

Robust Controllers in Disturbance Decoupling of Uncertain Max-Plus Linear Systems: An Application to a High Throughput Screening System for Drug Discovery

Ying Shang, Laurent Hardouin, Mehdi Lhommeau, and Carlos Andrey Maia

Abstract—This paper presents robust controllers solving the disturbance decoupling problem of a class of uncertain max-plus linear systems, whose system matrices vary between intervals with known lower and upper bounds. The disturbance decoupling problem in this paper is defined as finding a control interval such that the output trajectory interval for the disturbed controlled system belongs to the output trajectory interval for the undisturbed controlled system. In other words, the controls are able to absorb the effects of disturbances robustly for such uncertain systems by preserving the output trajectories in the same set with or without disturbances. Both of the open-loop control intervals and the output feedback control intervals solving the DDP are constructed using the residuation theory and proved to be optimal based on the just-in-time control criterion. The main results of this paper are illustrated by a high throughput screening system for drug discovery.

I. INTRODUCTION

Max-plus linear systems([1], [9]) are used to model timed discrete-event systems with system operations as discrete sequences of events in time. Many fundamental problems for max-plus linear systems have been studied by researchers, for example, controllability([15]), observability([7]), and model reference control([13]). However, in order to take in consideration of noises, system delays, and system malfunctions in discrete-event systems, this paper focuses on a class of uncertain max-plus linear systems, whose system matrices vary between intervals. This class of uncertain max-plus linear systems represents the timed event graphs (TEGs),¹ where time delays are varying and belong to intervals with known lower and upper bounds. There are some existing work for such uncertain max-plus linear systems, such as model predictive control([2], [12]) and optimal control([14], [10]). This paper, on the other hand, presents further investigations on the disturbance decoupling problem (DDP)([8], [16], [17]) in geometric control ([4], [18]) for such uncertain max-plus linear systems.

Corresponding author. Y. Shang is with the Department of Electrical and Computer Engineering, Southern Illinois University Edwardsville, Campus Box 1801, Edwardsville, IL 62026, USA yshang@siue.edu.

L. Hardouin is with the Laboratoire d'Ingénierie des Systèmes Automatisés of the University of Angers, 62 avenue Notre Dame du Lac, 49000 Angers, France laurent.hardouin@univ-angers.fr.

M. Lhommeau is with the Laboratoire d'Ingénierie des Systèmes Automatisés of the University of Angers, 62 avenue Notre Dame du Lac, 49000 Angers, France mehdi.lhommeau@univ-angers.fr.

C.A. Maia is with Departamento de Engenharia Elétrica, Universidade Federal de Minas Gerais (UFMG), Av. Antônio Carlos 6627, Pampulha, 31270-010, Belo Horizonte, MG, Brazil maia@cpdee.ufmg.br.

¹TEG is a subclass of timed Petri net models for discrete-event systems in which all places have a single transition upstream and a single one downstream.

Traditionally speaking, the uncertain max-plus linear system in Eq. (7) is called *disturbance decoupled* by an open-loop controller (or by a feedback controller) if and only if the system output signals will not be changed by any disturbances, for all the variations of the matrices. However, this definition is rather restrictive, especially, for uncertain max-plus linear systems, the solution to this problem are usually null sets. Instead, the DDP in this paper is defined as finding an control interval such that the output trajectory interval, generated by the disturbance interval and the control interval, belongs to the output trajectory interval, generated by the control interval only. In other words, that the DDP is solvable means that finding the interval of controls to absorb the effects of disturbances in a robust way and maintain the output trajectory interval same as before despite of disturbances' presence. For a manufacturing system, solving the DDP means that the outputs will be delayed more than the delays caused by the disturbances. From a practical point of view, it would be interesting to ask the question as whether there exists a controller such that the system is not disturbed more than the delays caused by the disturbances. With practical motivations, this paper presents the modified disturbance decoupling problem (MDDP), which is defined as finding a control interval such that the output trajectory interval, generated by the control and disturbance intervals, belongs to the output interval generated with disturbances only. In the past literature, the DDP can be categorized as the model reference control ([6]), whose the reference model is the uncertain max-plus linear system with only the controls. On the other hand, the MDDP can be categorized as the model reference control, whose the reference model is the uncertain max-plus linear system with disturbances only.

This paper is organized as the following. Section II introduces mathematical preliminaries needed in this paper. Section III defines the uncertain max-plus linear systems in this paper as well as the γ -domain analysis, i.e. the equivalent event-domain analysis similar as z -transform for discrete-time systems. Section IV defines the DDP and MDDP problems for the uncertain max-plus linear systems and presents the optimal open-loop control intervals solving the MDDP. Such a solution can solve the DDP with a necessary and sufficient condition. Section V presents the optimal output feedback control intervals solving the MDDP. Similarly, such an optimal solution can solve the DDP with the same necessary and sufficient condition as the open-loop case. Section VI illustrates the main results using a scheduling problem for a high throughput screening system in drug discovery.

II. MATHEMATICAL PRELIMINARIES

A. Idempotent Semirings and Residuation

Definition 1: A *semiring* is a set \mathcal{S} , equipped with two operations \oplus and \otimes , such that (\mathcal{S}, \oplus) is a commutative monoid (the zero element will be denoted ε), (\mathcal{S}, \otimes) is a monoid (the unit element will be denoted e), operation \otimes is right and left distributive over \oplus , and ε is absorbing for the product (i.e. $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a$).

A semiring \mathcal{S} is *idempotent* if $a \oplus a = a$ for all $a \in \mathcal{S}$. In an idempotent semiring \mathcal{S} , operation \oplus induces a partial order relation

$$a \succeq b \iff a = a \oplus b, \quad \forall a, b \in \mathcal{S}. \quad (1)$$

Then, $a \vee b = a \oplus b$. An idempotent semiring \mathcal{S} is *complete* if sums of infinite numbers of terms are always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all the elements of the idempotent semiring is denoted \top (for ‘‘top’’). In this paper, we denote $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, -\infty, +, 0)$ as the integer max-plus semiring, where $\varepsilon = -\infty$ is the neutral(zero) element for \max and $e = 0$ is the neutral(unit) element for $+$. A non empty subset \mathcal{B} of a semiring \mathcal{S} is a sub-semiring of \mathcal{S} if for all $a, b \in \mathcal{B}$ we have $a \oplus b \in \mathcal{B}$ and $a \otimes b \in \mathcal{B}$.

Definition 2: A mapping $f : \mathcal{S} \rightarrow \mathcal{S}$, where \mathcal{S} is a complete idempotent semiring, is residuated if and only if $f(\varepsilon) = \varepsilon$ and f is lower-semicontinuous, that is,

$$f\left(\bigoplus_{i \in I} a_i\right) = \bigoplus_{i \in I} f(a_i),$$

for any (finite or infinite) set I . The mapping f is said to be residuated and f^\sharp is called its residual.

Theorem 1: ([1]) When f is residuated, f^\sharp is the unique order preserving mapping such that

$$f \circ f^\sharp \preceq \text{Id} \quad f^\sharp \circ f \succeq \text{Id}, \quad (2)$$

where Id is the identity mapping from \mathcal{S} to \mathcal{S} .

