^i$ with $|x^i| \geq M$ for $i = 1, 2$. Since $B$ is extremely regular there are $g_1, g_2 \in B$ such that

$$g_i(y^i) = \frac{|x^i|}{|x^i|} \quad \text{for } i = 1, 2, \quad \|g_1| + |g_2|\| \leq 1 + \varepsilon'/3.$$

We have

$$\liminf_{x_{0 \in A}}\|Tf^i_{x_{0 \in A}} + g_i\| \geq 1 + M, \quad i = 1, 2;$$

hence, by the Proposition,

$$\liminf_{x_{0 \in A}}\|f^i_{x_{0 \in A}} + T^{-1}g_i\| \geq 1 + M - \varepsilon', \quad i = 1, 2.$$

Hence, by the definition of the peaking sequence, $|T^{-1}g_i(x_0)| \geq M - \varepsilon'$ for $i$
= 1, 2, and we get
\[ \max(\|T^{-1}g_1 + T^{-1}g_2\|, \|T^{-1}g_1 - T^{-1}g_2\|) \geq \sqrt{2}(M - \varepsilon) \]
(in the real case we even have \( \max(\ldots) \geq 2(M - \varepsilon) \)). Now, by the Proposition or just by (4) and since \( M > \sqrt{2}/2 + 2\varepsilon' \), we get
\[ \max(\|g_1 + g_2\|, \|g_1 - g_2\|) \geq \sqrt{2}(M - \varepsilon') - \varepsilon' > 1 + \varepsilon'/3, \]
which contradicts the assumption \( \|g_1\| + \|g_2\| \leq 1 + \varepsilon'/3 \).

In the remaining part of the proof we assume that \( \sqrt{2}/2 + 2\varepsilon' < M = 1 - \varepsilon - \varepsilon', \) and we define \( \varphi : \text{Ch} A \to Y \) by \( \varphi(x) = S^1_{x_0} \).

We denote by \( \Gamma \) the set of all scalars of modulus one; this means \( \Gamma \) is a two-point set or a circle.

**Step 3.** Fix \( x_0 \in \text{Ch} A \) and \( f_0 \in A \) with \( \|f_0\| = 1 = f_0(x_0) \). For any \( \lambda \in \Gamma \) we define \( \chi(\lambda) = T(\lambda f_0)(\varphi(x_0)) \). We have:

(i) \( \forall \lambda \in \Gamma \quad |\chi(\lambda)| \geq M. \)

(ii) \( \{ \chi(\lambda)/|\chi(\lambda)| : \lambda \in \Gamma \} = \Gamma. \)

**Proof.** Let \( (\lambda_k)_{\alpha \in A} \in P^1_{\lambda}(x_0) \). As in Step 1 we have \( \|\lambda \lambda_k + \lambda f_0\| = 2, \forall \alpha \in A, \) hence
\[ \|T(\lambda \lambda_k) + T(\lambda f_0)\| \geq 2 - \varepsilon', \quad \forall \alpha \in A, \]
and therefore
\[ \sup \{\|T(\lambda f_0)(y)\| : y \in Y, \|T(\lambda \lambda_k)(y)\| \geq 1 - \varepsilon' \} \geq 1 - \varepsilon' - \varepsilon. \]

It follows that there is a net \( (y_\alpha) \) such that \( |T(\lambda f_0)(y_\alpha)| \geq 1 - \varepsilon \) and \( |T(\lambda \lambda_k)(y_\alpha)| \geq 1 - \varepsilon' - \varepsilon. \) Hence, by the definition of \( S^1_{x_0} \), we have
\[ \sup \{\|T(\lambda f_0)(y)\| : y \in S^1_{x_0} \} \geq 1 - \varepsilon' - \varepsilon, \]
so, since \( S^1_{x_0} = \{ \varphi(x_0) \} \), we get \( |\chi(\lambda)| \geq M. \)

To prove (ii), note that by our assumption (4), \( \chi \) is continuous, and \( |\chi(\lambda) + \chi(-\lambda)| \leq \varepsilon' \); hence
\[ \tilde{\chi} : \Gamma \ni \lambda \mapsto \chi(\lambda)/|\chi(\lambda)| \in \Gamma \]
is a continuous function such that \( |\tilde{\chi}(\lambda) + \tilde{\chi}(-\lambda)| \leq 2(1 - 1/M) + \varepsilon' < 1, \) so \( \tilde{\chi} \) is surjective.

