Group actions, von Neumann algebras and fundamental groups

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Topics of the talk

Group action $\Gamma \curvearrowright (X, \mu)$
Probability measure preserving
Free : if $g \neq e$, then $g \cdot x \neq x$ a.e.
Ergodic : Γ-inv. $\Rightarrow$ meas. 0 or 1

$\downarrow$

Orbit equivalence relation
$\mathcal{R}(\Gamma \curvearrowright X)$: $x \sim y$ iff $x \in \Gamma \cdot y$

$\downarrow$

Von Neumann algebra $L^\infty(X) \rtimes \Gamma$
II_1 factor: trivial center, tracial

Isomorphism or equivalence
of $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$

Conjugacy :
Exist $\Delta : X \to Y$ and $\delta : \Gamma \to \Lambda$
s.t. $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$.

$\downarrow \quad \not\uparrow$ Obstruction: 1-cocycle ω
$\Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x)$

Orbit equivalence :
Exist $\Delta : X \to Y$
s.t. $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(y)$

$\downarrow \quad \not\uparrow$ Obstruction:
$\theta(L^\infty(X)) \neq L^\infty(Y)$

W^*-equivalence :
Exist iso $\theta : L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$

Rigidity : prove $\uparrow$. Weak equivalence $\Rightarrow$ strong equivalence.

Superrigidity

Standing assumptions: free, ergodic, p.m.p. actions.

Orbit equivalence superrigidity

We call $\Gamma \curvearrowright (X,\mu)$ **OE superrigid** if the following holds.

Whenever an arbitrary $\Lambda \curvearrowright (Y,\eta)$ is orbit equivalent with $\Gamma \curvearrowright (X,\mu)$, then both groups are isomorphic and the actions conjugate.

$\rightsquigarrow$ The equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ entirely remembers $\Gamma \curvearrowright X$.

W*-superrigidity

We call $\Gamma \curvearrowright (X,\mu)$ **W*-superrigid** if the following holds.

Whenever an arbitrary $\Lambda \curvearrowright (Y,\eta)$ is W*-equivalent with $\Gamma \curvearrowright (X,\mu)$, then both groups are isomorphic and the actions conjugate.

$\rightsquigarrow$ The II_1 factor $L^\infty(X) \rtimes \Gamma$ entirely remembers $\Gamma \curvearrowright X$.

Amenable groups : absence of rigidity

Theorem (Ornstein and Weiss, 1980)

All free ergodic pmp actions of all amenable groups are orbit equiv.

Recall. A group Γ is called amenable if the regular representation

$$\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma)) : \lambda_g e_h = e_{gh}$$

admits a sequence of almost invariant vectors :

$$\xi_n \in \ell^2(\Gamma) \; , \; \|\xi_n\|_2 = 1 \; , \; \|\lambda_g \xi_n - \xi_n\|_2 \to 0 \; \forall g \in \Gamma$$

Examples. The following groups are amenable.

- abelian groups, finite groups, solvable groups,
- stable by subgroups, quotients and extensions.

Away from amenability : a group Γ has Kazhdan's property (T) if every unitary representation with a sequence of almost invariant unit vectors, actually has an invariant unit vector.

Examples. $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$. Lattices in higher rank simple Lie groups. Random groups.

Orbit equivalence superrigidity

Recall. Orbit equivalence between $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ is iso $\Delta : X \to Y$ satisfying $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$.

$\rightsquigarrow$ Zimmer 1-cocycle $\Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x)$ with target group Λ.

- The action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$, $n \geq 3$, is OE superrigid. (Furman, 1999, using Zimmer's cocycle superrigidity)
- The Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is OE superrigid whenever Γ has property (T). (Popa, 2005, using his cocycle superrigidity with arbitrary countable target groups)
- For certain mapping class groups, any free ergodic pmp action is OE superrigid. (Kida, 2006)
- Any profinite action of a property (T) group is virtually OE superrigid. (Ioana, 2008) Profinite and ergodic : $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$.
- The linear action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n$, $n \geq 5$, is OE superrigid. (Popa – V, 2008).