It is straightforward that $L_a : \mathcal{S} \rightarrow \mathcal{S}, x \mapsto ax$ and $R_a : \mathcal{S} \rightarrow \mathcal{S}, x \mapsto xa$ are lower semi-continuous. Therefore these mappings are both residuated i.e., $L_a(x) \preceq b$ (resp. $R_a(x) \preceq b$) admits a greatest solution, then the following notations are considered :

$$\begin{aligned} L_a^\sharp(b) &= a \backslash b = \bigoplus \{x \mid ax \preceq b\} \text{ and} \\ R_a^\sharp(b) &= b / a = \bigoplus \{x \mid xa \preceq b\}, \quad \forall a, b \in \mathcal{S}, \end{aligned}$$

where L_a^\sharp and $R_a^\sharp(b)$ are the residual mappings.

Theorem 2: ([1]) Over a complete max-plus algebra, the implicit equation $x = ax \oplus b$ admits $x = a^*b$ as the least solution, where $a^* = \bigoplus_{i \in \mathbb{N}} a_i$ ($(\cdot)^*$ is the Kleene Star operator) with $a_0 = e$.

There are basic properties for star and residuation operations in the residuation theory ([1]), for example,

$$a^*(ba^*)^* = (a \oplus b)^* = (a^*b)^*a^*, \quad (3)$$

$$(ab)^*a = a(ba)^*, \quad (4)$$

$$a(a \backslash (ax)) = ax, \quad (5)$$

$$(ab) \backslash x = b \backslash (a \backslash x). \quad (6)$$

B. Semiring of Intervals

A semiring of interval may be constructed by considering an idempotent semiring of pairs. The set of pairs (x', x'') with $x', x'' \in \mathcal{S}$, and (y', y'') with $y', y'' \in \mathcal{S}$ endowed with the following coordinate-wise algebraic operations:

$$\begin{aligned} (x', x'') \oplus (y', y'') &\triangleq (x' \oplus y', x'' \oplus y'') \\ (x', x'') \otimes (y', y'') &\triangleq (x' \otimes y', x'' \otimes y'') \end{aligned}$$

is a semiring denoted by $\mathcal{C}(\mathcal{S})$ with $(\varepsilon, \varepsilon)$ as the zero element and (e, e) as the identify element. The set of pairs (x', x'') , where $x' \preceq x''$ is denoted by $C_0(\mathcal{S})$.

A closed interval in an idempotent semiring \mathcal{S} is a set of the form

$$\mathbf{x} = [\underline{x}, \bar{x}] = \{t \in \mathcal{S} \mid \underline{x} \preceq t \preceq \bar{x}\},$$

where $[\underline{x}, \bar{x}] = \overline{\mathbb{Z}}_{\max}^2$, and \underline{x} or \bar{x} is called to be the lower or the upper bound for the interval \mathbf{x} , respectively.

Definition 3: The set of interval, denoted as $I(\mathcal{S})$, is defined by two coordinate-wise algebraic operations:

$$\begin{aligned} \mathbf{x} \oplus \mathbf{y} &= [\underline{x} \oplus \underline{y}, \bar{x} \oplus \bar{y}], \\ \mathbf{x} \otimes \mathbf{y} &= [\underline{x} \otimes \underline{y}, \bar{x} \otimes \bar{y}], \end{aligned}$$

is a semiring, where the interval $\varepsilon = [\varepsilon, \varepsilon]$ is the zero element in $I(\mathcal{S})$ and $e = [e, e]$ is the unit element in $I(\mathcal{S})$. Moreover, the semiring $I(\mathcal{S})$ is isomorphic with $C_0(\mathcal{S})$.

Definition 4: Let $\{\mathbf{x}_\alpha\}$ be an infinite subset of $I(\mathcal{S})$, then the infinite sum of element of this subset is

$$\bigoplus \mathbf{x}_\alpha = \left[\bigoplus \underline{x}_\alpha, \bigoplus \bar{x}_\alpha \right].$$

The interval $I(\mathcal{S})$ can be endowed with a natural (partial) order: $\mathbf{a} \succeq_{I(\mathcal{S})} \mathbf{b} \Leftrightarrow \mathbf{a} = \mathbf{a} \oplus \mathbf{b} \Leftrightarrow \underline{a} \succeq \underline{b}$ and $\bar{a} \succeq \bar{b}$.

Proposition 1 ([11]): Mapping $L_{\mathbf{a}} : I(\mathcal{S}) \rightarrow I(\mathcal{S}), \mathbf{x} \mapsto \mathbf{a} \otimes \mathbf{x}$ is residuated. Its residual is equal to

$$L_{\mathbf{a}}^\sharp(\mathbf{b}) = \mathbf{a} \backslash \mathbf{b} = [\underline{a} \backslash \underline{b}, \bar{a} \backslash \bar{b}].$$

Proposition 2 ([11]): Mapping $R_{\mathbf{a}} : I(\mathcal{S}) \rightarrow I(\mathcal{S}), \mathbf{x} \mapsto \mathbf{a} \otimes \mathbf{x}$ is residuated. Its residual is equal to

$$R_{\mathbf{a}}^\sharp(\mathbf{b}) = \mathbf{a} / \mathbf{b} = [\underline{a} / \underline{b}, \bar{a} / \bar{b}].$$

III. UNCERTAIN MAX-PLUS LINEAR SYSTEMS

An uncertain max-plus linear system², is defined as

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k) \oplus Sq(k), \\ y(k) &= Cx(k), \end{aligned} \quad (7)$$

where system matrices lie in corresponding matrix intervals with known lower and upper bounds, specifically, $A \in \mathbf{A} = [A_l, A_u] \in I(\overline{\mathbb{Z}}_{\max})^{n \times n}$, $B \in \mathbf{B} = [B_l, B_u] \in I(\overline{\mathbb{Z}}_{\max})^{n \times p}$, $S \in \mathbf{S} = [S_l, S_u] \in I(\overline{\mathbb{Z}}_{\max})^{n \times r}$, and $C \in \mathbf{C} = [C_l, C_u] \in I(\overline{\mathbb{Z}}_{\max})^{q \times n}$. And the states are $x(k) \in X \cong \overline{\mathbb{Z}}_{\max}^n$, the inputs are $u(k) \in U \cong \overline{\mathbb{Z}}_{\max}^p$, the disturbances are $q(k) \in Q \cong \overline{\mathbb{Z}}_{\max}^r$, and the outputs are $y(k) \in Y \cong \overline{\mathbb{Z}}_{\max}^q$ and $k \in \mathbb{Z}$.

²In this paper, the class of uncertain max-plus linear systems are restricted to model TEGs with First In First Out (FIFO) places and transitions, where the ordering of events preserves linearity.

