Operators shrinking the Arveson spectrum

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Let $G$ be a locally compact abelian group, $X$ be a complex Banach space, and $\tau: G \to \text{inv}(\mathcal{B}(X))$ be a bounded and strongly continuous group homomorphism. For every $x \in X$, the Arveson spectrum of $x$ is defined as

$$\text{sp}(x) = \left\{ \gamma \in \hat{G}: \hat{f}(\gamma) = 0 \text{ for each } f \in L^1(G) \text{ with } \int_G f(t)\tau(t)x dt = 0 \right\}.$$
Example (Support of the Fourier transform)

Let $G$ be a locally compact abelian group and let $\tau$ be the so-called *regular representation* of $G$ on $L^1(G)$,

$$\tau: G \to B(L^1(G)), \quad [\tau(t)f](s) = f(t^{-1}s) \quad (s, t \in G, \, f \in L^1(G)).$$

Then

$$\text{sp}(f) = \text{supp} \hat{f} \quad (f \in L^1(G)).$$
Example (Local spectrum of an operator)

Let $X$ be a complex Banach space. Given an operator $T \in \mathcal{B}(X)$, the local resolvent set $\rho(T, x)$ of $T$ at the point $x \in X$ is defined as the union of all open subsets $U$ of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies

$$(T - z)f(z) = x$$ for each $z \in U$.

The local spectrum $\sigma(T, x)$ of $T$ at $x$ is then defined as

$$\sigma(T, x) = \mathbb{C} \setminus \rho(T, x).$$

1. Let $T \in \mathcal{B}(X)$ be a doubly power bounded operator and consider

$$\tau : \mathbb{Z} \to \mathcal{B}(X), \quad \tau(k) = T^k \quad (k \in \mathbb{Z}).$$

Then $\text{sp}(x) = \sigma(T, x)$ \quad ($x \in X$).

2. Let $T \in \mathcal{B}(X)$ be a hermitian operator and consider the one-parameter group

$$\tau : \mathbb{R} \to \mathcal{B}(X), \quad \tau(t) = \exp(itT) \quad (t \in \mathbb{R}).$$

Then $\text{sp}(x) = \sigma(T, x)$ \quad ($x \in X$).
Example (Support of a spectral measure)

Let $A : \mathcal{D} \subset H \to H$ be a selfadjoint operator on a Hilbert space $H$. Then there is a projection-valued measure $\mathcal{E}$ on $\mathbb{R}$ such that

$$A = \int_{-\infty}^{+\infty} t \, d\mathcal{E}(t).$$

For every $x \in H$, the map

$$\Delta \mapsto \|\mathcal{E}(\Delta)x\|^2$$

defines a measure $\mathcal{E}_x$ on the Borel subsets of $\mathbb{R}$.

Consider the one-parameter group

$$\tau : \mathbb{R} \to \mathcal{U}(H), \quad \tau(t) = \exp(itA) \quad (t \in \mathbb{R}).$$

Then

$$\text{sp}(x) = \text{supp}(\mathcal{E}_x) \quad (x \in H).$$
In many different contexts we find out operators $\Phi: X \rightarrow Y$, where $X$ and $Y$ are complex Banach spaces, which shrink the Arveson spectrum in the sense that

$$\text{sp}(\Phi x) \subset \text{sp}(x) \quad (x \in X)$$

for appropriate representations $\tau_X$ and $\tau_Y$ of a locally compact abelian group $G$ on $X$ and $Y$, respectively.

A typical example of operator $\Phi$ shrinking the spectrum is the one intertwining $\tau_X$ and $\tau_Y$

$$\Phi \circ \tau_X(t) = \tau_Y(t) \circ \Phi \quad (t \in G).$$

The standard problem consists in determining whether every operator shrinking the Arveson spectrum necessarily intertwines the representations.
Suppose that $\Phi : X \to Y$ shrinks the Arveson spectrum.