Classification of von Neumann algebras

Von Neumann algebra : weakly closed unital $*$-subalgebra of $B(H)$.

II_1 factor : infinite dim. von Neumann algebra, trivial center, tracial.

Group von Neumann algebra

Let Γ be a countable group and $g \mapsto \lambda_g$ its regular rep. on $\ell^2(\Gamma)$.

- $\mathrm{span}\{\lambda_g \mid g \in \Gamma\}$ is the group algebra $\mathbb{C}\Gamma$.
- Define $L(\Gamma)$ as the weak closure of $\mathbb{C}\Gamma$.

$\rightsquigarrow$ $L(\Gamma)$ is always tracial.
$L(\Gamma)$ is a factor iff Γ has infinite conjugacy classes (icc).

Extremely difficult problem : when is $L(\Gamma) \cong L(\Lambda)$?

- (Connes, 1975) All amenable II_1 factors are isomorphic.
  So, all $L(\Gamma)$ for Γ amenable and ICC are isomorphic.
- (Open problem) Are $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$?
- (Connes' conjecture)
  If Γ has property (T) and $L(\Gamma) \cong L(\Lambda)$, then $\Gamma \cong \Lambda$ (virtually).

Group measure space construction (M-vN 1943)

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic prob. measure preserving action.

The von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$ is generated by

- the subalgebra $\mathrm{L}^\infty(X)$,
- the group of unitaries $(u_g)_{g \in \Gamma}$,

with $\left| \begin{array}{l} u_g^* \, F \, u_g = F_g \text{ where } F_g(x) = F(g \cdot x), \quad \text{(for all } F \in \mathrm{L}^\infty(X) \text{ and } g \in \Gamma) \\ \tau(F) = \int_X F \, d\mu, \qquad \text{while} \qquad \tau(Fu_g) = 0 \text{ for } g \neq e. \end{array} \right.$

We now discuss:
Relation between orbit equivalence and von Neumann algebras.

Theorem (Singer 1955, Feldman-Moore 1977)

$\mathrm{L}^\infty(X) \to \mathrm{L}^\infty(Y) : F \mapsto F \circ \Delta^{-1}$ extends to iso $\mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(Y) \rtimes \Lambda$ if and only if Δ is an orbit equivalence (i.e. $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$ a.e.).

$\rightsquigarrow$ Orbit equivalence is the same as isomorphism $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$ sending $\mathrm{L}^\infty(X)$ onto $\mathrm{L}^\infty(Y)$.

Cartan subalgebras

Definition of a Cartan subalgebra A in a II_1 factor M

A von Neumann subalgebra $A \subset M$ is called a Cartan subalgebra if

- A is maximal abelian : $A' \cap M = A$,
- the unitaries $u \in M$ satisfying $uAu^* = A$, generate M.

Example : $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ for $\Gamma \curvearrowright X$ a free ergodic pmp action.

$\rightsquigarrow$ We call $\mathrm{L}^\infty(X)$ a group measure space Cartan subalgebra (and not all Cartan subalgebras are of this form).

An orbit equivalence of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ is the same as an isomorphism $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$ sending $\mathrm{L}^\infty(X)$ onto $\mathrm{L}^\infty(Y)$.

$\rightsquigarrow$ Importance to locate Cartan subalgebras in II_1 factors.

W*-superrigidity and uniqueness of Cartan subalg.

W*-superrigidity : $L^\infty(X) \rtimes \Gamma$ remembers $\Gamma \curvearrowright X$.

OE superrigidity : $\mathcal{R}(\Gamma \curvearrowright X)$ remembers $\Gamma \curvearrowright X$.