A trajectory of a timed event graph transition x is a firing date sequence $\{x(k)\} \in \overline{\mathbb{Z}}_{\max}$. For each increasing sequence $\{x(k)\}$, it is possible to define the transformation $X(\gamma) = \bigoplus_{k \in \mathbb{Z}} x(k)\gamma^k$ where γ is a backward shift operator in event domain (i.e., $Y(\gamma) = \gamma X(\gamma) \iff \{y(k)\} = \{x(k-1)\}$, (see [1], p. 228). This transformation is analogous to the z -transform used in discrete-time classical control theory and the formal series $X(\gamma)$ is a synthetic representation of the trajectory $x(k)$. The set of the formal power series in γ is denoted by $\overline{\mathbb{Z}}_{\max}[[\gamma]]$ and constitutes an idempotent semiring. Therefore, the state equation in Eq. (7) becomes a polynomial equation or a frequency domain representation,

$$\begin{aligned} X(\gamma) &= \overline{A}X(\gamma) \oplus BU(\gamma) \oplus SQ(\gamma), \text{ where } \overline{A} = \gamma A, \\ Y(\gamma) &= CX(\gamma), \end{aligned} \quad (8)$$

where the state series are $X(\gamma) \in (\overline{\mathbb{Z}}_{\max}[[\gamma]])^n$, the output series are $Y(\gamma) \in (\overline{\mathbb{Z}}_{\max}[[\gamma]])^q$, the input series are $U(\gamma) \in (\overline{\mathbb{Z}}_{\max}[[\gamma]])^p$, and the disturbance series are $Q(\gamma) \in (\overline{\mathbb{Z}}_{\max}[[\gamma]])^r$. Moreover, $\overline{A} \in [\gamma A_l, \gamma A_u] \triangleq [\overline{A}_l, \overline{A}_u] \in I(\overline{\mathbb{Z}}_{\max}[[\gamma]])^{n \times n}$, $B \in [B_l, B_u] \in I(\overline{\mathbb{Z}}_{\max}[[\gamma]])^{n \times p}$, $C \in [C_l, C_u] \in I(\overline{\mathbb{Z}}_{\max}[[\gamma]])^{q \times n}$ and $S \in [S_l, S_u] \in I(\overline{\mathbb{Z}}_{\max}[[\gamma]])^{n \times r}$ represent the link between transitions. Furthermore, according to the state equation (8), the evolution of the system is

$$\begin{aligned} X(\gamma) &= \overline{A}^*BU(\gamma) \oplus \overline{A}^*SQ(\gamma), \\ Y(\gamma) &= C\overline{A}^*BU(\gamma) \oplus C\overline{A}^*SQ(\gamma). \end{aligned} \quad (9)$$

The trajectories $U(\gamma)$ and $Y(\gamma)$ can be related ([1], p. 243) by the equation $Y(\gamma) = H(\gamma)U(\gamma)$, where the least solution $H(\gamma) = C\overline{A}^*B \in \mathbf{H} = \mathbf{C}\overline{\mathbf{A}}^*\mathbf{B} = [C_l\overline{A}_l^*B_l, C_u\overline{A}_u^*B_u] \in I(\overline{\mathbb{Z}}_{\max}[[\gamma]])^{q \times p}$ is called the transfer matrix of the TEG using Theorem 2, where \mathbf{H} represents the interval in which the transfer functions lie in, for all variations of the system matrices.

IV. SOLVING DDP AND MDDP USING OPEN-LOOP CONTROLLERS

This paper defines DDP as finding an interval of control U such that the output interval induced by such a control interval is unchanged by any disturbance interval Q . In another words, there always exists a control $U(\gamma)$ belonging to an interval U such that the output trajectory interval $Y(\gamma)$ with disturbances belongs to the output trajectory interval without disturbances, i.e., the controlled system is able to absorb the disturbances in a robust manner.

Definition 5: The uncertain max-plus linear system in Eq. (7) is called *disturbance decoupled* by an open-loop control interval if and only if the following equality

$$C\overline{A}^*BU \oplus C\overline{A}^*SQ = C\overline{A}^*BU, \quad (10)$$

holds due to equation (9), for some intervals of matrices A , B , C , S , the disturbance interval Q , and the control interval U .

The underlying meaning of Eq. (10) is that all output trajectories of a controlled uncertain max-plus system with disturbances belong to the same set of the output trajectories of the same system without disturbances. In the sense that the output trajectories are robustly invariant with and without disturbances. For a manufacturing system, a disturbance is an event which blocks the occurrence of an event (it could

be a machine breakdown or a delay in a component supply) and the control consists in choosing the date of an input event (e.g. when a job should be started on a machine). Hence, solving the DDP means finding input dates such that outputs will be delayed more than the delays caused by the disturbances. From a practical point of view, it should be more interesting to find a control such that the system is not delayed more than the delays caused by the disturbances. For example, when a system breakdown occurs, we can put the input parts of the manufacturing lines as late as possible to reduce the unnecessary waiting time in the network. Therefore, the modified disturbance decoupling problem (MDDP) in ([8], [16], [17]) is to find appropriate controls such that the output signals will not be delayed more than the outputs caused by the disturbances.

Definition 6: The max-plus linear system described in Eq. (7) is called *modified disturbance decoupled* by an open-loop control interval if and only if the following equality

$$C\overline{A}^*BU \oplus C\overline{A}^*SQ = C\overline{A}^*SQ, \quad (11)$$

due to equation (9), for some intervals of matrices A , B , C , S , the disturbance interval Q , and the control interval U .

Similarly as the DDP, the underlying meaning of Eq. (11) is that all the output trajectories of a controlled uncertain max-plus system with disturbances belong to the same set of the output trajectories of the uncontrolled system with only the disturbances, i.e. the output trajectories are robustly invariant with and without controls. This problem is reasonable because we don't want to impose longer delays than the disturbances have acted on the system. Therefore, the DDP can be categorized as the model reference control ([6]), where the reference model is the uncertain max-plus linear system without disturbance. On the other hand, the MDDP is the model reference control with the reference model as the uncertain max-plus linear system with disturbance only. Moreover, DDP can also be understood as the output trajectory interval induced by the disturbed control system is invariant despite of the disturbances. MDDP can be understood as the output trajectory interval induced by the disturbances is invariant with controls in order to achieve just-in-time criterion in manufacturing setting.

A. Solving MDDP by Open-Loop Controller

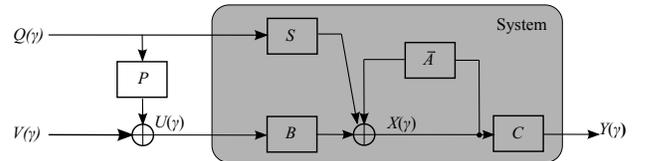


Fig. 1: The controller structure for MDDP.