**Step 4.** For any \( f_0 \in A \) with \( \|f_0\| \leq 2 \) and \( x \in \text{Ch} A \) we have
\[ |Tf_0(\varphi(x))| \leq |f_0(x)| + 2\varepsilon' + \varepsilon. \]

**Proof.** We have
\[ 1 + |f_0(x)| = \inf \{\sup \{\|f_0 + \lambda f\| : \lambda \in \Gamma\} : f \in A, \|f\| = f(x) \leq 1\}, \]
and by Step 3.

\[ M + |Tf_0(\varphi(x))| \leq \inf \{ \sup \{ ||Tf_0 + T(\lambda f)|| : f \in A, \|f\| = f(x) \leq 1 \} \}. \]

Hence, by the assumption that \( M = 1 - \epsilon - \epsilon' \) and the Proposition,

\[ |Tf_0(\varphi(x))| \leq |f_0(x)| + (1 - M) + \epsilon' = |f_0(x)| + 2\epsilon' + \epsilon. \]

**Step 5.** Fix \( x_0 \in \text{Ch} A \) and \( g \in B \) with \( g(\varphi(x_0)) = \|g\| = 1 \). For any \( \lambda \in \Gamma \) we define \( K(\lambda) = T^{-1}(\lambda g)(x_0) \). We have:

(i) \( \forall \lambda \in \Gamma \quad |K(\lambda)| \geq 1 - 2\epsilon' - \epsilon. \)

(ii) \( |K(\lambda)|/|K(\lambda)| : \lambda \in \Gamma' = \Gamma. \)

**Proof.** (i) is an immediate consequence of Step 4, and we get (ii) exactly as in the proof of (ii) in Step 3.

**Step 6.** For any \( f_0 \in A \) with \( \|f_0\| \leq 1 + 2\epsilon \) and \( x_0 \in \text{Ch} A \) we have

\[ |Tf_0(\varphi(x_0))| \geq |f_0(x_0)| - 4\epsilon'. \]

**Proof.** Since \( B \) is extremely regular, there is a \( g \in B \) such that \( \|g\| = 1 = g(\varphi(x_0)) \) and

\[ ||Tf_0 + |g|| \leq 1 + |Tf_0(\varphi(x_0))| + 4\epsilon. \]

By Step 5 there is a \( \lambda \in \Gamma \) such that

\[ |f_0 + T^{-1}(\lambda g)| \geq |f_0(x_0)| + (1 - 2\epsilon' - \epsilon). \]

Hence, by the Proposition,

\[ ||Tf_0 + \lambda g|| \geq |f_0(x_0)| + (1 - 2\epsilon' - \epsilon) - \epsilon'. \]

From (9) and (10) we get (8).

**Step 7.** \( \varphi \) can be extended to a continuous function from \( X \) into \( Y \). We denote the extended function by the same symbol.

**Proof.** Assuming the contrary, there is an \( x_0 \in X \) and two nets \( (x^i_y)_{\gamma \in \Gamma_i} \), \( i = 1, 2 \), in \( \text{Ch} A \), both converging to \( x_0 \), such that

\[ y^i_\gamma = \varphi(x^i_y) \rightarrow y^i, \]

where \( y^1 \neq y^2 \in Y \) (by Step 6, if \( Y \) is not compact, \( \varphi(x) \) still cannot be divergent to \( \infty \)). In particular, to prove the continuity of the original function \( \varphi \) we assume that \( (x^2_{\gamma})_{\gamma \in \Gamma_2} \) is a sequence constantly equal to \( x_0 \). By Step 4, for any \( g \in B \) with \( ||g|| \leq 1 \) we have

\[ \|g(y^2_\gamma)| \leq |T^{-1}g(x^1_\gamma)| + 2\epsilon' + \epsilon, \quad \text{for } i = 1, 2, \gamma \in \Gamma_i, \]
hence

\begin{equation}
|g(y^i)| \leq |T^{-1}g(x_0)| + 2\varepsilon + \varepsilon, \quad \text{for } i = 1, 2.
\end{equation}

Now, as in the proof of Step 2, we let \( g_1, g_2 \in B \) be such that \( g_i(y^i) = 1 \) for \( i = 1, 2 \) and \( \|g_1\| + \|g_2\| \leq 1 + \varepsilon \). From (11) we get

\[ |T^{-1}g_i(x_0)| \geq 1 - 2\varepsilon - \varepsilon, \quad i = 1, 2. \]

Hence \( \max \|T^{-1}g_1 \pm T^{-1}g_2\| \geq \sqrt{2}(1 - 2\varepsilon - \varepsilon) \), so

\[ \max \|g_1 \pm g_2\| \geq \sqrt{2}(1 - 2\varepsilon - \varepsilon) - \varepsilon, \]

which contradicts the assumption \( \|g_1 \pm g_2\| \leq 1 + \varepsilon \).

