IV. PERTURBATIONS OF OPERATOR ALGEBRAS

§ 11. Introduction.

This chapter is devoted to the study of some operator algebras from the point of view of whether the metric and algebraic perturbations produce the same class of algebras. First we prove that if $X$ and $Y$ are strictly convex Banach spaces and $T$ is an isometry from the algebra $K(X)$ of all compact operators from $X$ into itself, onto $K(Y)$ then $K(X)$ and $K(Y)$ are isomorphic in the category of Banach algebras. Next we show that this is no longer true, in general, for $c$-isometries but it remains true if we restrict ourselves to uniformly convex spaces.

For this we need some definitions and notation:

For Banach spaces $U$ and $V$:

- $B(V)$ denotes the closed unit ball of $V$;
- $E(V)$ denotes the set of extreme points of $B(V)$;
- $U \hat{\otimes} V$ denotes the injective tensor product of $U$ and $V$;
- $L(U,V)$ denotes the Banach space of all continuous linear operators from $U$ into $V$; if $U = V$ we write $L(U)$ in place of $L(U,U)$;
- $F(U,V)$ denotes the closure in $L(U,V)$ of the algebra of finite dimensional operators, if $U = V$ we write $F(U)$ in place of $F(U,U)$ (notice that if $V$ has the approximation property then $F(U,V)$ is the algebra of all compact operators);
- by $U \varphi V$ we mean that $U$ and $V$ are isometric.

Throughout this chapter we identify $F(U,V)$ with $U^* \hat{\otimes} V$ and we frequently view a Banach space $V$ as a subspace of $C(E(V^*))$ or $C(B(V^*))$ where $E(V^*) \subset B(V^*)$ are equipped with the weak * topology. The space $V \hat{\otimes} W$ is regarded as a subspace of $C(B(V^*) \times B(W^*))$.

A linear isometry $T$ from $F(U)$ onto $F(V)$ we call canonical if one of the following three possibilities holds:

a) $U \varphi V$ and

$$T = \tau_U \otimes \tau_V$$

where $\tau_U : U^* \varphi V^*$ and $\tau_V : U \varphi V$ are onto isometries;
to the study of some operator algebras.

whether the metric and algebraic perturbations of algebras. First we prove that if $X$

Banach spaces and $T$ is an isometry from compact operators from $X$ into itself, onto

are isomorphic in the category of Banach this is no longer true, in general, for true if we restrict ourself to uniformly

Definitions and notation:

- $V$:
  - unit ball of $V$;
  - of extreme points of $B(V)$;
  - injective tensor product of $U$ and $V$; such space of all continuous linear operators.

- If $U^* = V$ and $T$ is of the form
  
  $T(s) = F_0 \circ s^* \circ G_0$ for all $s \in F(U)$,

  where $F_0 : U^* \to V$ and $G_0 : V \to U^*$ are onto isometries;

- $U = V^*$ and $T^{-1}$ if of the form b).

Notice that if there is a canonical isometry from $F(U)$ onto $F(V)$
then $F(U)$ and $F(V)$ or $F(U)$ and $F(V^*)$ are isomorphic in the cate-
gory of Banach algebras and for finite dimensional Banach spaces $U$
and $V$ any canonical isometry $T$ from $L(U)$ onto $L(V)$ such that

$T(I^U_0) = I^V_0$ is an algebra isomorphism or antiisomorphism.

Notice also that if there exists a canonical onto isometry of the
form b) or c) then $U$ and $V$ have to be reflexive, so b) and
c) are equivalent.

For a Banach space $V$ we denote by $Multi(V)$ the multiplier alge-
bra of $V$. $Multi(V)$ consists of all linear and continuous operators
$S : V \to V$ such that every extreme functional is an eigenvector of the
conjugate operator $S^*$. By $Z(V)$ we denote the centralizer of $V$.

All fundamental results on multiplier algebra and centralizer are to be
found in Behrends [1].

§ 12. Isometries in operator algebras.

12.1. THEOREM. Let $X$ and $Y$ be Banach spaces such that $X^{**}$
and $Y^{**}$ are strictly convex, then any isometry from $F(X)$ onto
$F(Y)$ is canonical.

For finite dimensional Banach spaces we get the following stronger result:

12.2. THEOREM. Let $X$ and $Y$ be finite dimensional Banach spa-
ces and assume one of the spaces $X$, $X^*$, $Y$, $Y^*$ is strictly convex.
Then any isometry from $L(X)$ onto $L(Y)$ is canonical.
The above theorems are fulfilled as well in the real and the complex case.
We will prove both of the above theorems together.

12.3. Lemma. Let $U$ and $V$ be any linear spaces and assume
\[ u_1 \cdot v_1 + u_2 \cdot v_2 = u_3 \cdot v_3 \]
where $u_i \in U$, $v_i \in V$, $i = 1, 2, 3$; then either the vectors $u_1, u_2, u_3$ or the vectors $v_1, v_2, v_3$ are proportional.

*Proof.* Let $\psi^*$ be any linear functional on $V$. We have
\[ u_1 \psi^*(v_1) + u_2 \psi^*(v_2) = u_3 \psi^*(v_3) \]
hence, if $\psi^*(v_3) \neq 0$ then $u_3$ is a linear combination of $u_1$ and $u_2$; if $u_1, u_2, u_3$ were not proportional then the coefficients of this linear combination would be uniquely determined and this would mean
\[ \psi^*(v_1) = \text{const} \psi^*(v_3), \quad \psi^*(v_2) = \text{const} \psi^*(v_3) \]
for any $\psi^* \in V^*$.

Hence $v_1 \parallel v_2$ and $v_2 \parallel v_3$.

12.4. Lemma. Let $U$ and $V$ be Banach spaces then
\[ \mathcal{E}(U \oplus V)^* = \mathcal{E}(U)^* \oplus \mathcal{E}(V)^*. \]

*Proof.* We have
\[ U \oplus V \subseteq \{ f \in C(B(U^*) \times B(V^*)) : \forall u^* \in U^* \quad f(u^*, \cdot) \in V \quad \text{and} \quad \forall v^* \in V^* \quad f(\cdot, v^*) \in U^* \}. \]
So the lemma is an immediate consequence of the following fact.

Let $S$ be a compact Hausdorff space, let $A$ be a closed subspace of $C(S)$ which separates points of $S$ and let $s \in S$ be such that the functional $\psi_s \in A^* : \psi_s(f) = f(s)$ has norm one. Then $\psi_s$ is an
fulfilled as well in the real and the complex case above theorems together.

V be any linear spaces and assume $u_3$ proportional.

$v^* = v^*(v_3)$

then either the vectors $u_1, u_2, u_3$ are proportional.

* linear functional on V. We have $u^*_3(v_3) = \text{const} \cdot v^*(v_3)$ for any $v^* \in V^*$.

$V^*$ and V be Banach spaces then $E(V^*)$.

$E(V^*) : v^* \in U^*, f(v^*, \cdot) \in V$ and $v^* \in V^*, f(\cdot, v^*) \in U$.

The consequence of the following fact.