- W*-superrigidity arises as the sum of unique group measure space Cartan and OE superrigidity.
- Uniqueness of Cartan subalgebras is badly understood.
- Unique | up to unitary conjugacy : $A \rightsquigarrow uAu^*$
  | up to conjugacy by an automorphism : $A \rightsquigarrow \theta(A)$

Theorem

- $L^\infty(X) \rtimes \mathbb{F}_n$ with $\mathbb{F}_n \curvearrowright X$ profinite has unique Cartan up to unitary conjugacy. (Ozawa – Popa, 2007)
- $L^\infty(X) \rtimes \Gamma$ with $\Gamma \curvearrowright X$ profinite, $\Gamma = \Gamma_1 * \Gamma_2$ and Γ_1 non-Haagerup, $\Gamma_2 \neq \{e\}$, has unique group measure space Cartan up to unitary conjugacy. (Peterson, in preparation)

$\rightsquigarrow$ Peterson proves the existence of virtually W*-superrigid actions.

Uniqueness of Cartan for free product groups

Theorem (Popa – V, 2009)

Let $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1 has property (T) and $\Gamma_2 \neq \{e\}$. Let $\Gamma \curvearrowright (X, \mu)$ be an arbitrary free ergodic pmp action.

Then, $L^\infty(X) \rtimes \Gamma$ has a unique group measure space Cartan subalgebra, up to unitary conjugacy.

The theorem holds for a larger family of $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ where

- Γ_1 has a non-amenable subgroup with the relative property (T), or Γ_1 has two commuting non-amenable subgroups,
- Σ is amenable and $\Gamma_2 \neq \Sigma$,
- there exist $g_1, \ldots, g_n \in \Gamma$ s.t. $\bigcap_{i=1}^{n} g_i \Sigma g_i^{-1}$ is finite.

⇝ We now explain how to deduce examples of W*-superrigid actions.

W*-superrigid actions

Theorem (Popa – V, 2009)

For the following groups Γ, the Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ is W*-superrigid.

- $\Gamma = \mathrm{PSL}(n, \mathbb{Z}) *_\Sigma (\Sigma \times G)$, $n \geq 3$, for any $G \neq \{e\}$ and for Σ an infinite subgroup of the upper triangular matrices.
- $\Gamma = (H \times H) *_\Sigma (\Sigma \times G)$, where H is non-amenable with trivial center, $G \neq \{e\}$ and Σ is an infinite amenable subgroup of H embedded diagonally in $H \times H$.

So, whenever $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$, both actions are conjugate.

$\rightsquigarrow$ Uniqueness of group measure space Cartan follows from the theorem on the previous slide.

$\rightsquigarrow$ OE superrigidity follows from Popa's theorem.

Same methods cover other actions of the same groups: Gaussian actions, generalized Bernoulli actions, co-induced actions.

Method: deformation/rigidity

Theorem (Popa, 2004)

Let Γ be a property (T) group and $\Gamma \curvearrowright (X, \mu)$ free ergodic.
Let Λ be an ICC group and $\Lambda \curvearrowright (Y, \eta) = [0, 1]^\Lambda$ its Bernoulli action.
If $\mathrm{L}^\infty(X) \rtimes \Gamma \;\cong\; \mathrm{L}^\infty(Y) \rtimes \Lambda$, then
the groups Γ and Λ are isomorphic and their actions conjugate.

First theorem ever deducing conjugacy out of iso of II_1 factors.

- Rigidity assumption on one side : Γ has property (T).
- Deformation property on the other side : action is Bernoulli.

Joint work with Popa on the previous slides :

- All conditions on one side $\rightsquigarrow$ W*-superrigid actions.
  Important ingredient : a transfer of rigidity lemma.
- Deformation of $M = \mathrm{L}^\infty(X) \rtimes (\Gamma_1 * \Gamma_2)$ given by
  $\varphi_\rho : M \to M : \varphi_\rho(Fu_g) = \rho^{|g|} Fu_g$, $\quad (F \in \mathrm{L}^\infty(X), g \in \Gamma_1 * \Gamma_2)$.

Non-uniqueness of Cartan subalgebras

Theorem (Connes – Jones, 1982)

Let Γ be a non-amenable group and Σ a non virtually abelian group acting profinitely and faithfully on (X_0, μ_0).