The block diagram of an open-loop controlled max-plus linear system is illustrated in Fig. 1. We can solve the MDDP problem by finding an optimal interval $\mathbf{P}_{\text{opt}}^{\text{MDDP}} = [P_{l\text{opt}}, P_{u\text{opt}}]$ of the pre-controller $U = \mathbf{P}\mathbf{Q} \oplus \mathbf{V}$, where $U = [U_l, U_u]$, $\mathbf{P} = [P_l, P_u]$, \mathbf{Q} is an interval of the disturbances, and \mathbf{V} is an interval of arbitrary external inputs, such that

Eq. (11) holds. Without loss of generality, assuming $\mathbf{V} = \varepsilon$, Eq. (11) becomes

$$\begin{aligned} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} \oplus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} &= \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}, \\ \iff \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} &\preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}. \end{aligned} \quad (12)$$

Notice that Eq. (12) indicates that $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}}$ and $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$ satisfy the order relation $\preceq_{I(\overline{\mathbf{z}}_{\max}[\gamma])}$ in the semiring of intervals of series.

Proposition 3: The optimal interval satisfying $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$, is given by $\mathbf{P}_{\text{opt}}^{\text{MDDP}} \triangleq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B} \setminus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} = [P_{\text{lopt}}^{\text{MDDP}}, P_{\text{uopt}}^{\text{MDDP}}]$, where

$$\begin{aligned} P_{\text{lopt}}^{\text{MDDP}} &= C_l \overline{A}_l^* B_l \setminus C_l \overline{A}_l^* S_l \wedge C_u \overline{A}_u^* B_u \setminus C_u \overline{A}_u^* S_u, \\ P_{\text{uopt}}^{\text{MDDP}} &= C_u \overline{A}_u^* B_u \setminus C_u \overline{A}_u^* S_u. \end{aligned} \quad (13)$$

And the control interval $\mathbf{U} = \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}$ solves the MDDP. Moreover, any interval $\mathbf{P} \preceq \mathbf{P}_{\text{opt}}^{\text{MDDP}}$ satisfies Eq. (12).

Proof: The construction of the optimal pre-filter interval $\mathbf{P}_{\text{opt}}^{\text{MDDP}}$ is the direct extension of proposition 1, where $\mathbf{a} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}$ and $\mathbf{b} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$. For any choice of disturbance \mathbf{Q} , Eq. (12) is satisfied, i.e. $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$, that is

$$[C_l \overline{A}_l^* B_l P_{\text{lopt}}^{\text{MDDP}}, C_u \overline{A}_u^* B_u P_{\text{uopt}}^{\text{MDDP}}] \preceq [C_l \overline{A}_l^* S_l, C_u \overline{A}_u^* S_u].$$

The right hand side interval is the output trajectory interval induced by all the disturbances (where \mathbf{Q} is omitted), and the left hand side interval is the output trajectory interval induced by the controls. Therefore, the MDDP is solved by the control $\mathbf{U} = \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}$. ■

B. Solving DDP by Open-Loop Controller

If the optimal pre-controller $\mathbf{P}_{\text{opt}}^{\text{MDDP}}$ is a candidate for solving the DDP, then the next reasonable question is, under what condition, the same optimal pre-controller $\mathbf{P}_{\text{opt}}^{\text{MDDP}}$, solving the MDDP, solves the DDP as well, i.e. the inequality holds

$$\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}}.$$

By Proposition 3, $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$ is already satisfied. Combining the two inequalities, we just need to show whether we can have the following equality:

$$\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}.$$

Proposition 4: The pre-controller interval $\mathbf{U} = \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}$, where $\mathbf{P}_{\text{opt}}^{\text{MDDP}} \triangleq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B} \setminus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$, solves the DDP, i.e. the equality $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$ holds, if and only if $\text{Im} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} \subseteq \text{Im} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}$, i.e. there exists an interval \mathbf{L} , such that $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{L} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$.

Proof: “Sufficiency \Leftarrow .” If $\text{Im} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} \subseteq \text{Im} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}$, i.e. there exists an interval \mathbf{L} , such that $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{L} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$, then $\mathbf{L} \preceq \mathbf{P}_{\text{opt}}^{\text{MDDP}} \triangleq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B} \setminus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$. Hence we have

$$\begin{aligned} &\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} \oplus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} \\ &= \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} \oplus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{L} \\ &= \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}}. \end{aligned}$$

Then, DDP is solvable.

“Necessity \Rightarrow .” The pre-controller interval $\mathbf{U} = \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}$, where $\mathbf{P}_{\text{opt}}^{\text{MDDP}} \triangleq \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B} \setminus \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$, solves the DDP, i.e. the equality $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{P}_{\text{opt}}^{\text{MDDP}} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$ holds, it

implies $\text{Im} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S} \subseteq \text{Im} \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}$, i.e. there exists an interval $\mathbf{L} = \mathbf{P}_{\text{opt}}^{\text{MDDP}}$, such that $\overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{B}\mathbf{L} = \overline{\mathbf{C}}\overline{\mathbf{A}}^*\mathbf{S}$. This implies that Eq. (10) holds, therefore, the DDP is solved by $\mathbf{U} = \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}$. ■

V. SOLVING MDDP AND DDP USING OUTPUT FEEDBACK CONTROLLERS

This section proposes an output feedback control structure shown in Fig. 2. In order to solve the DDP and MDDP using the output feedback control interval $\mathbf{U} = \mathbf{F}\mathbf{Y} \oplus \mathbf{V} = \mathbf{F}\mathbf{C}\mathbf{X} \oplus \mathbf{V}$, where \mathbf{V} in this structure is proposed to be $\mathbf{P}\mathbf{Q} \oplus \mathbf{W}$. Without loss of generality, we assume $\mathbf{W} = \varepsilon$ in this paper. In order to solve the DDP and MDDP in terms of the output feedback controls, first, we design a robust feedback control interval \mathbf{F} such that it preserves the behaviors of the system with the open-loop control interval \mathbf{V} only; second, we choose \mathbf{V} as the optimal open-loop controller solving DDP and MDDP, respectively. We will prove such an output feedback control interval solves MDDP. With the same necessary and sufficient condition in Proposition 4, such an output feedback control solves DDP as well.

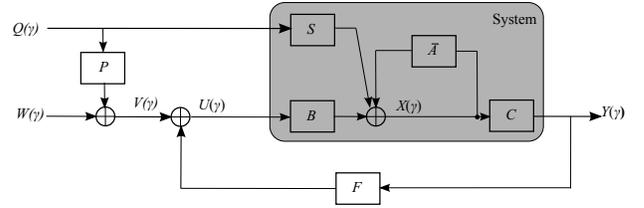


Fig. 2: The output feedback structure.