Hausdorff space, let A be a closed subspace points of S and let $s \in S$ be such that $(f) = f(s)$ has norm one. Then $v_s$ is an extreme point of the unit ball of $A^*$ if and only if the measure concentrated in $(s)$ is the unique probability measure $\mu$ on S such that $\int f d\mu = f(s)$ for all $f \in A$.

12.5. Lemma. Let $X, Y, \tilde{Y}$ be Banach spaces with $Y^*$ being strictly convex and let $T : X \otimes \tilde{X} + Y \otimes \tilde{Y}$ be a linear onto isometry. Then for any $\tilde{v}_o \in E(\tilde{Y}^*)$ one of the following possibilities holds:

a) there is an $\tilde{x}_o \in E(\tilde{X}^*)$ and a continuous, linear map $\psi : X \rightarrow Y$ such that $T^*(y^* \otimes \tilde{v}_o) = \psi^*(y^*) \otimes \tilde{x}_o$ for all $y^* \in Y^*$,

b) there is an $\tilde{x}_o \in E(\tilde{X}^*)$ and a continuous, linear map $\psi : \tilde{X} \rightarrow \tilde{Y}$ such that $T^*(y^* \otimes \tilde{v}_o) = \psi^*(y^*) \otimes \tilde{x}_o$ for all $y^* \in Y^*$.

Moreover $\psi^* : Y^* \rightarrow X^*$ and $\psi^* : Y^* \rightarrow \tilde{X}^*$ are isometric embeddings.

Proof. The map $T^* : (X \otimes \tilde{X}^*)^* \rightarrow (X \otimes \tilde{X})^*$ is an onto isometry so it maps the extreme points of the unit ball onto the extreme points of the unit ball and by Lemma 12.4 we get for any $y_1, y_2, y_3 \in E(Y^*)$ there are $x_1, x_2, x_3 \in E(X^*)$ and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in E(\tilde{X}^*)$ such that

$T^*(y_1 \otimes \tilde{v}_o) = x_1 \otimes \tilde{x}_1$ for $i = 1, 2, 3$.

Assume $y_1^* \neq y_2^* \neq 0$. The space $Y^*$ being strictly convex it follows that $(y_1^* + y_2^*)/\|y_1^* + y_2^*\|$ is an extreme point of $B(Y^*)$, so

$x_1^* \otimes \tilde{x}_1 + x_2^* \otimes \tilde{x}_2 = T^*((y_1^* + y_2^*) \otimes \tilde{v}_o) = (y_1^* \otimes \tilde{x}_1 + y_2^* \otimes \tilde{x}_2)$

for some $x_1^* \in E(X^*)$ and $\tilde{x}_i \in E(\tilde{X}^*)$. Hence by Lemma 12.3 we have $x_1^* \|x_1^*\| \otimes \tilde{x}_1 + x_2^* \|x_2^*\| \otimes \tilde{x}_2$. The same arguments show that $x_1^* \|x_1^*\| \otimes \tilde{x}_1 + x_2^* \|x_2^*\| \otimes \tilde{x}_2$ and $x_3^* \|x_3^*\| \otimes \tilde{x}_3 + x_4^* \|x_4^*\| \otimes \tilde{x}_4$. Hence the strict convexity of $Y^*$ imply now that for any $\tilde{v}_o \in E(\tilde{Y}^*)$ one of the following two possibilities takes place:
a. There is an $x^* \in E(\bar{X}^*)$ and a linear, weak * continuous into isometry $\hat{\psi} : Y^* \rightarrow \bar{X}^*$ such that

$$T^*(y^* \circ \hat{\psi}) = \hat{\psi}(y^*) \circ \bar{x}^*_0$$

for all $y^*$ in $Y^*$;

b. There is an $x^* \in E(X^*)$ and a linear, weak * continuous into isometry $\hat{\psi} : Y^* \rightarrow \bar{X}^*$ such that

$$T^*(y^* \circ \hat{\psi}) = x^*_0 \circ \hat{\psi}(y^*)$$

for all $y^*$ in $Y^*$.

To end the proof of the lemma we put $\hat{\psi} = \hat{\psi} \big|_X$ and $\hat{\varphi} = \hat{\psi} \big|_{X^*}$.

Notice now that if $X, \bar{X}, Y, \bar{Y}$ are of the same, finite dimension then maps $\hat{\psi}$ and $\hat{\varphi}$ defined in Lemma 12.5 have to be onto isometries so in this case if $Y^*$ is strictly convex then also one of the spaces $X^*$ or $\bar{X}^*$ have to be strictly convex. The above shows that if there is an onto isometry from $L(X) = X \otimes X^* = L(X^*)$ onto $L(Y) = Y \otimes Y^* = L(Y^*)$ with $X, Y$ being finite dimensional and if $X$ or $X^*$ is strictly convex then also $Y$ or $Y^*$ is strictly convex. So to end, it is sufficient to restrict ourselves to proving Theorem 12.1.

Let $X$ and $Y$ be as in Theorem 12.1 and let $T$ be an onto linear isometry from $F(X) = X \otimes X^*$ onto $F(Y) = Y \otimes Y^*$. We can assume $\dim(Y) \geq 2$ since otherwise the theorem is trivial. By Lemma 12.5 for any $y^{**} \in E(Y^{**})$ one of the following two possibilities holds:

a. There is an $x^{**} \in E(X^{**})$ and a linear, continuous map $\hat{\varphi} : X^{**} \rightarrow Y^{**}$ such that

$$T^*(y^{**} \circ \hat{\varphi}) = x^{**} \circ \hat{\varphi}(y^{**})$$

for all $y^{**} \in Y^{**}$;

b. There is an $x^{**} \in E(X^{**})$ and a linear, continuous map $\hat{\varphi} : X^{**} \rightarrow Y^{**}$ such that

$$T^*(y^{**} \circ \hat{\varphi}) = \hat{\varphi}(y^{**}) \circ x^{**}_0$$

for all $y^{**} \in Y^{**}$.

By Lemma 12.5 applying to both $T$ and $T^{-1}$ we get that $\hat{\varphi}$, and hence $\hat{\psi}$ too, are onto isometries. We shall show that $\hat{\varphi}$ also has to be an onto isometry. Assume to the contrary, let $y^{**}_0, x^{**}_0, \hat{\varphi}$ be as in the point b. above, and let $x^{**} \in E(X^{**}) \setminus E(Y^{**})$. Fix...
and a linear, weak * continuous into such that
\[ y^* \in Y^* \text{ for all } y^* \text{ in } Y^*. \]

and a linear, weak * continuous into such that
\[ \phi(y^*) \text{ for all } y^* \text{ in } Y^*. \]

In the lemma we put \( \phi = \delta^*|_X \) and \( \psi = \psi^*|_X \).

\( X, Y, \bar{X}, \bar{Y} \) are of the same, finite dimensional. The above shows that if there is \( X \cong X \cong X^* \cong L(X^*) \) onto \( L(Y) \cong Y \cong Y^* \) finite dimensional and if \( X \) or \( X^* \) is \( Y \) or \( Y^* \) is strictly convex, so to end, let ourselves to proving Theorem 12.1.