(e.g. $\Sigma = \Sigma_0^{(\mathbb{N})} \curvearrowright X_0 = \Sigma_0^{\mathbb{N}}$ for Σ_0 finite non-abelian)

Then, $\mathrm{L}^\infty(X_0^\Gamma) \rtimes (\Sigma \times \Gamma)$ has at least two Cartan subalgebras that are non conjugate by an automorphism of M.

$\rightsquigarrow$ You cannot 'see' the other Cartan subalgebra.

Example (Ozawa – Popa, 2008)

Put $K = \mathbb{Z}_p^n$ and consider $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z}) \curvearrowright K$.

Then, $\mathrm{L}^\infty(K) \rtimes (\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})) \cong \mathrm{L}^\infty(\mathbb{T}^n) \rtimes (\hat{K} \rtimes \mathrm{SL}(n, \mathbb{Z}))$.

- With $n = 2$: Haagerup property is not stable under W*-equiv.
- With $n \geq 3$: property (T) is not stable under W*-equivalence.

Rigidity of invariants

Rigidity paradigm : weak isomorphism $\Rightarrow$ strong isomorphism.

In its strongest form :
every weak 'morphism' is itself equivalent to a 'strong morphism'.

$\rightsquigarrow$ Classification results and computations of invariants.

Definition (Murray – von Neumann, 1943)

The fundamental group $\mathcal{F}(M)$ of a II_1 factor M is
the subgroup of $\mathbb{R}_+$ given by $\mathcal{F}(M) = \left\{ \frac{\tau(p)}{\tau(q)} \;\middle|\; pMp \cong qMq \right\}$.

Note : $\tau(p)$ ranges over the whole of $[0,1]$.

- $\mathcal{F}(R) = \mathbb{R}_+$ for the hyperfinite II_1 factor R, e.g. $R = \mathrm{L}(S_\infty)$ (Murray – von Neumann, 1943).
- $\mathcal{F}(\mathrm{L}(\Gamma))$ is countable when Γ is an icc property (T) group (Connes, 1980).

Computations of fundamental groups

- $\mathcal{F}(\mathrm{L}(\mathbb{F}_\infty)) = \mathbb{R}_+$ (Voiculescu 1989, Rădulescu 1991)
  - When $\tau(p) = 1/k$, then $p\mathrm{L}(\mathbb{F}_{1+n})p \cong \mathrm{L}(\mathbb{F}_{1+k^2 n})$ (Voiculescu)
  - Similar for arbitrary $\tau(p) \in \mathbb{R}_+$: interpolated free group factors (Dykema, Rădulescu)
- The II_1 factor $M = \mathrm{L}^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ has $\mathcal{F}(M) = \{1\}$ (Popa 2001).
  - Deformation : $\mathrm{SL}(2,\mathbb{Z})$ has Haagerup property
    Rigidity : $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ has relative property (T).
  - It follows that $\mathcal{F}(M)$ equals $\mathcal{F}$(orbit equivalence relation),
  - which is trivial by Gaboriau's cost or ℓ^2-Betti numbers.
- $\mathcal{F}(M)$ can by any countable subgroup of $\mathbb{R}_+$ (Popa 2003).
  But these II_1 factors cannot be written as $M = \mathrm{L}^\infty(X) \rtimes \Gamma$.

Questions that remained open :

$\rightsquigarrow$ Can $\mathcal{F}(M)$ be uncountable without being $\mathbb{R}_+$?

$\rightsquigarrow$ What about $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes \Gamma)$? Up to now : $\{1\}$ or $\mathbb{R}_+$.

Uncountable fundamental groups

Theorem (Popa - V, 2008)

Let Γ be of the form $\Gamma = G^{*\infty} * \Sigma$ with Σ infinite amenable, e.g. $\mathbb{F}_\infty$.

For all subgroups $\mathcal{F} \subset \mathbb{R}_+$ belonging to a large class $\mathcal{S}$, there exist uncountably many free ergodic p.m.p. actions $\sigma_i : \Gamma \curvearrowright (X_i, \mu_i)$ s.t.

- $\mathrm{L}^\infty(X_i) \rtimes_{\sigma_i} \Gamma$ has fundamental group $\mathcal{F}$,
- the II_1 factors $\mathrm{L}^\infty(X_i) \rtimes_{\sigma_i} \Gamma$ are non-isomorphic for different i.

