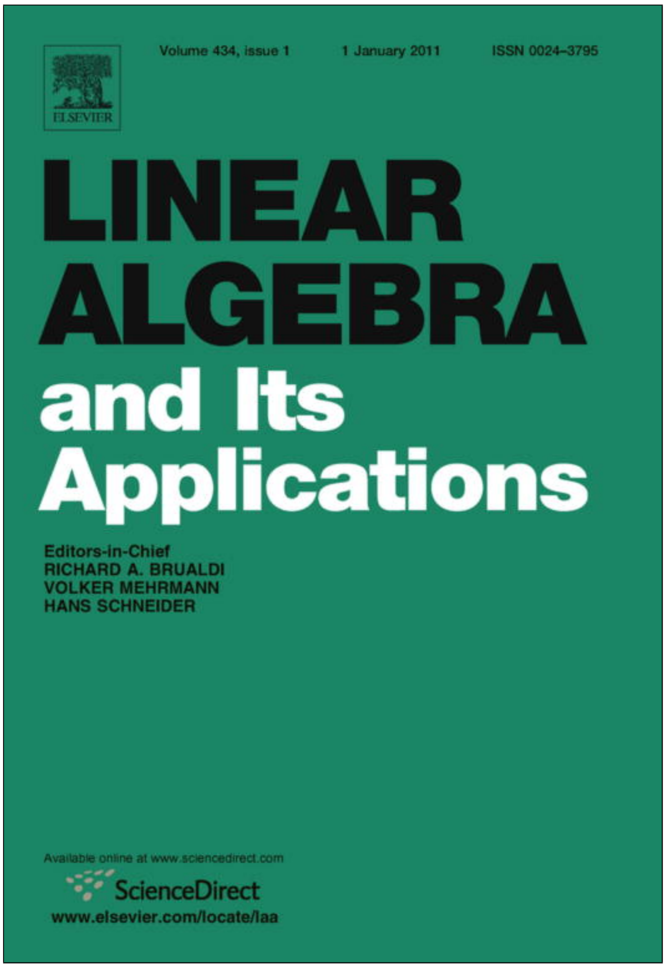
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Fixed zeros in the model matching problem for systems over semirings [☆]

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ABSTRACT

This paper studies "fixed zeros" of solutions to the model matching problem for systems over semirings. Such systems have been used to model queueing systems, communication networks, and manufacturing systems. The main contribution of this paper is the discovery of two fixed zero structures, which possess a connection with the extended zero semimodules of solutions to the model matching problem. Intuitively, the fixed zeros provides an essential component that is obtained from the solutions to the model matching problem. For discrete-event dynamic systems modeled in max-plus algebra, a common Petri net component constructed from the solutions to the model matching problem can be discovered from the fixed zero structure.

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1. Introduction

A semiring can be understood as a set of objects not all of which have inverses with respect to the corresponding operators. There are many examples of this special algebraic structure, such as the max-plus algebra [3], the min-plus algebra [4], and the Boolean semiring [9]. Systems over semirings are systems evolving with variables taking values in semimodules over a semiring. Intuitively, such

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systems are not equipped with "additive inverses" and are used in many applications. For instance, systems over the max-plus algebra model queueing systems [5], systems over the min-plus algebra model communication networks [4], and systems over the Boolean semiring model hysteretic discrete structural systems [19].

This paper, as a companion paper of [20], continues to study the model matching problem (MMP) for systems over semirings. The goal of the MMP is to design a compensator for a given plant, or to design a plant for a given compensator, so that the response of the closed-loop system matches a given reference model. The MMP is indeed a well-studied problem for systems over a field. In 1951, just after the mid-century, Aaron [1] presented the MMP for single input and single output systems. Two decades later, work on the MMP began to intensify. Wolovich [25] precisely defined the MMP via linear feedback as the problem of finding a feedback pair such that the transfer function matrix of the closed-loop system is identical to a prescribed transfer function matrix. The solution existence condition was obtained using coordinate transformations and a factorization of the transfer function matrix. At about the same time, the MMP with static and dynamic state feedback controllers was studied using various methods, for instance, by solving linear algebraic equations [23], by geometric methods [14], and by solving rational matrix equations [22]. Continuing on into the next decade, MMP was addressed by solving matrix Diophantine equations [21], by utilizing the theory of inverses and state-space algorithms, and so forth. Later, in the 1990s, the MMP was studied for more general classes of systems, such as linear systems with delays [17], periodic discrete-time systems [6], and nonlinear systems [11,12]. With this type of study proceeding into the new century, recently, the MMP for systems over a field has been applied to various application areas, for example, adaptive flight control systems [15], maneuver vehicles [16], and propulsion control aircraft [10].