In Theorem 12.1 and let \( T \) be an onto \( \mathbb{R}^n \cong X \cong X^* \) onto \( F(Y) \cong Y \cong Y^* \). We can assume the theorem is trivial. By Lemma 12.5 for the following two possibilities holds:

and a linear, continuous map \( \phi : X^* \to Y^* \)
\[ \phi(y^*) \text{ for all } y^* \in Y^*. \]

and a linear, continuous map \( \psi : X^* \to Y^* \)
\[ \psi(y^*) \text{ for all } y^* \in Y^*. \]

To both \( T \) and \( T^{-1} \) we get that \( \psi^* \), \( \phi^* \), \( \psi^{-1} \phi \) to isometries. We shall show that \( \psi \) also has properties. Assume to the contrary, let \( y^* \in \mathbb{R}^n \), \( x^*, y^* \in X^* \), and let \( x^* \in E(X^*) \setminus Y^*(Y^*) \). Fix

\[ y^* \in E(Y^*). \] The space \( X^* \) is strictly convex so \( X^* \) is smooth and hence there are numbers \( a, b \neq 0 \) such that \( ax^* + by^* \) is an extreme point of the unit ball of \( X^* \). By Lemma 12.5 there are \( y^*_1 \in \mathbb{R}^n \), \( y^*_2 \in E(Y^*) \), \( i = 1, 2 \) such that

\[ (T^{-1})^*(ay^* + by^*) = y^*_1 \oplus y^*_2 \]

and also

\[ (T^{-1})^*(ay^* + by^*) \in y^*_2 \circ y^*_3 \in Y^* \circ Y^*. \]

Hence, by Lemma 12.3 we have \( y^*_1 \parallel y^*_2 \parallel y^*_3 \parallel y^*_4 \parallel y^*_5 \), but \( y^*_1 \parallel y^*_2 \parallel y^*_3 \parallel y^*_4 \parallel y^*_5 \), contradicting our assumption that \( x^* \notin E(Y^*) \). Hence \( y^*_1 \parallel y^*_2 \parallel y^*_3 \parallel y^*_4 \parallel y^*_5 \). Since \( y^*_1 \parallel y^*_2 \parallel y^*_3 \parallel y^*_4 \parallel y^*_5 \) is an arbitrary point of \( E(Y^*) \) we have proven that \( y^*_1 \parallel y^*_2 \parallel y^*_3 \parallel y^*_4 \parallel y^*_5 \) is absurd since \( Y^* \) is not one dimensional.

We have shown that both \( \phi \) and \( \psi \) have to be onto isometries, so \( T \) being one to one proves that for all \( y^*_2 \in E(Y^*) \) the same possibility \( a \) or \( b \) always holds. If the second one holds then \( X \) and \( Y \) are isometric and also by symmetry \( X \) and \( Y \) are isometric so \( X \) and \( Y \) are reflexive and composing \( T \) with the isometry \( F(Y) \cong F(Y^*) \) we have situation \( a \). So without loss of generality we can assume that for any \( y^*_2 \in E(Y^*) \) a. holds.

Let us recapitulate what we have proven:

There is a map \( A : E(Y^*) \cong E(X^*) \) and a map \( B : E(Y^*) \times X^* \to Y^* \) such that

\[ T^*(y^* \circ y^**) = A(y^*) \circ B(y^* \circ y^**) \text{ for all } y^* \in E(Y^*) \]

and \( y^* \in Y^* \) (43)

where by \( B(y^*, \cdot) \) we have denoted the map conjugate to \( B(y^*, \cdot) \).

By symmetry there is also a map \( A : E(X^*) \cong E(Y^*) \) and a map \( B : E(X^*) \times Y^* \to X^* \) such that

\[ (T^{-1})^*(x^* \circ x^**) = A(x^*) \circ B^*(x^* \circ x^**) \text{ for all } x^* \in E(X^*) \]

and \( x^* \in X^* \). (44)

By (43) and (44) the equality \( T \circ T^{-1} = \text{Id} \) gives
\[ y^* \circ y^{**} = \bar{A} \cdot A(y^*) \cdot \bar{B}^* \{ A(y^*), B(y^*, y^{**}) \} \quad \text{for any } y^* \in E(Y^*) \quad \text{and } y^{**} \in Y^{**}. \]

Hence
\[ \bar{A} \cdot A(y^*) = a(y^*) \cdot y^* \quad \text{for all } y^* \in E(Y^*), \quad (45) \]

where \( a \) is a function defined on \( E(Y^*) \) whose values are scalar of modulus one.

12.6. Lemma. For any \( y_1^*, y_2^* \in E(Y^*) \) the linear maps \( B(y_1^*, \cdot) \) and \( B(y_2^*, \cdot) \) are proportional that is, there is a number \( \lambda \neq 0 \) such that \( B(y_1^*, x^*) = \lambda B(y_2^*, x^*) \) for all \( x^* \in X^* \).

Assume we have the lemma. This shows that multiplying \( A(y^*) \) and \( B(y^*, \cdot) \) for any \( y^* \in E(Y^*) \) by an appropriate scalar of modulus one we can assume without loss of generality that the function \( B \) does not depend on the first coordinate that is, there is a map \( A_0 : E(Y^*) \to E(X^*) \) and a map \( B_0 : X^* \to Y^* \) such that
\[ T^*(y^* \circ y^{**}) = \bar{A}_0(y^*) \cdot \bar{B}_0^*(y^{**}) \quad \text{for all } y^* \in E(Y^*) \quad \text{and} \quad y^{**} \in Y^{**}. \]

The above proves that \( \bar{A}_0 \) can be extended to a weak \( * \) continuous linear isometry from \( Y^* \) onto \( X^* \) and hence \( T \) is a tensor product of \( A_0 \) \( \otimes \) \( X \) and \( B_0 \). So to end we should prove Lemma 12.6. The proof of the lemma is based on the following statement.

12.7. THEOREM. Let \( X \) be a Banach space and assume \( X \) is strictly convex then \( \dim(\text{Mult}(X)) = 1 \).

The above theorem may be interesting in itself and its proof is not very short so before passing to the proof of this theorem we first deduce from it our lemma and so shall end the proof of theorems 12.1 and 12.2. Notice please that if \( X^{**} \) is strictly convex then \( X \) is also a strictly convex Banach space.

Proof of Lemma 12.6. To prove that two linear operators \( \bar{b}_1 : U \to V \) and \( \bar{b}_2 : U \to V \) are proportional it is sufficient to show that for any
for any \( \mathbf{x} \in E(Y^*) \) and \( \mathbf{y} \in Y^* \).

\[ (45) \]

This shows that multiplying \( A(\mathbf{y}) \) and \( B(\mathbf{y}) \) by an appropriate scalar of modulus one of generality that the function \( B \) does not dilate that is, there is a map \( A_0 : E(Y^*) \rightarrow Y^* \) such that

- \( B(\mathbf{y})^* \) for all \( \mathbf{y} \in E(Y^*) \)

and \( \mathbf{y} \in Y^* \).

can be extended to a weak * continuous functions to \( X^* \) and hence \( T \) is a tensor product and we should prove Lemma 12.6. The proof is as follows.

\begin{itemize}
  \item A Banach space and assume \( X \) is strict-
\end{itemize}

Interesting in itself and its proof is not to the proof of this theorem we first de-

\( \mathbf{x} \in X^* \) is strictly convex then \( X \) is also convex.