A. Robust Feedback Control Interval Preserving Open-loop Behaviors

First, we choose $U(\gamma) = FCX(\gamma) \oplus V(\gamma)$ in the control interval $\mathbf{U} = \mathbf{F}\mathbf{C}\mathbf{X} \oplus \mathbf{V}$, such that the output signals are the same as the output signals controlled by the open-loop controller $V(\gamma)$, for any variations of system matrices $\overline{\mathbf{A}} \in [\overline{A}_l, \overline{A}_u]$, $\mathbf{B} \in [B_l, B_u]$, $\mathbf{C} \in [C_l, C_u]$, $\mathbf{S} \in [S_l, S_u]$ and disturbance $Q(\gamma)$. In summary, that is, the following equality holds

$$C(\overline{\mathbf{A}} \oplus \mathbf{B}\mathbf{F}\mathbf{C})^* \tilde{\mathbf{B}} \begin{pmatrix} V(\gamma) \\ Q(\gamma) \end{pmatrix} = \overline{\mathbf{C}}\overline{\mathbf{A}}^* \tilde{\mathbf{B}} \begin{pmatrix} V(\gamma) \\ Q(\gamma) \end{pmatrix}. \quad (14)$$

where $\tilde{\mathbf{B}} = [B \mid S]$. Obviously, $\mathbf{F} = \varepsilon$ is solution, then the greatest solution of

$$C(\overline{\mathbf{A}} \oplus \mathbf{B}\mathbf{F}\mathbf{C})^* \tilde{\mathbf{B}} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^* \tilde{\mathbf{B}}, \quad (15)$$

leads to equality. Noting that $\mathbf{B} = \tilde{\mathbf{B}} \begin{pmatrix} e \\ \varepsilon \end{pmatrix}$, we have

$$C \left(\overline{\mathbf{A}} \oplus \tilde{\mathbf{B}} \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \mathbf{F}\mathbf{C} \right)^* \tilde{\mathbf{B}} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^* \tilde{\mathbf{B}}, \quad (16)$$

Due to Eq. (3) and Eq. (4), we have that

$$\begin{aligned} &C \left(\overline{\mathbf{A}} \oplus \tilde{\mathbf{B}} \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \mathbf{F}\mathbf{C} \right)^* \tilde{\mathbf{B}} \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^* \tilde{\mathbf{B}} \iff \\ &\overline{\mathbf{C}}\overline{\mathbf{A}}^* \tilde{\mathbf{B}} \left(\begin{pmatrix} e \\ \varepsilon \end{pmatrix} \mathbf{F}\mathbf{C} \overline{\mathbf{A}}^* \tilde{\mathbf{B}} \right)^* \preceq \overline{\mathbf{C}}\overline{\mathbf{A}}^* \tilde{\mathbf{B}}. \end{aligned} \quad (17)$$

Define $\tilde{H} = C\bar{A}^*\tilde{B}$, hence, the objective of the robust feedback synthesis is to find a controller F which achieves the following inequality,

$$\left(\tilde{H} \begin{pmatrix} e \\ \varepsilon \end{pmatrix} F\right)^* \tilde{H} \preceq \tilde{H}, \quad (18)$$

for all $\tilde{H} \in \tilde{\mathbf{H}} = [C_l\bar{A}_l^*\tilde{B}_l, C_u\bar{A}_u^*\tilde{B}_u]$, $\tilde{B}_l = [B_l | S_l]$ and $\tilde{B}_u = [B_u | S_u]$. Moreover, the problem is equivalent to finding the interval of output feedback controls $\tilde{\mathbf{F}} = [F_l, F_u]$ such that the following equality of intervals is satisfied:

$$C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\tilde{\mathbf{B}} = C\bar{\mathbf{A}}^*\tilde{\mathbf{B}}. \quad (19)$$

The right hand side is clearly less than the left hand side of the equality, so essentially, we only need to show

$$C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\tilde{\mathbf{B}} \preceq C\bar{\mathbf{A}}^*\tilde{\mathbf{B}}. \quad (20)$$

Next, we find an efficient way to compute the greatest interval denoted as $\hat{\tilde{\mathbf{F}}}$, which ensures the output trajectories are lower than $\tilde{\mathbf{H}} \in I(\mathbb{Z}_{\max}[\gamma])^{q \times (p+r)}$ for all $\tilde{H} \in \tilde{\mathbf{H}}$, i.e. computing the upper bound of the following set

$$\left\{ \tilde{\mathbf{F}} \in I(\mathbb{Z}_{\max}[\gamma])^{(p+r) \times q} \mid (\tilde{\mathbf{H}}\tilde{\mathbf{F}})^*\tilde{\mathbf{H}} \preceq_{I(\mathbb{Z}_{\max}[\gamma])} \tilde{\mathbf{H}} \right\}, \quad (21)$$

$$\text{where } \tilde{\mathbf{F}} = \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \mathbf{F} = \left[\begin{pmatrix} e \\ \varepsilon \end{pmatrix} F_l, \begin{pmatrix} e \\ \varepsilon \end{pmatrix} F_u \right].$$

Proposition 5: Define a mapping $M_{\tilde{\mathbf{H}}} : I(\mathbb{Z}_{\max}[\gamma])^{(p+r) \times q} \rightarrow I(\mathbb{Z}_{\max}[\gamma])^{q \times (p+r)} : \tilde{\mathbf{F}} \mapsto (\tilde{\mathbf{H}}\tilde{\mathbf{F}})^*\tilde{\mathbf{H}}$. The greatest interval $\hat{\tilde{\mathbf{F}}}$ such that $M_{\tilde{\mathbf{H}}} \preceq \tilde{\mathbf{H}}$, is given by

$$\begin{aligned} \hat{\tilde{\mathbf{F}}} &= \overline{\bigoplus \left\{ \tilde{\mathbf{F}} \in I(\mathbb{Z}_{\max}[\gamma])^{(p+r) \times q} \mid (\tilde{\mathbf{H}}\tilde{\mathbf{F}})^*\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}} \right\}} \\ &= \tilde{\mathbf{H}} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}}. \end{aligned} \quad (22)$$

Proof: In order to find the greatest $\tilde{\mathbf{F}}$, denoted by $\hat{\tilde{\mathbf{F}}}$, satisfying $(\tilde{\mathbf{H}}\hat{\tilde{\mathbf{F}}})^*\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}}$, it is equivalent as finding the greatest $\tilde{\mathbf{F}}$ satisfying $(e \oplus \tilde{\mathbf{H}}\tilde{\mathbf{F}} \oplus \tilde{\mathbf{H}}\tilde{\mathbf{F}}\tilde{\mathbf{H}} \oplus \dots)\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}}$. Equivalently, we need to find the greatest $\tilde{\mathbf{F}}$ satisfying the set of the following inequalities:

$$\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}} \quad (23)$$

$$\tilde{\mathbf{H}}\tilde{\mathbf{F}}\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}} \quad (24)$$

$$\tilde{\mathbf{H}}\tilde{\mathbf{F}}\tilde{\mathbf{H}}\tilde{\mathbf{F}}\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}} \quad (25)$$

⋮

inequality (23) is clearly satisfied. The greatest solution to inequality to Inequality (24) is $\hat{\tilde{\mathbf{F}}} = \tilde{\mathbf{H}} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}}$ thanks to Proposition 1 and Proposition 2. In addition,

$$\tilde{\mathbf{H}} \hat{\tilde{\mathbf{F}}} \tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}} \hat{\tilde{\mathbf{F}}} \tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}},$$

Inequality (25) is also satisfied, so on and so forth. Therefore, $\hat{\tilde{\mathbf{F}}}$ is also the greatest solution for the rest of the inequalities. Hence $\hat{\tilde{\mathbf{F}}}$ is the greatest solution $\tilde{\mathbf{F}}$ satisfying $(\tilde{\mathbf{H}}\hat{\tilde{\mathbf{F}}})^*\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}}$. ■