Pick $t \in G$ and $x \in X$. We define a continuous bilinear map

$$
\varphi : A(\mathbb{T}) \times A(\mathbb{T}) \to Y, \quad \varphi(f, g) = \sum_{j, k \in \mathbb{Z}} \hat{f}(j) \hat{g}(k) \tau_Y(t^j) \left( \Phi \left( \tau_X(t^k)x \right) \right) (f, g \in A(\mathbb{T})).
$$

We check that

$$
\text{sp}(\varphi(f, g)) \subset \left\{ \gamma \in \text{sp}(x) : \gamma(t) \in \text{supp}(f) \cap \text{supp}(g) \right\}.
$$
Our approach to the problem:
Disjointness vanishing bilinear maps on $A(\mathbb{T})$

Accordingly, $\varphi$ satisfies the property

$$ f, g \in A(\mathbb{T}), \text{ supp}(f) \cap \text{ supp}(g) = \emptyset \ \Rightarrow \ \varphi(f, g) = 0. $$


The bilinear map $\varphi$ induces a continuous linear operator

$$ \psi: A(\mathbb{T}^2) \to X, \quad \psi(f) = \sum_{j,k \in \mathbb{Z}} \hat{f}(j,k) \varphi(z^j, z^k) \quad \left( f \in A(\mathbb{T}^2) \right) $$

with the property that

$$ f \in A(\mathbb{T}^2), \text{ supp}(f) \cap \{ (z, z) : z \in \mathbb{T} \} = \emptyset \ \Rightarrow \ \psi(f) = 0. $$

At this point the synthesis is coming into the scene!
The set $\Delta$ is a set of synthesis for $A(\mathbb{T}^2)$. This means that every function $f \in A(\mathbb{T}^2)$ vanishing at $\Delta$ can be approximated by a sequence $(f_n)$ in $A(\mathbb{T}^2)$ with the property that $f_n$ vanishes on a neighborhood of $\Delta$.

We then consider the function $f \in A(\mathbb{T}^2)$ defined by

$$f(z, w) = z - w \quad (z, w \in \mathbb{T})$$

and there exists a sequence $(f_n)$ in $A(\mathbb{T}^2)$ with

$$f = \lim f_n \quad \text{and} \quad \text{supp}(f_n) \cap \Delta = \emptyset \quad (n \in \mathbb{N}).$$

Accordingly, we have

$$0 = \lim \Psi(f_n) = \Psi(f) = \underbrace{\varphi(z, 1)}_{\tau_Y(t)(\Phi x)} - \underbrace{\varphi(1, z)}_{\Phi(\tau_X(t)x)}.$$
Our contribution

Theorem

Let $G$ be a locally compact abelian group. Let $X$ and $Y$ be Banach spaces and let $\tau_X : G \to B(X)$ and $\tau_Y : G \to B(Y)$ bounded and strongly continuous group homomorphisms. If $\Phi \in B(X, Y)$ shrinks the Arveson spectrum, i.e.

$$\text{sp}(\Phi x) \subset \text{sp}(x) \quad (x \in X),$$

then $\Phi$ intertwines $\tau_X$ and $\tau_Y$, i.e.

$$\Phi \circ \tau_X(t) = \tau_Y(t) \circ \Phi \quad (t \in G).$$
The secret of our success

**Theorem**

Let

\[ \varphi: A(\mathbb{T}) \times A(\mathbb{T}) \to X \]

be a continuous bilinear map into some Banach space \( X \) with the property that

\[ f, g \in A(\mathbb{T}), \quad \text{supp}(f) \cap \text{supp}(g) = \emptyset \implies \varphi(f, g) = 0, \]

then

\[ \varphi(z, 1) = \varphi(1, z). \]
The theory of the Arveson spectrum still works for a **non-quasianalytic** representation $\tau : G \to B(X)$. This means that the weight function $\omega$ given by

$$\omega(t) = \max\{1, \|\tau(t)\|\} \quad (t \in G)$$

satisfies the **Beurling-Domar condition**

$$\sum_{k=-\infty}^{+\infty} \frac{\log \omega(t^k)}{1 + k^2} < \infty \quad (t \in G).$$

For the definition of the Arveson spectrum of $\tau$ at a point $x \in X$ we now replace the group algebra $L^1(G)$ by the weighted group algebra $L^1(G, \omega)$.

$$\text{sp}(x) = \left\{ \gamma \in \hat{G} : \hat{f}(\gamma) = 0 \text{ for each } f \in L^1(G, \omega) \text{ with } \int_G f(t)\tau(t)xdt = 0 \right\}.$$
Our theorem may fail in the case where the representation is unbounded.