To prove that two linear operators \( \mathbf{x}_1 : U \rightarrow V \) and \( \mathbf{x}_2 : U \rightarrow V \) are proportional, moreover \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are proportional if the conjugate operators are so. Hence,

\( \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3 \) are proportional if the conjugate operators are so. Hence,

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\( \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3 \) are proportional if the conjugate operators are so. Hence,
where the closure is taken in the weak * topology. Put
\[ G = W_0(M), \]
\[ f_0(z) = \sup \{ |x^*(x_0)| : x^* \in M_0^1(z) \} \quad \text{for } z \in G \]
\[ r_0 = \inf \{ r \geq 0 : r(2 - \Re z) \geq f_0(z) \} \quad \text{for any } z \in G. \]

We have
\[ \mathbb{G} = \{ z \in \mathbb{E} : |z| \leq 1 \}, \]
\[ 0 < f_0(z) < 1 \quad \text{for } z \in G, \]
\[ 0 < r_0 < \infty. \]

Let \( z_0 \in G \) be such that
\[ f_0(z_0) = f_0(z_0) \quad (47) \]
(such a \( z_0 \) exists provided \( f_0 \) is upper semicontinuous and for any sequence \( z_n \in G \) such that \( z_n \to z_0 \in \partial G \) we have \( f_0(z_n) \to 0 \)). Put
\[ k(z) = k(z - w_0)^2 \quad \text{for } z \in \mathbb{E} \]
where \( k \) and \( w_0 \) are such that the plane in \( \mathbb{E} \times \mathbb{R} = \mathbb{E}^3 \) given by \( z - r_0(2 - \Re z) \) is tangential to the surface \( z = |k(z)| \) in the point \( (z_0, r_0(2 - \Re z_0)) \).

Since the function \( z \to |k(z)| \) is real analytic there is a positive integer \( p \) such that
\[ \lim_{z \to z_0} \inf \frac{|k(z)| - r_0(2 - \Re z)}{|z - z_0|^p} = \rho > 0 \quad (48) \]
(by a direct computation we get \( p = 2 \) and \( \rho = |k(z_0)| \)); hence for any sufficiently small complex number \( \epsilon \) we have
\[ |k(z) + \epsilon(z - z_0)|^p \geq r_0(2 - \Re z) \quad \text{for } z \in G. \quad (49) \]

We have assumed that \( x_0 \) and \( Tz_0 \neq 0 \) are not proportional so there is a \( z_1 \in G \) such that \( z_1 \neq z_2 \) and \( f_0(z_1) = \sup \{ |x^*(x_0)| : x^* \in M_0^1(z_1) \} > 0 \). Put
in the weak * topology. Put

\[ \| x \| = \sup \{ | x^* x(z) | : x^* x \in xz \} \quad \text{for} \quad z \in G \]

for \( z \in G \),

\[ 2 = \Re z \geq f_0(z) \quad \text{for any} \quad z \in G. \]

for \( z \in G \),

and assume \( e \) is such that \( \kappa_e(z_1) \neq \kappa_e(z_2) \) and (49) is satisfied.

Put

\[ T_1 = \frac{1}{\kappa_e(T)}, \quad T_2 = \frac{1}{\kappa_e(T)}, \quad X_1 = T_1(x_0), \quad X_2 = T_2(x_0). \]

We have

\[ N_{T_1} = \frac{1}{\kappa_e(M_1)} \quad \text{and} \quad N_{T_2} = \frac{1}{\kappa_e(M_2)} \]

so \( X_1 \neq X_2 \). We get

\[ \| x_1 \| = \sup \{ | x^* x(z) | : x^* x \in xz \} = \sup \sup \frac{| x^* x(z) |}{\kappa_e(z)} = f_0(z) = \sup \frac{| x^* x(z) |}{\kappa_e(z)} = 1. \]

\[ \| x_1 + x_2 \| \geq \| x_1 \| + \| x_2 \| \]

In the same manner we get \( \| x_2 \| = 1 \). Moreover

\[ \| x_1 + x_2 \| \geq \| x_1 \| + \| x_2 \| \]

so we have the following situation

\[ \| x_1 \| = \| x_2 \| = \| \frac{x_1 + x_2}{2} \| = 1 \quad \text{and} \quad x_1 \neq x_2 \]

and this contradicts the assumption \( X \) is strictly convex.


13.1. Example. Let \( X_0 = (R^2, \| \cdot \|_1) \) be a two dimensional real Banach space with the norm defined as a supremum of the modulus of coordinates. For any \( \epsilon > 0 \) fix a strictly convex, smooth two dimensional real Banach space \( X_\epsilon = (R^2, \| \cdot \|_1) \) such that the Banach-Mazur distance between \( X_\epsilon \) and \( X_0 \) is less than \( 1 + \epsilon \). Elements of \( B(X_0) \) can we identified with \( 2 \times 2 \) matrices. For \( \epsilon > 0 \) we define \( T_\epsilon : B(X_0) \to B(X_\epsilon) \) by

\[ T_\epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{a + b + c + d}{2} \begin{bmatrix} a + b - c - d \\ a - b + c - d \end{bmatrix} \]
We have \( \| T_\epsilon \| = \| T_\epsilon^{-1} \| = 1 \) so \( \lim_{\epsilon \to 0} \| T_\epsilon \| \| T_\epsilon^{-1} \| = 1 \) and \( T_\epsilon \circ \text{Id}_Y = \text{Id}_X \). But

\[
\| T_\epsilon \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) - T_\epsilon \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \|_\epsilon = \| \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \|_\epsilon \| \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \|_\epsilon = 1
\]

so \( T \) is not \( O(\epsilon) \)-almost multiplicative.

The above example shows that Theorem 12.1 is no longer true for \( \epsilon \)-isometries. We shall now prove that it remains true for uniformly convex spaces.

13.2. THEOREM. Let \( X, \bar{X}, Y, \bar{Y} \) be Banach spaces with uniformly convex duals. Then there is an \( \epsilon_0 > 0 \) such that for any \( \epsilon \leq \epsilon_0 \) and any linear isomorphism \( T \) from \( X \) onto \( Y \) with \( \| T \| \| T^{-1} \| \leq 1 + \epsilon \), there are linear isomorphisms \( \Theta : X \to Y \) and \( \Phi : \bar{X} \to \bar{Y} \) or \( \Theta : X \to \bar{Y} \) and \( \Phi : \bar{X} \to \bar{Y} \) with \( \| \Theta \| \| \Theta^{-1} \| \leq 1 + c(\epsilon) \) and \( \| \Phi \| \| \Phi^{-1} \| \leq 1 + c(\epsilon) \) such that \( \| T - \Theta \circ \Phi \| \leq c(\epsilon) \). The constant \( \epsilon_0 \) and the function \( c \) depend only on the modulus of convexity of the considered Banach spaces and \( \lim_{\epsilon \to 0^+} c(\epsilon) = 0 \).