The above theorem has to be opposed to:

Theorem (Popa – V, 2008)

Let $\Gamma = G * \Sigma$ be a free product of finitely generated, infinite groups.

- Either assume G an ICC group with property (T),
- or, $G = G_1 \times G_2$ a non-amenable product of ICC groups.

Then, $\mathcal{F}(M) = \{1\}$ for all $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ and $\Gamma \curvearrowright (X, \mu)$ free ergodic.

Uncountable fundamental groups

Theorem (Popa - V, 2008)

Let Γ be of the form $\Gamma = G^{*\infty} * \Sigma$ with Σ infinite amenable, e.g. $\mathbb{F}_\infty$.

For all subgroups $\mathcal{F} \subset \mathbb{R}_+$ belonging to a large class $\mathcal{S}$, there exist uncountably many free ergodic p.m.p. actions $\sigma_i : \Gamma \curvearrowright (X_i, \mu_i)$ s.t.

- $\mathrm{L}^\infty(X_i) \rtimes_{\sigma_i} \Gamma$ has fundamental group $\mathcal{F}$,
- the II_1 factors $\mathrm{L}^\infty(X_i) \rtimes_{\sigma_i} \Gamma$ are non-isomorphic for different i.

The large class $\mathcal{S}$ includes

- all countable subgroups of $\mathbb{R}_+$,
- $\mathcal{F} \subset \mathbb{R}_+$ of prescribed Hausdorff dimension,
- Certain $\left\{ e^x \in \mathbb{R}_+ \;\middle|\; \sum_{n=1}^{\infty} \gamma_n \|\alpha_n x\| < \infty \right\}$, where $\|x\| = \mathrm{d}(x, \mathbb{Z})$.

$\rightsquigarrow$ Non-pathological uncountable subgroups of $\mathbb{R}_+$.

Ergodic measures and definition of $\mathcal{S}$

Definition (Aaronson, Nadkarni, 1987)

An ergodic measure ν on $\mathbb{R}$ is a σ-finite measure on the Borel sets of $\mathbb{R}$ such that

- for every $x \in \mathbb{R}$, either $\nu \circ \lambda_x = \nu$ or $\nu \circ \lambda_x \perp \nu$,
- writing $H_\nu = \{x \in \mathbb{R} \mid \nu \circ \lambda_x = \nu\}$, every H_ν-invariant Borel function is ν-almost everywhere constant.

- Define $\mathcal{S} := \{\exp(H_\nu) \mid \nu$ is ergodic measure on $\mathbb{R}\}$.
- We get in $\mathcal{S}$, certain $\left\{e^x \in \mathbb{R}_+ \;\middle|\; \sum_{n=1}^{\infty} \gamma_n \|\alpha_n x\| < \infty\right\}$ by using a Cantor measure construction.

Some speculations (staying on separable Hilbert spaces)

Conjecture : every $\mathcal{F}(M)$ is realizable as $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty)$.

Open problem : characterize intrinsically which subgroups of $\mathbb{R}_+$ are of the form $\mathcal{F}(M)$.

- Every $\mathcal{F}(M)$ is a Borel subset of $\mathbb{R}_+$.
- Every $\mathcal{F}(M)$ is Polishable, i.e. carries a (unique) Polish topology that generates the Borel σ-algebra inherited from $\mathbb{R}_+$.

Remember : we realize all groups in $\mathcal{S} = \{\exp(H_\nu)\}$ as $\mathcal{F}(M)$.

All H_ν are

- a countable union of compact subsets of $\mathbb{R}$,
- such that H_ν is the eigenvalue group of a non-singular flow $\mathbb{R} \curvearrowright (Y, \eta)$, i.e. $H_\nu = \{t \in \mathbb{R} \mid \exists F : Y \to \mathbb{T} : F(s \cdot y) = e^{ist} F(y)\}$.

We expect that none of both properties holds for all $\log(\mathcal{F}(M))$.