**Step 8. If \( X \) is noncompact then neither is \( Y \), and \( \varphi \) can be extended to a continuous map from \( X^* = X \cup \{\infty\} \) (the one-point compactification of \( X \)) into \( Y^* = Y \cup \{\infty\} \) (the one-point compactification of \( Y \)).**

**Proof.** Assuming the contrary, there is a net \( (x_\gamma)_{\gamma \in I} \) in \( X \) tending to \( \infty \) such that \( \varphi(x_\gamma) = y_\gamma \) tends to some point \( y_0 \) of \( Y \). By Step 4 and since \( \text{Ch} A \) is dense in \( X \), for any \( g \in B, \|g\| \leq 1 \), we get

\[ |g(y_{0})| = \lim_{\gamma} |g(y_{\gamma})| \leq \lim_{\gamma} |T^{-1}g(x_{\gamma})| + 2\varepsilon' + \varepsilon = 2\varepsilon' + \varepsilon < 1, \]

which contradicts the assumption that \( B \) is extremely regular.

**Step 9. \( \varphi \) maps \( X \) onto \( Y \).**

**Proof.** By the previous step \( \varphi \) is a closed map, so it is sufficient to show that \( \varphi(X) \) is dense in \( Y \). Assuming the contrary, there is a \( g \in B \) with \( \|g\| = 1 \) such that

\[ \sup \|g(\varphi(x))\|: x \in X < \varepsilon. \]

But on the other hand, by Step 6 and the Proposition,

\[ \sup \|g(\varphi(x))\|: x \in X \geq \sup \|T^{-1}g(x)\|: x \in X = 1 - 4\varepsilon' - \|g\|-4\varepsilon' = 1 - 3\varepsilon'. \]

**Step 10. \( \varphi \) is injective.**

**Proof.** Let \( x_1 \in X_0 := \text{Ch} A, x_2 \in X, x_1 \neq x_2 \), and let \( f \in A \) be such that \( \|f\| = f(x_1) = 1 \) and \( |f(x_2)| < \varepsilon \). By Steps 3 and 4,

\[ |Tf(\varphi(x_1))| \geq 1 - \varepsilon' - \varepsilon, \quad |Tf(\varphi(x_2))| \leq 2(\varepsilon' + \varepsilon). \]

Hence \( \varphi(x_1) \neq \varphi(x_2) \), so in particular \( \varphi|_{X_0} \) is injective.

Let us now pick one point from each set \( \varphi^{-1}(y), y \in Y \), i.e. let us consider a function \( \psi: Y \rightarrow X \) such that \( \varphi \circ \psi \) is the identity map. By Step 8, \( \varphi \) is a closed map, so \( \psi \) is continuous. To prove that \( \varphi \) is injective we have
to prove that $\psi$ is surjective. Since $\varphi|_{X_0}$ is injective $\psi(Y)$ contains $X_0$, so is a dense subset of $X$. Hence if $Y$ is compact then we are done. If $Y$ is not compact, then by Step 8, $\psi$ can be extended to a continuous function from $Y^*$ into $X^*$ with $\psi(\infty) = \infty$; now the domain is again compact and so is the image.

Now by Steps 7–10, $\varphi$ is a homeomorphism from $X$ onto $Y$ and (**) follows from Steps 4 and 6 with $c(\varepsilon) = 4\varepsilon' = 400\varepsilon^{1/10}$.

§ 3. Remark. Note that the complex Banach space $C_0(X)$ is isometric to $C_0^b(X) \otimes H_2$ (the injective tensor product of the real $C_0(X)$ space and the real two-dimensional Hilbert space) as well as to $C_0^b(X, H_2)$ (the real Banach space of $H_2$-valued continuous functions on $X$ vanishing at infinity). The inspection of the proof of Theorem 1 immediately shows that we have only used the following properties of $A \subset C_0^b(X) \otimes H_2$, $B \subset C_0^b(Y) \otimes H_2$:

(i) $A$ is almost extremely regular and $B$ is extremely regular.

(ii) $\exists c > 1 \ \forall h_1, h_2 \in H_2, \ ||h_1|| = 1 = ||h_2||,$

\[ \max(||h_1 + h_2||, ||h_1 - h_2||) \geq c. \]

(iii) The unit ball of $H_2$ is compact.

Hence by exactly the same arguments we can get the following more general result.

Theorem 4. Let $X$, $Y$ be locally compact Hausdorff spaces, let $A$ be an almost extremely regular subspace of $C_0(X)$ and $B$ an extremely regular subspace of $C_0(Y)$, and let $E$ be a strictly convex finite-dimensional Banach space. Assume $T$ is an $\varepsilon$-bi-Lipschitz map from $A \otimes E \subset C_0(X, E)$ onto $B \otimes E \subset C_0(Y, E)$ with $T(0) = 0$ and $\varepsilon \leq \varepsilon_0$. Then there is a homeomorphism $\varphi: X \to Y$ such that

\[ ||T(\varphi(x))|| - ||f(x)|| \leq c(\varepsilon)||f||, \quad \forall f \in A \otimes E, \]

where $\varepsilon_0 > 0$ as well as the function $c(\cdot)$, with $c(\varepsilon) \to 0$ as $\varepsilon \to 0$, depend on the modulus of convexity of $E$ only.

References


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