13.1. Corollary. Let \( X, Y \) be Banach spaces with the approximation property and such that \( X, X^*, Y \) and \( Y^* \) are uniformly convex. Then there is an \( \epsilon_0 > 0 \) such that if the Banach-Mazur distance between \( K(X) \) and \( K(Y) \) is less than \( 1 + \epsilon_0 \), then \( K(X) \) and \( K(Y) \) or \( K(X) \) and \( K(Y^*) \) are isomorphic in the category of Banach algebras. The constant \( \epsilon_0 \) depends only on the modulus of convexity of \( X, X^*, Y \) and \( Y^* \).
13.4. Corollary. Let $X, Y$ be finite dimensional Banach spaces such that $X, X^*, Y$ and $Y^*$ are strictly convex. Then there is an $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ and any linear map $T$ from $L(X)$ onto $L(Y)$ with $||T|| ||T^{-1}|| \leq 1 + \epsilon$ and $T(\text{Id}_X) = \text{Id}_Y$, there is an algebra isomorphism or an algebra antilinear isomorphism $T$ from $L(X)$ onto $L(Y)$ such that $||T^{-1}|| \leq c'(\epsilon)$, where $\epsilon_0$ and $c'$ depend only on the modulus of convexity of $X, X^*, Y, Y^*$.

Proof of the theorem. We assume, without loss of generality, that $||T|| \leq 1 + \epsilon$ and $||T^{-1}|| \leq 1 + \epsilon$.

At various points of the proof we shall use the inequalities involving $\epsilon$ which are valid only if $\epsilon$ is sufficiently small, in those cases we will merely assume that $\epsilon$ is near 0 and this assumption gives rise to the constant $\epsilon_0$.

13.5. Lemma. Let $U$ and $V$ be normed, linear spaces, let $\delta$ be positive and assume that

\[ ||u_1 \bullet v_1 + u_2 \bullet v_2 + u_3 \bullet v_3|| \leq \delta \]  \hspace{1cm} (50)

where $u_1, u_2, u_3 \in U$, $v_1, v_2, v_3 \in V$ and $||u_1|| = ||u_2|| = 1 = ||v_1|| = ||v_2|| = ||v_3||$. Then there is a number $\lambda$ of modulus one such that $||u_1 - \lambda u_2|| \leq 3/\delta$ or $||v_1 - \lambda v_2|| \leq 3/\delta$.

Proof of the lemma. If $\inf_{|\lambda| = 1} ||\lambda v_1 - v_2|| \leq \frac{3}{\sqrt{\delta}}$ for both $i = 1$ and 2 then we get $||v_i - \lambda v_2|| \leq 3/\delta$ for some $\lambda$ of modulus one, so we can assume that $\inf_{|\lambda| = 1} ||\lambda v_1 - v_2|| \leq \frac{3}{\sqrt{\delta}}$.

Assume there is an $a \in E$ with $|av_2 - v_1| \leq \frac{3}{\sqrt{\delta}}$. We get $1 - \frac{3}{4\sqrt{\delta}} \geq |a| \geq 1 - \frac{3}{4\sqrt{\delta}}$ and hence, by (51)
\[ \left\| \frac{a}{|a|} v_3 - v_1 \right\| \leq \left( \frac{|a|}{|a|} - a \right) + \left\| av_3 - v_1 \right\| \leq \frac{|a(1-|a|)|}{|a|} + \frac{3/2}{3/2} \leq \frac{3}{2}. \]

The above contradicts (51) and we get

\[ \inf_{a \in \mathbb{C}} \left\| av_3 - v_1 \right\| \geq \frac{3}{2}. \quad (52) \]

We define a functional \( v^* \) on \( \text{span}(v_1, v_3) \) by

\[ v^*(av_3 + \delta v_3) = \frac{3}{2} \delta a. \]

From (52) we have \( \|v^*\| \geq 1 \). Let \( \mathbb{V}^* \) be a norm preserving extension of \( v^* \) from \( \text{span}(v_1, v_3) \) to \( V \). From (50) we get

\[ \|u_1 \mathbb{V}^*(v_1) + u_2 \mathbb{V}^*(v_3)\| \leq \delta, \]

so

\[ \left\| u_1 + \frac{u_2}{\mathbb{V}^*(v_1)} \frac{\mathbb{V}^*(v_3)}{(u_1 \mathbb{V}^*(v_1))} \right\| \leq \frac{4}{3}\delta. \]

Hence, in the same manner as before we get

\[ \left\| u_1 + \frac{u_2}{\mathbb{V}^*(v_1)} \frac{\mathbb{V}^*(v_3)}{(u_1 \mathbb{V}^*(v_1))} \mathbb{V}^*(v_1) \right\| \leq 2 \cdot \frac{4}{3}\delta < 3\delta. \]

For the next lemmas we need the following observations. The first one is easy to check by direct computation.

13.6. Proposition. Let \( V \) be a Banach space with uniformly convex dual and let \( v \in V \), \( \|v\| = 1 \) then

\[ \text{diam} \{ v^* \in B(V^*) : \text{Re}(v^*(v)) \geq 1 - \delta \} \leq \delta^* \mathcal{B}(2\delta). \]

13.7. Proposition. Let \( V, U \) be Banach spaces with uniformly convex duals and let \( v \in V \), \( u \in U \), \( \|v\| = 1 = \|u\| \) then
\[ -a \| v_2 \| - v_1 \| \leq \frac{2(1-2\theta)}{9} + \frac{3\sqrt{6}}{2} \leq \frac{3\sqrt{6}}{2} \]

and we get

(52)

\[ v^* \text{ on } \text{span}(v_1, v_2) \text{ by} \]

(53)

\[ \| \| \leq \delta, \]

\[ \frac{\sqrt{6}}{2}. \]

as before we get

\[ \| v \| \leq \delta. \]

\[ \frac{\sqrt{6}}{2}. \]

\[ \| v \| \leq \delta. \]

\[ \frac{\sqrt{6}}{2}. \]

we need the following observations. The first direct computation.

13.8. Proposition. Let S be a compact Hausdorff space, let A be a closed subspace of C(S) and let F be a norm one functional on A. We denote by \( S_0 \) the subset of S consisting of all points s from S such that the norm of functional A \( f \) of f(s) is equal to one. Assume that for any \( s \in S \) and any number \( \lambda \) of modulus one there is exactly one \( s_0 \in S \) such that \( f(s) = \lambda f(s_0) \) for all f in A. Then there is a probability measure \( \mu \) on S which is norm preserving extension of F from A to C(S). Furthermore for any such \( \mu \) we have

\[ \mu(S_0) = \mu(S) = 1. \]
Proof. Let $v$ be a norm one extension of $F$ from $A$ to $C(S)$. Denote by $K_r$ the subset of $S$ consisting of all points $s \in S$ such that the norm of functional $Af - f(s)$ is not greater than $r$.

