The James–Schreier Space

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In this talk: $x = (\alpha_i) \subset \mathbb{R}^\mathbb{N}$;

$$\nu_p(x, A) := \left(\sum_{n \in A} |\alpha_n|^p\right)^{1/p}$$

Schreier norm: $\|x\|_{S_p} := \sup\{\nu_p(x, A) : A \subseteq \mathbb{N} \text{ admissible}\}$

$$\mu_p(x, A) := \left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p\right)^{1/p}$$

James norm: $\|x\|_{J_p} := \sup\{\mu_p(x, A) : A \subseteq \mathbb{N}\}$

James–Schreier norm: $\|x\|_{V_p} := \sup\{\mu_p(x, A) : A \subseteq \mathbb{N} \text{ permissible}\}$
Definitions

Two ways of defining $J_p$:

**Definition 1**
Let $J_p$ be completion of $c_{00}$ with respect to the $J_p$ norm.

**Definition 2**
Let $J_p$ be those sequences in $c_0$ which have finite $J_p$ norm.

These two definitions describe the same space as

$$\|(I - P_n)x\|_{J_p} \to 0 \iff \|x\|_{J_p} < \infty,$$

that is, the unit vectors $(e_i)$ are a Schauder basis of $J_p$ in Definition 2.
Definitions

**Definition - $V_p$**

Let $V_p$ be completion of $c_{00}$ with respect to the $\| \cdot \|_{V_p}$-norm.

**Definition - $W_p$**

Let $W_p$ be those sequences in $c_0$ which have finite $\| \cdot \|_{V_p}$-norm.

Here, the spaces $V_p$ and $W_p$ are not equal or isomorphic.

**Equivalent Definition - $V_p$**

Let $V_p$ to be the closure of the linear span of the basic sequence $(e_i)$ in $W_p$ with respect to the $\| \cdot \|_{V_p}$-norm.

These two definitions of $V_p$ are trivially equivalent. However, unlike the James space, now the basic sequence $(e_i)$, the basis by definition, of $V_p$, does not span $W_p$. 
Similarly for the Schreier space:

**Definition - \( Z_p \)**

Let \( Z_p \) be those sequences in \( c_0 \) which have finite \( \| \cdot \|_{S_p} \)-norm.

**Definition - \( S_p \)**

Let \( S_p \) to be the closure of the linear span of the basic sequence \((e_i)\) in \( Z_p \) with respect to the \( S_p \) norm.

**e.g.** \((1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)\) has finite \( \| \cdot \|_{S_1} \)-norm, so is in \( Z_1 \), but is not in \( S_1 \).

**Claim:** \( Z_p \) is the second dual of the Schreier space \( S_p \).
Let \((b_n)\) be a basis for a Banach space \(X\).

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<th>Definition and Theorem</th>
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<td>Let ((f_n)) be the sequence of coordinate functionals for ((b_n)), that is, (f_n : \sum \alpha_mb_m \mapsto \alpha_n); then each (f_n) is bounded.</td>
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<td>A basis ((b_n)) is <strong>shrinking</strong> if and only if the coordinate functionals ((f_n)) are a basis for (X^*).</td>
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<td>For (p &gt; 1) the unit vectors ((e_i)) are a shrinking basis for (V_p).</td>
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Define the natural map $\kappa : V_p \rightarrow V_{p}^{**}$ by $\kappa(x) : f_i \mapsto x_i$.

The shrinking basis for $V_p$ allows us to construct an isometric isomorphism from $V_{p}^{**}$ to

$$X_{V_p} := \left\{(\alpha_n) \subseteq \mathbb{C}^{\mathbb{N}} : \sup_{m} \left\| \sum_{n=1}^{m} \alpha_n e_n \right\|_{V_p} < +\infty \right\} \cong W_p \oplus \mathbb{C}1.$$
The James–Schreier Banach Algebra

For $1 \leq p < \infty$ the space $X_{V_p}$ is a commutative *-algebra under the norm $\|\cdot\|_{V_p}$ equipped with pointwise multiplication and pointwise complex conjugation as involution, and has separately continuous product.

The Banach space $X_{V_p}$ is a commutative Banach *-algebra with identity $e_0 = (1, 1, \ldots)$ under the $\|\cdot\|_{V_p}$-norm equipped with pointwise multiplication.

$W_p$ is a *-subalgebra of $X_{V_p}$.

$V_p$ is a *-ideal of $W_p$ and $X_{V_p}$. 
We define $\chi_n$ to be $\sum_{i=1}^{n} e_i = (1, 1, \ldots, 1, 0, \ldots)$. The sequence $(\chi_n)$ is a bounded approximate identity of projections of $\|\cdot\|_{V_p}$-norm 1 in the Banach *-algebra $V_p$, and they are contained in $c_{00}$.

The commutative Banach *-algebra $V_p$ is weakly amenable, but not amenable.

$V_p$ is Arens regular. Hence, the multiplier algebra of $V_p$, $(V_p^{**}, \square), X_{V_p}, W_p \oplus \mathbb{C}1$ are all isometrically isomorphic.
James’ Theorem - 1950

A Banach space with an unconditional basis, is reflexive if and only if it has no embedded copies of \( c_0 \) or \( l_1 \).

Corollary

The James space has no copies of \( c_0 \) or \( l_1 \), but is not reflexive. So \( J_p \) has no unconditional basis.