Example

Consider the representations $\tau_1, \tau_2 : \mathbb{Z} \to \mathcal{B}(\mathbb{C}^2)$ given by

$$\tau_1(k) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k$$
and
$$\tau_2(k) = l_{\mathbb{C}^2} \quad (k \in \mathbb{Z}).$$

Then

$$\text{sp}_{\tau_2}(x) \subset \text{sp}_{\tau_1}(x) \quad (x \in \mathbb{C}^2)$$

and

$$l_{\mathbb{C}^2} \circ \tau_1(k) \neq \tau_2(k) \circ l_{\mathbb{C}^2} \quad (k \neq 0).$$

In this case we have

$$\sqrt{1 + k^2} \leq \|\tau_1(k)\| \leq \sqrt{2 + k^2} \quad (k \in \mathbb{Z}).$$
For dealing with representations with polynomial growth we are required to involve disjointness vanishing bilinear maps

\[ \varphi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to X \]

on the weighted Fourier algebra

\[ A_\alpha(\mathbb{T}) = \left\{ f \in C(\mathbb{T}): \|f\| = \sum_{k \in \mathbb{Z}} \left| \hat{f}(k) \right| (1 + |k|)^\alpha < \infty \right\} \]

for some \( \alpha \geq 0 \).
Theorem

Let

\[ \varphi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \to X \]

be a continuous bilinear map into some Banach space \( X \) with the property that

\[ f, g \in A_\alpha(\mathbb{T}), \quad \text{supp}(f) \cap \text{supp}(g) = \emptyset \quad \Rightarrow \quad \varphi(f, g) = 0. \]

Then

\[
\sum_{i=0}^{N} \binom{N}{i} (-1)^i \varphi(z^{N-i}, z^i) = 0
\]

for each \( N > 2\alpha \).

This time, \( \varphi \) gives rise to a continuous linear operator \( \Psi : A_{2\alpha}(\mathbb{T}^2) \to X \). However, the diagonal \( \Delta \) may fail to be a set of synthesis for \( A_{2\alpha}(\mathbb{T}^2) \) because of the weight. We are thus required to take care of the function \( f \) at which we intend to apply \( \Psi \). The so-called Beurling-Pollard type theorems assert, roughly speaking, that a function \( f \) admits synthesis if its growth is appropriate. We show that this is just the case for the function \( f(z, w) = (z - w)^N \).
An improvement to our contribution

Theorem

Let $G$ be a locally compact abelian group. Let $X$ and $Y$ be Banach spaces and let $\tau_X : G \to \mathcal{B}(X)$ and $\tau_Y : G \to \mathcal{B}(Y)$ strongly continuous group homomorphisms with polynomial growth, i.e.

$$\|\tau_Y(t^k)\|, \|\tau_X(t^k)\| = O(|k|^{\alpha}) \text{ as } |k| \to \infty$$

for some $\alpha \geq 0$. If $\Phi \in \mathcal{B}(X, Y)$ is such that

$$\text{sp}(\Phi x) \subset \text{sp}(x) \ (x \in X),$$

then

$$\sum_{i=0}^{N} \binom{N}{i} (-1)^i \tau_Y(t^{N-i}) \circ \Phi \circ \tau_X(t^i) = 0 \quad (t \in G),$$

whenever $N > 2\alpha$. 
Applications:
Translation invariant operators

Corollary

Let $G$ be a locally compact abelian group. Then $T : L^1(G) \to L^1(G)$ is translation invariant if and only if

$$\text{supp}(\hat{Tf}) \subset \text{supp}(\hat{f}) \quad (f \in L^1(G)).$$

Proof.

We consider the regular representation of $G$ on $L^1(G)$. Then

$$\text{supp}(\hat{Tf}) \subset \text{supp}(\hat{f}),$$

with

$$\text{sp}(Tf) \subset \text{sp}(f),$$

and therefore $T$ shrinks the Arveson spectrum. 

\qed
Applications:
Determining operators through the spectrum

Corollary (Colojoară-Foiaş, 1968)

Let $X$ be a complex Banach space and $S, T \in \mathcal{B}(X)$ be invertible operators with polynomial growth. Suppose that

$$\sigma(S, x) \subset \sigma(T, x) \quad \forall x \in X.$$  

Then

$$\sum_{i=0}^{N} \binom{N}{i} (-1)^i T^{N-i} S^i = 0,$$

whenever $N > 2\alpha$, where $\alpha$ is such that

$$\|S^k\|, \|T^k\| = O(|k|^\alpha) \text{ as } |k| \to \infty$$

Consequently,

1. If $S$ and $T$ are doubly power bounded, then $S = T$.
2. If $S$ and $T$ commute, then $(S - T)^N = 0$. 
Proof.

We consider the representations $\tau_S, \tau_T: \mathbb{Z} \rightarrow \mathcal{B}(X)$ given by $\tau_S(k) = S^k$ and $\tau_T(k) = T^k$ ($k \in \mathbb{Z}$). Then