For any $f \in A$ with $\|f\| = 1$ we have

$$|F(f)| = \int_{K_r} |f(s)| = \int_{K_r} |f(s)| = \int_{K_r} |f(s)| \leq \sup_{s \in K_r} \|v(s)\| \cdot \|v\| \cdot (1 - \|K_r\| - r).$$

Hence $|v| \leq 1$ for any $r < 1$.

Put $h = \frac{d}{d|v|}$. We can assume $|h| = 1$ on $S$. By our assumption there is a map $\psi : S \rightrightarrows S$ such that

$$h(s)f(s) = f \cdot \psi(s) \quad \text{for } f \in A, \quad s \in S.$$

If $h$ is continuous then the corresponding function $\psi$ defined by the above equality is also continuous. Hence it is standard to prove that if $h$ is a Borel function then $\psi$ is also Borel. To end the proof we define $\mu$ by

$$\mu(K) = |\psi^{-1}(K)| \quad \text{for any Borel subset } K \text{ of } S.$$

13.9. Lemma. Let $X, \overline{X}, Y, \overline{Y}$ be Banach spaces with uniformly convex duals and let $T$ be a linear isomorphism from $X \otimes \overline{X}$ onto $Y \otimes \overline{Y}$ with $\|T\| \leq 1 + c$, $\|T^{-1}\| \leq 1 + c$. Then for any $y^* \in E(Y^*)$, $\overline{y^*} \in E(\overline{Y}^*)$ there are $x^* \in E(X^*)$, $\overline{x^*} \in E(\overline{X}^*)$ such that

$$\|T^*(y^* \otimes \overline{y}^*) - x^* \otimes \overline{x^*}\| \leq \alpha(c),$$

where $\alpha(c) \to 0$ and the function $\alpha$ depends only on the modulus of convexity of $X^*, \overline{X}^*, Y^*, \overline{Y}^*$.

Proof of the lemma. Fix $y^* \in E(Y^*)$, $\overline{y^*} \in E(\overline{Y}^*)$ and let $\mu$ be a measure on $B(X^*) \times B(\overline{X}^*)$ which is a norm preserving extension of the functional $T^* (y^* \otimes \overline{y}^*)$ from $X \otimes \overline{X}$ to $C(B(X^*) \times B(\overline{X}^*))$. By Proposition 13.8 we can assume that $\mu$ is positive and we have

$$\|\mu\| = \mu(B(X^*) \times B(\overline{X}^*)) = \mu(E(X^*) \times E(\overline{X}^*))$$
Let $Y$, $Z$, and $S$ be Banach spaces with uniformly convex norms. For any $y \in Y$ and $z \in Z$, let $\varphi : X \times Y \to X \otimes Y^*$ be a continuous linear operator such that $\varphi(x, y) = x \otimes y$. Then, for any $x \in X$, $y \in Y$, and $z \in Z$, we have

$$\varphi(x, y) = x \otimes y = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i = \sum_{i=1}^{n} \lambda_i \langle x_i, y \rangle z_i$$

where $\lambda_i$ are scalars.

By Proposition 11.1, we get

$$\|x \otimes y + z\| = \|x \otimes y\| + \|z\| \geq 1 - 2\varepsilon$$

for $\varepsilon > 0$ sufficiently small.

Hence, we have

$$\|x \otimes y + z\| = \|x \otimes y\| + \|z\| \geq 1 - 2\varepsilon$$

for $\varepsilon > 0$ sufficiently small.

We shall show that

$$\|x \otimes y + z\| = \|x \otimes y\| + \|z\| \geq 1 - 2\varepsilon$$

for $\varepsilon > 0$ sufficiently small.

We have

$$\|x \otimes y + z\| = \|x \otimes y\| + \|z\| \geq 1 - 2\varepsilon$$

for $\varepsilon > 0$ sufficiently small.
which in view of previous inequalities leads to

\[ \text{Re}(T(x_i^* \otimes \bar{x}_i)) \geq 1 - 3\sqrt{\varepsilon} - 3(1 + \varepsilon)[\delta_1^*(4\sqrt{\varepsilon}) + \delta_2^*(6\sqrt{\varepsilon})] = \gamma(\varepsilon) \]

for \( i = 1, 2, \)

so

\[ \|x_1^* \otimes \bar{x}_1 + x_2^* \otimes \bar{x}_2\| \geq 2\gamma(\varepsilon). \]

Hence there is an \( x^* \otimes \bar{x}^* \in E(X^*) \otimes E(\bar{X}^*) \) such that

\[ \text{Re}(x_i^* \otimes \bar{x}_i)(x^* \otimes \bar{x}^*) \geq 2\gamma(\varepsilon) - 1 \]

for both \( i = 1, 2. \)

By Proposition 13.7

\[ \|x_i^* \otimes \bar{x}_i - x_i^* \otimes \bar{x}_i\| \leq \delta_i^*(4 - 4\gamma(\varepsilon)) + \delta_i^*(4 - 4\gamma(\varepsilon)) = a^*(\varepsilon). \]

Fix \( (x_i^* \otimes \bar{x}_i) \in S. \) To end the proof we observe that for any \( f \in X \otimes \bar{X} \) with \( \|f\| \leq 1 \), it follows from (53) and (54) that

\[ |f(x_i^* \otimes \bar{x}_i) - T^*(y_i^* \otimes \bar{y}_i)(f)| = |f(x_i^* \otimes \bar{x}_i) - \int f \, d\mu| \leq \]

\[ \leq 4\sqrt{\varepsilon} + \int_{S} |f - f(x_i^* \otimes \bar{x}_i)| \, d\mu + \left| 1 - \mu(S) \right| \leq \]

\[ \leq 4\sqrt{\varepsilon} + a^*(\varepsilon)(1 + \varepsilon) + 4\sqrt{\varepsilon} = a(\varepsilon). \]

13.10. Lemma. Let \( x, \bar{x}, y, \bar{y}, T, \varepsilon, a \) be as in Lemma 13.9. Assume

\( y_i^* \in E(Y^*), \quad \bar{y}_i^*, \bar{y}_j^* \in E(\bar{Y}^*), \quad x_i^*, x_j^*, x_3^* \in E(X^*), \quad \bar{x}_i^*, \bar{x}_j^*, \bar{x}_3^* \in E(\bar{X}^*) \)

are such that

\[ \|T^*(y_i^* \otimes \bar{y}_i^*) - x_i^* \otimes \bar{x}_i^*\| \leq a(\varepsilon) \quad \text{for } i = 1, 2, 3, \]

then there are numbers \( \lambda_{i,j} \) for \( i, j = 1, 2, 3 \) of modulus one such that

\[ \|x_i^* - \lambda_{i,j} x_j^*\| \leq \beta(\varepsilon) \quad \text{for } i, j = 1, 2, 3 \]

or

\[ \|\bar{x}_i^* - \lambda_{i,j} \bar{x}_j^*\| \leq \beta(\varepsilon) \quad \text{for } i, j = 1, 2, 3 \]

where \( \beta(\varepsilon) = 24\sqrt{\varepsilon} a(\varepsilon). \)