Remark 1: Using Proposition 1 and Proposition 2, we can easily compute the interval

$$\begin{aligned} \hat{\tilde{\mathbf{F}}} &= \tilde{\mathbf{H}} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}} \\ &= \left[\tilde{H}_l \backslash \tilde{H}_l \phi \tilde{H}_l \wedge \tilde{H}_u \backslash \tilde{H}_u \phi \tilde{H}_u, \tilde{H}_u \backslash \tilde{H}_u \phi \tilde{H}_u \right]. \end{aligned} \quad (26)$$

Corollary 1: The greatest interval \mathbf{F} , where $\tilde{\mathbf{F}} = \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \mathbf{F}$,

such that $M_{\tilde{\mathbf{H}}} \preceq \tilde{\mathbf{H}}$, is given by

$$\begin{aligned} \hat{\mathbf{F}} &= \overline{\bigoplus \left\{ \mathbf{F} \in I(\mathbb{Z}_{\max}[\gamma])^{(p) \times q} \mid \left(\tilde{\mathbf{H}} \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \mathbf{F}\right)^* \tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}} \right\}} \\ &= \mathbf{H} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}}. \end{aligned} \quad (27)$$

Proof: Based on Proposition 5, $\hat{\tilde{\mathbf{F}}} = \tilde{\mathbf{H}} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}}$ is the greatest solution satisfying $(\tilde{\mathbf{H}}\hat{\tilde{\mathbf{F}}})^*\tilde{\mathbf{H}} \preceq \tilde{\mathbf{H}}$, where $\hat{\tilde{\mathbf{F}}} = \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \hat{\mathbf{F}}$. Hence, $\hat{\mathbf{F}}$ is

$$\begin{aligned} \hat{\mathbf{F}} &= \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \backslash \hat{\tilde{\mathbf{F}}} = \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \backslash \left((C\bar{\mathbf{A}}^*\tilde{\mathbf{B}}) \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}} \right) \\ &= \left(C\bar{\mathbf{A}}^*\tilde{\mathbf{B}} \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \right) \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}} \text{ due to Eq.(6)} \\ &= \left(C\bar{\mathbf{A}}^*\mathbf{B} \right) \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}} \\ &= \mathbf{H} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}}. \end{aligned}$$

■

Remark 2: Because $\tilde{\mathbf{H}}$ is in the image of $M_{\tilde{\mathbf{H}}}$, therefore, $\hat{\tilde{\mathbf{F}}}$ is the greatest solution such that $\left(\tilde{\mathbf{H}} \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \hat{\tilde{\mathbf{F}}}\right)^*\tilde{\mathbf{H}} = \tilde{\mathbf{H}}$, which is the same as Eq. (19).

Remark 3: For calculation, we have

$$\begin{aligned} \hat{\tilde{\mathbf{F}}} &= \mathbf{H} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}} = \left[\hat{F}_{l\text{opt}}, \hat{F}_{u\text{opt}} \right] \\ &= \left[H_l \backslash \tilde{H}_l \phi \tilde{H}_l \wedge H_u \backslash \tilde{H}_u \phi \tilde{H}_u, H_u \backslash \tilde{H}_u \phi \tilde{H}_u \right]. \end{aligned} \quad (28)$$

Hence, $\hat{\mathbf{F}}$ is the greatest interval \mathbf{F} satisfying the equality

$$C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\tilde{\mathbf{B}} = C\bar{\mathbf{A}}^*\tilde{\mathbf{B}}.$$

B. Solving MDDP and DDP using Output Feedback Controls

Solving MDDP implies that we need to find intervals of the feedback control \mathbf{F} and the open loop control \mathbf{V} satisfying have the following equality

$$C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\mathbf{B}\mathbf{V} \oplus C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\mathbf{S}\mathbf{Q} = C\bar{\mathbf{A}}^*\mathbf{S}\mathbf{Q}. \quad (29)$$

Similarly, solving DDP implies that we need to find intervals of the feedback control \mathbf{F} and the open loop control \mathbf{V} satisfying have the following equality

$$\begin{aligned} C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\mathbf{B}\mathbf{V} \oplus C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\mathbf{S}\mathbf{Q} = \\ C(\bar{\mathbf{A}} \oplus \mathbf{B} \mathbf{F} C)^*\mathbf{B}\mathbf{V}. \end{aligned} \quad (30)$$

If we pick $\hat{\mathbf{F}}$ as the feedback control interval in Eq. (29), we only need to find an open-loop control \mathbf{V} solving the DDP and MDDP, respectively, then the pair would solve the DDP and MDDP as well.

Proposition 6: If we choose the output feedback control interval $\mathbf{U} = \hat{\mathbf{F}}\mathbf{C}\mathbf{X} \oplus \mathbf{V}$, where $\hat{\mathbf{F}} = \mathbf{H} \backslash \tilde{\mathbf{H}} \phi \tilde{\mathbf{H}}$ and $\mathbf{V} = \mathbf{P}_{\text{opt}}^{\text{MDDP}}\mathbf{Q}$ (w.l.o.g. assuming $\mathbf{W} = \varepsilon$), then this output feedback control interval solves the MDDP. This output feedback control solves the DDP as well if and only if $\text{Im } C\bar{\mathbf{A}}^*\mathbf{S} \subseteq \text{Im } C\bar{\mathbf{A}}^*\mathbf{B}$.

Proof: Because $\hat{\mathbf{F}} = \mathbf{H} \overline{\mathbf{H}} \tilde{\mathbf{H}} \tilde{\mathbf{H}}$ solves Eq. (19), i.e.

$$\mathbf{C}(\overline{\mathbf{A}} \oplus \mathbf{B} \hat{\mathbf{F}} \mathbf{C})^* \mathbf{B} \mathbf{V} \oplus \mathbf{C}(\overline{\mathbf{A}} \oplus \mathbf{B} \hat{\mathbf{F}} \mathbf{C})^* \mathbf{S} \mathbf{Q} = \overline{\mathbf{C}} \mathbf{A}^* \mathbf{B} \mathbf{V} \oplus \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S} \mathbf{Q}. \quad (31)$$

If we pick $\mathbf{V} = \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}$, then the preceding equation holds, because

$$\overline{\mathbf{C}} \mathbf{A}^* \mathbf{B} \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q} \oplus \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S} \mathbf{Q} \preceq \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S} \mathbf{Q}. \quad (32)$$

Therefore, $\hat{\mathbf{F}}$ and \mathbf{V} solve the MDDP.