$$
\sigma(S, x) \subset \sigma(T, x),
$$

$$
\text{sp}_{\tau_S}(x) \subset \text{sp}_{\tau_T}(x),
$$

which shows that the identity operator $I_X: X \rightarrow X$ shrinks the spectrum. Consequently,

$$
\sum_{i=0}^{N} \binom{N}{i} (-1)^i \underbrace{\tau_T(N - i) I_X \tau_S(i)}_{T^{N-i}S^i} = 0 \quad (t \in G),
$$
Corollary (I. Gelfand, 1941)

Let $T$ be a bounded linear operator on a complex Banach space $X$ such that $\sigma(T) = \{1\}$. If

$$\sup_{k \in \mathbb{Z}} \| T^k \| < \infty,$$

then

$$T = I_X.$$

Example

It should be pointed out that Gelfand theorem fails in the case where the operator is not doubly power bounded. As a matter of fact, the operator $T \in B(\mathbb{C}^2)$ corresponding to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is different from the identity operator and nevertheless $\sigma(T) = \{1\}$. In this case

$$T^k \equiv \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sqrt{1 + k^2} \leq \| T^k \| \leq \sqrt{2 + k^2} \quad \forall k \in \mathbb{Z}.$$
Corollary (E. Hille, 1944)

Let $T$ be a bounded linear operator on a complex Banach space $X$ such that $\sigma(T) = \{1\}$. If

$$\|T^k\| = O(|k|^n) \text{ as } |k| \to \infty$$

for some $n \geq 0$, then

$$(T - I_X)^N = 0.$$ for each integer $N$ with $N > n$.

Example

It should be pointed out that both Gelfand and Hille theorems fail for comparing the given operator with an operator different from the identity. Indeed, let $S, T \in B(\mathbb{C}^2)$ be the operators with matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively. Then

$$\sigma(T) = \sigma(S) = \{1\}$$

and $T - S$ is far from being nilpotent.
Theorem (Local version of Gelfand-Hille theorem)

Let $X$ be a complex Banach space and let $T$ be an invertible operator on $X$ such that $\|T^k\| = O(|k|^{\alpha})$ as $|k| \to \infty$ for some $\alpha \geq 0$. If $x \in X$ is such that

$$\sigma(T, x) = \{\lambda\}$$

for some $\lambda \in \mathbb{C}$, then

$$(T - \lambda I_X)^N x = 0$$

whenever $N \in \mathbb{N}$ is such that $N > 2\alpha$. In particular, if $T$ is doubly power bounded, then

$$Tx = \lambda x.$$
Proof.
First of all, it should be pointed out that
\[ \lambda \in \sigma(T,x) \subset \sigma(T) \subset \mathbb{T}. \]

We define \( \varphi : A_\alpha(\mathbb{T}) \times A_\alpha(\mathbb{T}) \rightarrow X \) by
\[
\varphi(f,g) = g(\lambda) \sum_{k \in \mathbb{Z}} \hat{f}(k) T^k x.
\]

Then \( \varphi \) is a disjointness vanishing continuous bilinear map. Our seminal result gives
\[
0 = \sum_{i=0}^{N} \binom{N}{i} (-1)^i \varphi(z^{N-i}, z^i)\underbrace{\lambda^i T^{N-i} x}_{\lambda^i T^{N-i} x} = (T - \lambda I_x)^N x.
\]
Theorem (An improvement of the Colojoară-Foiaş theorem)

Let $X$ be a complex Banach space, $S, T_1, \ldots, T_n, \in \mathcal{B}(X)$ invertible operators with polynomial growth with the property that

$$
\sigma(S, x) \subset \bigcup_{j=1}^{n} \sigma(T_j, x) \quad \forall x \in X
$$

and that $T_1, \ldots, T_n$ are pairwise commuting Then there exists $N \in \mathbb{N}$ such that

$$
C(S, T_1)^N \cdots C(S, T_n)^N I_X = 0.
$$

Here the intertwiner $C(S, T)$ of $S, T \in \mathcal{B}(X)$ is defined by

$$
C(S, T) : \mathcal{B}(X) \to \mathcal{B}(X), \quad C(S, T)A = SA - AT \quad \forall A \in \mathcal{B}(X).
$$
Theorem (A. C. Zaanen, 1975)