Proof of the lemma. Since \( \bar{Y}^* \) is uniformly convex, by Lemma 13.9, there are \( x_4^* \in E(X^*) \) and \( \bar{x}_4^* \in E(\bar{X}^*) \) such that
\[ \| T(y^*_o \circ (y^*_1 + y^*_2)) - kx^*_4 \circ \bar{x}^*_4 \| \leq k\alpha(\varepsilon), \]

where \( k = \| y^*_1 + y^*_2 \| \leq 2 \). Hence

\[ \| x^*_1 \circ \bar{x}^*_1 + x^*_2 \circ \bar{x}^*_2 - kx^*_4 \circ \bar{x}^*_4 \| \leq (k\alpha(\varepsilon) + 2d(\varepsilon)) \leq 4\alpha(\varepsilon), \]

and by Lemma 13.5 we have

\[ \| x^*_1 - kx^*_2 \| \leq 12\sqrt{\alpha(\varepsilon)} \]

or

\[ \| x^*_1 - kx^*_2 \| \leq 12\sqrt{\alpha(\varepsilon)} \]

for some \( \lambda \) of modulus one.

Considering successively the pairs of indices \((1,2)\), \((2,3)\) and \((1,3)\) we get the assertion of the lemma.

From Lemmas 13.9 and 13.10 we get that for any \( y^*_o \in E(Y^*) \) we have exactly two possibilities:

a) there is an \( x^*_o \in E(X^*) \) and a function \( \psi : E(Y^*) \to E(X^*) \) such that

\[ \| T^*(y^*_o \circ \bar{y}^*) - x^*_o \circ \psi(\bar{y}^*) \| \leq \alpha(\varepsilon) + \beta(\varepsilon) = \gamma(\varepsilon) \]

for all \( \bar{y}^* \in E(Y^*) \). (55)

or

b) there is an \( \bar{x}^*_o \in E(X^*) \) and a function \( \psi : E(Y^*) \to E(X^*) \) such that

\[ \| T^*(y^*_o \circ \bar{y}^*) - \psi(\bar{y}^*) \circ x^*_o \| \leq \gamma(\varepsilon) \]

for all \( \bar{y} \in E(Y^*) \). (56)

By the same arguments applied to the map \( T^{-1} \) in place of \( T \), we get by symmetry (replacing the space \( X \) by \( \bar{X} \) and \( Y \) by \( \bar{Y} \)) and by Lemma 13.10 that

\[ \sup (\inf \| \psi(\bar{y}^*) - x^* \|: \bar{y}^* \in E(\bar{Y}^*)) : x^* \in E(\bar{X}^*) \leq \gamma(\varepsilon) \]

\[ \sup (\inf \| \psi(\bar{y}^*) - x^* \|: \bar{y}^* \in E(\bar{Y}^*)) : x^* \in E(X^*) \leq \gamma(\varepsilon). \] (57)

For any \( y^*_o \in E(Y^*) \) we define, depending on which of the above possibilities takes place, a function

\[ \psi : \bar{X} \to \bar{Y} \]

or

\[ \psi : X \to \bar{Y} \]
as follows:

a) fix $x_0 \in B(X)$ such that $x_0^*(x_0) = 1$ and define $\phi$ by

$$\tilde{y}^*(\tau(x)) = y_0^* \circ \tilde{y}^*(T(x_0 \circ \tilde{x})) \quad \text{for } \tilde{y}^* \in \tilde{Y}^*, \ x \in \tilde{X};$$

b) fix $x_0 \in B(X)$ such that $x_0^*(x_0) = 1$ and define $\psi$ by

$$\tilde{y}^*(\tau(x)) = y_0^* \circ \tilde{y}^*(T(x_0 \circ \tilde{x})) \quad \text{for } \tilde{y}^* \in \tilde{Y}^*, \ x \in X.$$

The above definitions may depend on the choice of $x_0$ ($\tilde{x}_0$) and we assume that we have fixed some $\phi$ ($\psi$) as above, for any $y_0^* \in E(Y^*)$. We have

$$\|\phi\| \leq 1 + \epsilon, \quad \|\psi\| \leq 1 + \epsilon,$$

and

$$|\tilde{y}^*(\tau(x)) - \phi(\tilde{x})| \leq \gamma(\epsilon) \|\tilde{x}\| \quad \text{for all } \tilde{y}^* \in \tilde{Y}^*, \ x \in \tilde{X},$$

$$|\tilde{y}^*(\tau(x)) - \psi(\tilde{x})| \leq \gamma(\epsilon) \|x\| \quad \text{for all } \tilde{y}^* \in \tilde{Y}^*, \ x \in X,$$

so from (57) we enter that $\phi$ and $\psi$ are one to one, onto isometries with

$$\|\phi^{-1}\| \leq 1 + \gamma(\epsilon), \quad \|\psi^{-1}\| \leq 1 + \gamma(\epsilon),$$

and

$$|\phi^*(\tilde{y}^*) - \phi(\tilde{y}^*)| \leq \gamma(\epsilon)$$

$$|\psi^*(\tilde{y}^*) - \psi(\tilde{y}^*)| \leq \gamma(\epsilon)$$

for all $\tilde{y}^* \in \tilde{Y}^*$.

To end the proof we show that for all $y_0^* \in E(Y^*)$ one of the two possibilities a) or b) takes place and the map assigning to $y_0^* \in E(Y^*)$ a $\phi \in L(\tilde{X}, \tilde{Y})$ ($\forall \in L(X, Y)$) is "$\epsilon$-almost" constant.

To this end assume that $y_1^*, y_2^* \in E(Y^*)$, $x_1^* \in E(X^*)$, $x_2^* \in E(X^*)$, $\phi_1 \in L(\tilde{X}, \tilde{Y})$, $\phi_2 \in L(\tilde{X}, \tilde{Y})$ are such that

$$\|T^*(y_1^* \circ \tilde{y}^*) - x_1^* \circ \phi_1^*(\tilde{y}^*)| \leq 2\gamma(\epsilon)$$

and

$$\|T^*(y_2^* \circ \tilde{y}^*) - x_2^* \circ \phi_2^*(\tilde{y}^*)| \leq 2\gamma(\epsilon)$$

for all $\tilde{y}^* \in E(Y^*)$.