If and only if the condition $\text{Im } \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S} \subseteq \text{Im } \overline{\mathbf{C}} \mathbf{A}^* \mathbf{B}$ in Proposition 4, is satisfied, we have $\overline{\mathbf{C}} \mathbf{A}^* \mathbf{B} \mathbf{P}_{\text{opt}}^{\text{MDDP}} = \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S}$. Hence,

$$\overline{\mathbf{C}} \mathbf{A}^* \mathbf{B} \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q} \oplus \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S} \mathbf{Q} = \overline{\mathbf{C}} \mathbf{A}^* \mathbf{B} \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q} \preceq \mathbf{C}(\overline{\mathbf{A}} \oplus \mathbf{B} \hat{\mathbf{F}} \mathbf{C})^* \mathbf{B} \mathbf{P}_{\text{opt}}^{\text{MDDP}} \mathbf{Q}. \quad (33)$$

Therefore, the output feedback control interval $\mathbf{U} = \hat{\mathbf{F}} \mathbf{C} \mathbf{X} \oplus \mathbf{V}$ solves the DDP as well. ■

VI. APPLICATION TO A HIGH THROUGHPUT SCREENING SYSTEM IN DRUG DISCOVERY

High throughput screening (HTS) is a standard technology in drug discovery. In HTS systems, optimal scheduling is required to finish the screening in the shortest time, as well as to preserve the consistent time spending on each activity. The HTS system in this section, adapted from [3], consists of three activities: activity 1, executed on the resource *Pipettor*, is filling the chemical compound A into the wells of a microplate, which lasts from 3 to 5 time units. Next, the microplate is transferred to a storage or incubator for at least 1 time unit. Before the waiting time is done for the compound A, activity 3, executed on the resource *Robot*, is that the robot picks up another compound B, which takes 1 to 3 time units. Once the 1 unit waiting time for the compound A is over, the robot moves the compound B to the microplate carrying the substance A. Activity 2 is that the microplate containing the compound AB is transferred to the pipettor again for 4 to 6 time units.

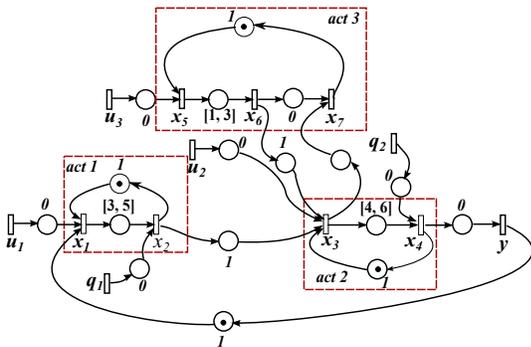


Fig. 3: The TEG model of the HTS system.

If we are interested in the start and release event time of each activity, we can model the HTS system as a TEG model,

shown in Fig. 3, in which x_1 and x_2 denote the start and release time of the activity 1 on the pipettor, denoted as *act 1*, x_3 and x_4 denote the start and release time of the activity 2 on the same pipettor, denoted as *act 2*, and x_5 and x_7 denote the start and release time of the activity 3 on the robot, denoted as *act 3*. x_6 is the event time that the compound B is ready to be transferred to the microplate containing the compound A. The inputs u_1 , u_2 , and u_3 , are the starting time of the activity 1, 2, and 3, respectively, which the users can decide when to load the chemical compounds. The disturbance q_1 delays the release time of the pipettor after activity 1, and the disturbance q_2 delays the release time of the pipettor after activity 2, e.g. uncontrollable system delays or uncontrollable transition time delays. The output y is the release time of the pipettor. The circles represent places and the bars represent the transitions x_i . Each black token in the places represents that the corresponding resource is available, i.e. the activity is ready to start.

For the TEG model of a HTS system shown in Fig. 3, the system over the max-plus algebra $\mathbb{Z}_{\max}[\gamma]$ is described as the following:

$$\begin{aligned} X(\gamma) &= \overline{\mathbf{A}} X(\gamma) \oplus \mathbf{B} U(\gamma) \oplus \mathbf{S} Q(\gamma) \\ Y(\gamma) &= \mathbf{C} X(\gamma), \end{aligned}$$

where

$$\overline{\mathbf{A}} = \begin{bmatrix} \varepsilon & 1\gamma & \varepsilon & 1\gamma & \varepsilon & \varepsilon & \varepsilon \\ [3, 5] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & 1\gamma & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & [4, 6] & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1\gamma \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & [1, 3] & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} [e, e] & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & [e, e] & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & [e, e] \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \varepsilon & \varepsilon \\ [e, e] & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & [e, e] \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix},$$

$$\mathbf{C} = [\varepsilon \ \varepsilon \ \varepsilon \ [e, e] \ \varepsilon \ \varepsilon \ \varepsilon].$$

The example has been computed by using toolbox MinMaxGD, a C++ library allowing to handle periodic series as introduced in ([5]), and it can be noted that this library is also interfaced with Scilab and MATLAB. We obtain the transfer function intervals between the output $Y(\gamma)$ and disturbance $U(\gamma)$ and the input $Q(\gamma)$ are given, respectively, as

$$\begin{aligned} \overline{\mathbf{C}} \mathbf{A}^* \mathbf{B} &= [C_l \overline{A}_l^* B_l, C_u \overline{A}_u^* B_u], \text{ where} \\ C_l \overline{A}_l^* B_l &= [8(9\gamma)^*, 4(9\gamma)^*, 6(9\gamma)^*] \\ C_u \overline{A}_u^* B_u &= [12(13\gamma)^*, 6(13\gamma)^*, 10(13\gamma)^*] \\ \overline{\mathbf{C}} \mathbf{A}^* \mathbf{S} &= [C_l \overline{A}_l^* S_l, C_u \overline{A}_u^* S_u], \text{ where} \\ C_l \overline{A}_l^* S_l &= [5(9\gamma)^*, (9\gamma)^*], \\ C_u \overline{A}_u^* S_u &= [7(13\gamma)^*, (13\gamma)^*], \end{aligned}$$

in which each component of these matrices consists of periodic series. Essentially, the γ -periodic series represent the

output sequence when an infinity of tokens is put in the system at time 0 (impulse input). For instance, $C_l \bar{A}_l^* B_l(1, 1) = 8(9\gamma)^*$ represents the impulse response of the pipettor as $y(0) = 8, y(1) = 17, y(2) = 26$, etc.

The remaining session presents the pre-filter intervals and feedback controls solving MDDP and DDP, respectively, for any disturbances. Using Proposition 3, the pre-controller interval $\mathbf{P}_{opt}^{MDDP} \triangleq \overline{\mathbf{CA}}^* \mathbf{B} \setminus \overline{\mathbf{CA}}^* \mathbf{S} = [P_{lopt}^{MDDP}, P_{uopt}^{MDDP}]$, where

$$P_{lopt}^{MDDP} = \begin{bmatrix} -5 \oplus 6\gamma(9\gamma)^* & -12 \oplus 1\gamma(9\gamma)^* \\ 1(9\gamma)^* & -6 \oplus 5\gamma(9\gamma)^* \\ -3 \oplus 8\gamma(9\gamma)^* & -10 \oplus 3\gamma(9\gamma)^* \end{bmatrix},$$

$$P_{uopt}^{MDDP} = \begin{bmatrix} -5(13\gamma)^* & -12(13\gamma)^* \\ 1(13\gamma)^* & -6(13\gamma)^* \\ -3(13\gamma)^* & -10(13\gamma)^* \end{bmatrix}.$$

Moreover, the condition $\text{Im } \overline{\mathbf{CA}}^* \mathbf{S} \subseteq \text{Im } \overline{\mathbf{CA}}^* \mathbf{B}$ in Proposition 4 is satisfied, hence, the pre-controller interval $\mathbf{U} = \mathbf{P}_{opt}^{MDDP} \mathbf{Q}$ solves the DDP as well.