But the James–Schreier space, \( V_p \) does contain copies of \( c_0 \)—it is \( c_0 \)-saturated! So a different approach is necessary.
A Banach space $X$ has Pełczyński’s property (u) if for all weak Cauchy sequences $(x_n) \subset X$ there exists a sequence $(y_n) \subset X$ such that for all $f \in X^*$

$$\left\langle x_n - \sum_{i=1}^{n} y_i, f \right\rangle \to 0 \text{ as } n \to \infty$$

and $\sum_{n=1}^{\infty} |\left\langle y_n, f \right\rangle|$ is convergent.
Pełczyński’s Property (u)

**Reminder:** $X$ has Pełczyński’s property (u) if for all weak Cauchy sequences $(x_n) \subset X$ there exists a weakly unconditionally convergent (WUC) series $\sum_{n=1}^{\infty} y_n$ such that $(x_n - \sum_{i=1}^{n} y_i)_{n \in \mathbb{N}}$ is weakly null.

Every subspace of a space with Pełczyński’s property (u) also has Pełczyński’s property (u).

Every Banach space with an unconditional basis has property (u). In particular $c_0$ and $S_p$ have it.

To show a Banach space does not have an unconditional basis it is enough to show it doesn’t have property (u).
Pełczyński’s Property (u) - James Space

Theorem - Bessaga and Pełczyński

Every weak unconditionally convergent (WUC) series in a Banach space $X$ is unconditionally convergent if and only if $X$ contains no copy of $c_0$.

Proposition: $J_p$ does not have Pełczyński’s property (u).

Proof (by contradiction):
- Assume that property (u) holds.
- Then $(\chi_n)$ is weakly Cauchy in $J_p$ and has no weak limit as $e_0 = (1, 1, \ldots) \in J_p^{**} \setminus J_p$.
- So there is sequence $(y_n) \subset J_p$ with $\sum_{n=1}^{\infty} y_n$ WUC such that $\chi_n - \sum_{i=1}^{n} y_n \rightharpoonup 0$.
- If $\sum_{n=1}^{\infty} y_n$ converges unconditionally then it must do so to $e_0 = (1, 1, \ldots)$, but this is not in $J_p$.
- By the Theorem above, $J_p$ contains $c_0$. Contradiction!
Pełczyński’s Property (u)

This proof still depends on $J_p$ not containing copies of $c_0$; so a new idea is needed for a successful proof. Instead of going for an abstract approach, we can view the proof as a simple game with concrete sequences:

1. We supply a weak Cauchy sequence $(x_n)$.
2. Our opponent counters with a sequence $(y_n)$ such that $(x_n - \sum_{i=1}^{n} y_i)_{n \in \mathbb{N}}$ is weakly null.
3. We win if we can find an $f \in V_p^*$ such that $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$ is divergent. If none exists, we lose.

If we can show that a winning strategy exists for us, this proves that $V_p$ does not have Pełczyński’s property (u).
As $V_p$ has a shrinking basis for $p > 1$, a sequence is weak Cauchy if and only if it is bounded and $(\langle x_n, f_k \rangle)_{n \in \mathbb{N}}$ converges for all $k$.

If $(x_n)$ is a weak Cauchy sequence that weakly converges then the conditions for Pełczyński’s Property (u) to hold are trivially satisfied.

So we need $(x_n)$ not weakly convergent in $V_p$.

A natural candidate for our sequence is, again, the bounded approximate identity $x_n = \chi_n$. 
As $V_p$ is a vector subspace of $c_0$, if $(x_n)$ is weakly Cauchy in $V_p$, then it is in $c_0$ too. Hence for all $f \in l_1 = c_0^*$, the sum $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$ converges for all possible returned sequences $(y_n)$.

To have any chance of winning, we must find $f \in V_p^* \setminus l_1$.

A candidate functional, not defined on $l_1$:

$$\sum_n \frac{1}{n} f_n \notin V_p^*.$$ 

Evaluation against $\chi_n$ gives

$$\left\langle \chi_n, \sum_k \frac{1}{k} f_k \right\rangle = \sum_{k=1}^{n} \frac{1}{k} \to \infty$$
Choosing $x_n = \chi_n$, forces $(y_n)$ to have weights in each coordinate eventually summing to 1. We want to choose $f$ that picks out these large weights.

- Sum of alternating harmonic series converges

$$\sum_n \frac{(-1)^n}{n} = -\log 2,$$

but its absolute values, the harmonic series diverges

$$\sum_n \frac{1}{n} = \infty.$$ 

- We do have

$$\xi := \sum_n \frac{(-1)^n}{n} f_n \in V_p^*.$$
Pełczyński’s Property (u) - Match Point

Want to show that

$$\sum |\langle y_m, f \rangle| \quad (\star)$$

diverges for some choice of $f$.

- If $y_n = e_n$ then we win with $\xi$ as defined.
- If faced with a block basic sequence

$$y_n = \sum_{i=\sigma(n)}^{\sigma(n+1)-1} \alpha_i e_i,$$

(with increasing $\sigma(n)$), then we win by playing:

$$\xi^\sigma := \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n} f_{\sigma(n)} \in V^*_p.$$

- We can ignore any terms of $(y_n)$ and prove that for a subsequence, $(\star)$ diverges.
- Add terms and show sum diverges. ($|a| + |b| \geq |a + b|$)
We aim to exploit this fact by summing consecutive terms \( y_n \) to construct an ’approximate block basic sequence’ \((z_n)\), with small weight on the initial co-ordinates and tail, and approximately one on the non-overlapping ’blocks’.

- Approximate blocks: \((z_n)\)
- Perfect blocks: \((u_n)\)
- These are (in some sense) close:

\[
\|u_n - z_n\| < \epsilon.
\]

Choosing these approximate blocks is a delicate process.
Pełczyński’s Property (u)

Theorem

$V_p$ does not have Pełczyński’s property (u).

Corollary

$V_p$ doesn’t embed in any space that has an unconditional basis.

Conclusion

Hence $V_p$ is not isomorphic to any $S_q$ for $q \geq 1$. 