Since $|\phi_1^{-1}(\tilde{y}^*) - 1 + \gamma(\epsilon)| \leq 1 + \gamma(\epsilon)$, $|\phi_2^{-1}(\tilde{y}^*) - 1 + \gamma(\epsilon)| \leq 1 + \gamma(\epsilon)$ there are

$$\tilde{y}_1^*, \tilde{y}_2^* \in E(Y^*)$$

such that $\|\phi_1^*(\tilde{y}^*_1) - \tilde{X}_2\| \leq \gamma(\epsilon)$, $\|\phi_2^*(\tilde{y}^*_2) - x_2^*\| \leq \gamma(\epsilon)$;

so we get

$$\|x_1^* \circ \phi_1^*(\tilde{y}^*) - x_2^* \circ \phi_2^*(\tilde{y}^*) - \tilde{X}_2\| \leq \gamma(\epsilon)$$

and hence

$$\|y_1^* \circ \phi_1^*(\tilde{y}^*) - y_2^* \circ \phi_2^*(\tilde{y}^*) - \tilde{Y}^*\| \leq \gamma(\epsilon)$$

leading to the inequality

$$\|y_1^* - y_2^*\| \leq \gamma(\epsilon)$$

which contradicts the assumption.

Thus without loss of generality we can assume the possibility that a) takes place.

Fix $y_0^* \in E(Y^*)$ and $\phi \in L(\tilde{X}, \tilde{Y})$ with

$$\|T^*(y_0^* \circ \tilde{y}^*) - x_0^* \circ \phi^*(\tilde{y}^*)| \leq (1 + \epsilon) \gamma(\epsilon)$$

by symmetry, there are $x_1^*, x_2^* \in E(X^*)$ such that

$$\|x_1^* \circ \phi^*(\tilde{y}^*) - x_2^* \circ \phi^*(\tilde{y}^*)| \leq (1 + \epsilon) \gamma(\epsilon)$$

Moreover replacing $\phi$ by $\phi^*$ it is always possible to assume that $\phi^*$ is constant.

Let us compose $\phi^*$ to show the following:

13.8. Lemma. Let $T$ and $\phi$ be as above, then there is an $\epsilon$-isometry $\tilde{T}$ such that

$$\|T - \phi\| \leq \epsilon$$

if $T$ is a linear isometry.

13.9. Lemma. Let $T$ and $\phi$ be as above, then there is an $\epsilon$-isometry $\tilde{T}$ such that

$$\|T - \phi\| \leq \epsilon$$

and $T$ is a linear isometry.

13.10. Lemma. Let $T$ and $\phi$ be as above, then there is an $\epsilon$-isometry $\tilde{T}$ such that

$$\|T - \phi\| \leq \epsilon$$

then $\|T - \phi\| \leq 6\epsilon$. 

so we get
\[ \|Y^i \circ \tilde{X} - T^i(Y^i \circ \tilde{X})\| \leq (1 + c)\gamma(c) + 2\gamma(c) \text{ for } i = 1, 2 \]
and hence
\[ \|Y^i \circ \tilde{X} - Y^i \circ \tilde{X}\| \leq 2(1 + c)(1 + c)\gamma(c) \leq 7\gamma(c) \]
leading to the inequality
\[ \|Y^i - Y^i\| \leq 7\gamma(c) \]
which contradicts (58) and (59).

Thus without loss of generality we can assume that it is the first possibility that always holds.

Fix \( y^*_o \in E(Y^*) \) and \( \tilde{y}^*_o \in E(\tilde{Y}^*) \). There is an \( x^*_o \in E(X^*) \) and \( \phi_o \in L(\tilde{X}^*, X) \) with \( \|\phi_o\| \leq \gamma(c) \) such that
\[ \|T^*(y^*_o \circ \tilde{X}^*) - x^*_o \circ \phi^*_o(\tilde{X}^*)\| \leq 2\gamma(c) \text{ for all } \tilde{y}^*_o \in E(\tilde{Y}^*), \]  \hspace{1cm} (60)
by symmetry, there is an \( \tilde{x}^*_o \in E(\tilde{X}^*) \) and \( \psi_o \in L(X^*, Y) \) with \( \|\psi_o\| \leq \gamma(c) \) such that
\[ \|T^*(y^*_o \circ X^*) - \psi^*_o(y^*_o) \circ \tilde{x}^*_o\| \leq 2\gamma(c) \text{ for all } y^*_o \in E(Y^*). \] \hspace{1cm} (61)
Moreover replacing \( 2\gamma(c) \) in (60) and (61) by \( 4\gamma(c) \) we can assume \( \tilde{x}^*_o = \phi^*_o(\tilde{y}^*_o) \) and \( x^*_o = \psi^*_o(y^*_o) \).

Let us compose \( T \) with \( \phi^{-1} \circ \psi^{-1} \). To end the proof it is enough to show the following lemma:

13.8. Lemma. Let \( X, \tilde{X} \) be Banach spaces with uniformly convex duals, then there is an \( \epsilon_o > 0 \) such that for all \( \epsilon < \epsilon_o \) the following implication holds:

if \( T \) is a linear isomorphism from \( X \circ \tilde{X} \) onto itself with \( ||T \circ ||T^{-1}|| \leq 1 + \epsilon \) and if there exist \( x^*_o \in E(X^*) \) and \( \tilde{x}^*_o \in E(\tilde{X}^*) \) such that
\[ T^*(x^*_o \circ \tilde{X}^*) = x^*_o \circ \tilde{x}^*_o \text{ for all } \tilde{x}^*_o \in \tilde{X}^* \]
and
\[ T^*(x^*_o \circ \tilde{x}^*_o) = x^*_o \circ \tilde{x}^*_o \text{ for all } x^*_o \in X^* \]
then \( ||T - Id|| \leq 6\gamma(c) \).
Proof of the lemma. Let $x^*_1 \in E(X^*)$, $\overline{x}^*_1 \in E(\overline{X}^*)$. It follows from the assumption and our previous considerations that there are isomorphisms $\psi \in \mathbb{I}(X)$ and $\psi^* \in \mathbb{I}(X)$ such that

$$\|T^*(x^*_1 \circ \overline{x}^*_1) - x^*_1 \circ \psi^*(\overline{x}^*_1)\| \leq 2\gamma(\varepsilon) \quad \text{for all } \overline{x}^* \in E(\overline{X}^*)$$

and

$$\|T^*(x^* \circ \overline{x}^*_1) - \psi^*(x^*) \circ \overline{x}^*_1\| \leq 2\gamma(\varepsilon) \quad \text{for all } x^* \in E(X^*).$$

Substituting $\overline{x}^* = \overline{x}^*_1$ and $x^* = x^*_1$ we get

$$\|T^*(x^*_1 \circ \overline{x}^*_1) - x^*_1 \circ \psi^*(\overline{x}^*_1)\| \leq 2\gamma(\varepsilon)$$

and

$$\|T^*(x^*_1 \circ \overline{x}^*_1) - \psi^*(x^*_1) \circ \overline{x}^*_1\| \leq 2\gamma(\varepsilon).$$

Hence $\|\psi(\overline{x}^*_1) - \overline{x}^*_1\| \leq 4\gamma(\varepsilon)$ and $\|x^*_1 - \overline{x}^*(x^*_1)\| \leq 4\gamma(\varepsilon)$, so

$$\|T^*(x^*_1 \circ \overline{x}^*_1) - x^*_1 \circ \overline{x}^*_1\| \leq 6\gamma(\varepsilon)$$

ending the proof of the lemma.

The results of this section are based on the author's papers [6,9].