Let $\Omega$ be a locally compact Hausdorff space and let $T : C_0(\Omega) \to C_0(\Omega)$ be a bounded linear operator with the property that

$$f \in C_0(\Omega), \quad E \subset \Omega, \quad f = 0 \text{ on } E \Rightarrow Tf = 0 \text{ on } E.$$

Then there exists $g \in C_b(\Omega)$ such that

$$Tf = fg \quad \forall f \in C_0(\Omega).$$
Theorem (Our non-commutative version)

Let $A$ be a (unital) $C^*$-algebra and let $T : A \rightarrow A$ be a bounded linear operator with the property that
\[ \forall x \in A, \quad p x = 0 \Rightarrow p T x = 0. \]

Then there exists $a \in A$ such that
\[ T x = x a \quad \forall x \in A. \]
Proof.

For every unitary $u \in A$, the map $\varphi : A(\mathbb{T}) \times A(\mathbb{T}) \to A$ defined by

$$\varphi(f, g) = \sum_{j, k \in \mathbb{Z}} \hat{f}(j) \hat{g}(k) u^j T(u^k)$$

is disjointness vanishing. This entails that

$$T(u) = u T(1).$$

Since $A$ is generated by the unitaries, it follows that

$$T(x) = x T(1) \quad \forall x \in A.$$
If we remove the requirement of $A$ being unital, then the theorem still works though the element $a$ defining the operator $T$ lies in $\mathcal{M}(A)$, the multiplier algebra of $A$ (i.e., the largest $C^*$-algebra containing $A$ as an essential ideal).

- Of course, if $A$ is unital, then $\mathcal{M}(A) = A$.
- If $A = C_0(\Omega)$, then $\mathcal{M}(A) = C_b(\Omega)$.
- If $A = \mathcal{K}(H)$, then $\mathcal{M}(A) = \mathcal{L}(H)$. 

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Theorem (A. C. Zaanen, 1975)

Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, \(1 \leq p \leq \infty\), and let \(T : L^p(\mu) \to L^p(\mu)\) be a bounded linear operator with the property that

\[
f \in L^p(\mu), \ E \in \Sigma, \ f = 0 \ a.e. \ on \ E \ \Rightarrow \ Tf = 0 \ a.e. \ on \ E.
\]

Then there exists \(g \in L^\infty(\mu)\) such that

\[
Tf = fg \quad \forall f \in L^p(\mu).
\]
Theorem (Our noncommutative version)

Let $\mathcal{M}$ be a von Neumann algebra with a (normal semifinite faithful) trace $\tau$, $1 \leq p \leq \infty$, and let $T : L^p(\mathcal{M}, \tau) \to L^p(\mathcal{M}, \tau)$ be a bounded linear operator with the property that

$$x \in L^p(\mathcal{M}, \tau), \ p \text{ projection in } \mathcal{M}, \ px = 0 \implies pTx = 0.$$  

Then there exists $a \in \mathcal{M}$ such that

$$Tx = xa \ \forall x \in L^p(\mathcal{M}, \tau).$$

- Let $\mathcal{M} = L^\infty(\mu)$. Then integration with respect to $\mu$ gives a n.s.f. trace and $L^p(\mathcal{M}) = L^p(\mu)$ ($1 \leq p \leq \infty$).
- Let $\mathcal{M} = \mathcal{L}(H)$ and $\tau$ the usual trace on $\mathcal{L}(H)$. Then $L^p(\mathcal{M})$ is the Schatten class $S^p(H)$ ($1 \leq p < \infty$).
Applications:

Determining observables in quantum mechanics through the support

1. Every physical observable in quantum mechanics is mathematically represented by a selfadjoint operator.

2. The outcome of a measurement of an observable is of statistical nature. The value obtained in a measurement of the observable $A$ on the state $\psi$ is a random variable $\hat{A}\psi$ whose probability distribution is given by the spectral measure $\mathcal{E}_{\psi}$ of $A$ at $\psi$, so that the probability to obtain a value in $\Delta \subset \mathbb{R}$ as the result of this measure is given by

$$
\mathbb{P}[\hat{A}\psi \in \Delta] = \|\mathcal{E}(\Delta)\psi\|^2.
$$

**Corollary**

*Let $A$ and $B$ be observables on a quantum physical system with the property that

$$
\mathbb{P}[\hat{A}\psi \in \Delta] = 1 \Rightarrow \mathbb{P}[\hat{B}\psi \in \Delta] = 1.
$$

Then $A = B$.***