This pre-filter interval \mathbf{P}_{opt}^{MDDP} is not causal because there are negative coefficients in the matrix. The canonical injection from the causal elements of $\mathbb{Z}_{\max}[\gamma]$ (denoted $\mathbb{Z}_{\max}[\gamma]^+$) in $\mathbb{Z}_{\max}[\gamma]$ is also residuated (see [6] for details). Its residual is given by $\text{Pr}(\bigoplus_{k \in \mathbb{Z}} s(k)\gamma^k) = \bigoplus_{k \in \mathbb{Z}} s_+(k)\gamma^k$ where

$$s_+(k) = \begin{cases} s(k) & \text{if } (k, s(k)) \geq (0, 0), \\ \varepsilon & \text{otherwise.} \end{cases}$$

The causal pre-filter interval is $\mathbf{P}_{opt+}^{MDDP} = [P_{lopt+}^{MDDP}, P_{uopt+}^{MDDP}]$, where

$$P_{lopt+}^{MDDP} = \begin{bmatrix} 6\gamma(9\gamma)^* & 1\gamma(9\gamma)^* \\ 1(9\gamma)^* & 5\gamma(9\gamma)^* \\ 8\gamma(9\gamma)^* & 3\gamma(9\gamma)^* \end{bmatrix},$$

$$P_{uopt+}^{MDDP} = \begin{bmatrix} 8\gamma(13\gamma)^* & 1\gamma(13\gamma)^* \\ 1(13\gamma)^* & 7\gamma(13\gamma)^* \\ 10\gamma(13\gamma)^* & 3\gamma(13\gamma)^* \end{bmatrix}.$$

Furthermore, by implementing Corollary 1, we can find the greatest interval $\hat{\mathbf{F}} = [\hat{F}_{lopt}, \hat{F}_{uopt}]$, such that the feedback controls would preserve the behaviors of the open-loop systems, where

$$\hat{F}_{lopt} = \begin{bmatrix} -12 \oplus 1\gamma(9\gamma)^* \\ -6 \oplus 5\gamma(9\gamma)^* \\ -10 \oplus 3\gamma(9\gamma)^* \end{bmatrix}, \hat{F}_{uopt} = \begin{bmatrix} -12(13\gamma)^* \\ -6(13\gamma)^* \\ -10(13\gamma)^* \end{bmatrix}.$$

The causal pre-filter interval $\hat{\mathbf{F}}_+$ is given by

$$\hat{F}_{lopt+} = \begin{bmatrix} 1\gamma(9\gamma)^* \\ 5\gamma(9\gamma)^* \\ 3\gamma(9\gamma)^* \end{bmatrix}, \hat{F}_{uopt+} = \begin{bmatrix} 1\gamma(13\gamma)^* \\ 7\gamma(13\gamma)^* \\ 3\gamma(13\gamma)^* \end{bmatrix}.$$

The lower bound P_{lopt+}^{MDDP} of the pre-filter \mathbf{P}_{opt}^{MDDP} and the lower bound \hat{F}_{lopt+} the feedback control $\hat{\mathbf{F}}$ are realized in the TEG model of the HTS system, shown in Fig. 4. For example, in Fig. 4, the first element of the causal pre-filter $P_{lopt+}^{MDDP}(1, 1) = 6\gamma(9\gamma)^*$ is represented by a cyclic transition due to $(9\gamma)^*$, where γ indicates one token inside of the place and 9 units are the holding time of the token, and 6γ is represented by an additional transition with one token and

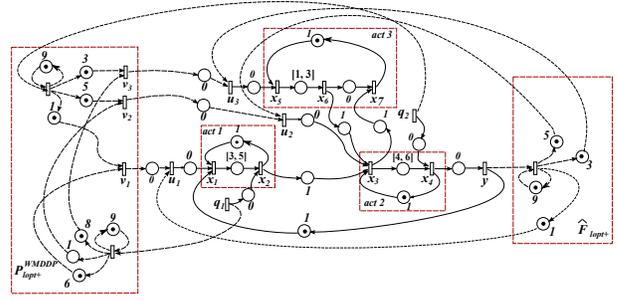


Fig. 4: The TEG model of the HTS system with controllers.

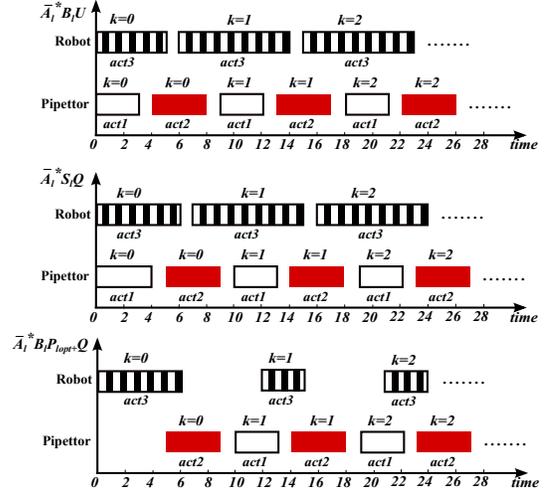


Fig. 5: The Gantt chart of the HTS system.

6 units holding time. Same analogy works for the feedback $\hat{F}_{lopt+} : X \rightarrow U$.

The Gantt chart of the scheduling is illustrated in Fig. 5, which describes the start and release time for each activity as the system evolves. For instance, without the disturbance, the system is running on its own, then it will repeat the same scheduling every 9 time units according to the TEG model. For instance, the disturbance is $Q(\gamma)(1, 1) = 4(9\gamma)^*$, which means the incubator will finish the activity 1 time unit late every event cycle. In this case shown in the second figure in Fig. 5, the process time of the activity 1 at $k = 0$ is 4 time unit with 1 unit delay than the original 3 time units duration, and all other events shifted 1 time unit comparing to the undisturbed system. If we implement the lower bound P_{lopt+} of the pre-filter interval \mathbf{P}_{opt}^{MDDP} on the system, then the scheduling is shown in the bottom figure in Fig. 5. The advantage of the pre-filter is that, according to the just-in-time control principle, the activity 3, executed on the robot, does not wait 8 time units to finish the task, and it only needs 3 time units to transport the compound B to the pipettor, without affecting the output dates. Moreover, such a pre-filter P_{lopt+} proposed in this paper is designed no matter what disturbances are, which can be implemented for online scheduling processes.

VII. CONCLUSION

This paper presents robust controllers solving disturbance decoupling problems of a class of uncertain max-plus linear systems, whose system matrices belong to intervals with known bounds. Both of the open-loop control intervals and the output feedback control intervals solving the DDP are constructed using the residuation theory and proved to be optimal for the just-in-time control criterion. Future research can be extended to other geometric control problems, such as block decoupling problem and non-interacting control problem in uncertain max-plus linear systems.

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