The MMP for linear systems over a semiring, however, has not been investigated as well as for traditional linear systems. The MMP for systems over a special semiring, the max-plus algebra, has been studied in [8,13], in which residuation theory is used to characterize solutions. This paper studies the MMP from a frequency-domain point of view. The challenge for the frequency-domain method in systems over a semiring is that defining poles and zeros in the frequency domain requires operation inverses. The focus of this paper switches to fixed zeros instead of fixed poles of the MMP in [20]. In [20], two fixed pole structures are introduced for the MMP, and relationships are established between these fixed poles and the pole semimodules of the solutions to the MMP. These fixed pole structures characterize common components in the solutions to MMP. Fixed zeros, on the other hand, are more complicated than fixed poles, because there is no direct relation between the fixed zero semimodule and the zero semimodule of a solution. Fixed zero semimodules, however, are related to two different extended zero semimodules, the Γ-zero semimodule and the Ω-zero semimodule. The relations, embedded in short exact sequences, imply that fixed zero structures in the MMP characterize the essential information in the solutions to MMP. This result is a generalization of the study of the MMP for traditional linear systems over fields in [18] to systems over semirings.

The paper is organized as follows. Section 2 introduces some mathematical preliminaries. Section 3 defines the model matching problem for systems over semirings. Section 4 defines zero semimodules and extended zero semimodules. Relationships are established between these zero semimodules. Section 5 defines two fixed zero structures and studies the relationships between the fixed zero semimodules and the zero semimodules of the solutions to the MMP. These fixed zero semimodules provide essential information about the realizations of solutions. Section 6 presents a discrete event system application. Fixed zero structures provide essential information in the solutions to the MMP. Section 7 is the conclusion.

2. Mathematical preliminaries

A monoid R is a semigroup $(R, \boxplus)$ with an identity element e_R with respect to the binary operation $\boxplus$. The term *semiring* means a set, $R = (R, \boxplus, e_R, \boxtimes, 1_R)$ with two binary associative operations, $\boxplus$ and $\boxtimes$, such that $(R, \boxplus, e_R)$ is a commutative monoid and $(R, \boxtimes, 1_R)$ is a monoid, which are connected by a two-sided distributive law of $\boxtimes$ over $\boxplus$. Moreover, $e_R \boxtimes r = r \boxtimes e_R = e_R$, for all r in R. Let $(R, \boxplus, e_R, \boxtimes, 1_R)$ be a semiring, and $(M, \square_M, e_M)$ be a commutative monoid, where the subscript M

denotes the semimodule associated with the operator $\Box_M$. M is called a *left R-semimodule* if there exists a map $\mu : R \times M \to M$, denoted by $\mu(r, m) = rm$, for all $r \in R$ and $m \in M$, such that the following conditions are satisfied:

(1) $r(m_1 \ \Box_M \ m_2) = rm_1 \ \Box_M \ rm_2$;
(2) $(r_1 \boxplus r_2)m = r_1 m \ \Box_M \ r_2 m$;
(3) $r_1(r_2 m) = (r_1 \boxtimes r_2)m$;
(4) $1_R \ m = m$;
(5) $r \ e_M = e_M = e_R \ m$,

for any $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. In this paper, e denotes the unit semimodule. A *sub-semimodule* K of M is a submonoid of M with $rk \in K$, for all $r \in R$ with $k \in K$. A sub-semimodule K of M is called *subtractive* if $k \in K$ and $k \ \Box_M \ m \in K$ imply $m \in K$, for $m \in M$.

An R-morphism between two semimodules $(M, \Box_M, e_M)$ and $(N, \Box_N, e_N)$ is a map $f : M \to N$ satisfying

(1) $f(m_1 \ \Box_M \ m_2) = f(m_1) \ \Box_N \ f(m_2)$;
(2) $f(rm) = rf(m)$,

for all m, m_1, $m_2 \in M$ and $r \in R$. The kernel of an R-semimodule morphism $f : M \to N$ is defined as $\mathrm{Ker}\, f = \{x \in M | f(x) = e_N\}$. An R-semimodule M is called a free R-semimodule if it has a linearly independent subset N of M which generates M and then N is called a basis of M. If N has a finite number of elements, M is called a finitely generated R-semimodule.

The Bourne relation is introduced in [9, pp. 164] for an R-semimodule. If K is a sub-semimodule of an R-semimodule M, then the *Bourne relation* is defined by setting $m \equiv_K m'$ if and only if there exist two elements k and k' of K such that $m \ \Box_M \ k = m' \ \Box_M \ k'$. The factor semimodule $M/\equiv_K$ induced by $\equiv_K$ is also written as M/K. With the Bourne relation, we can define the cokernel of an R-morphism $f : M \to N$ as $N/f(M)$. If K is equal to the kernel of an R-semimodule morphism $f : M \to N$, then $m \equiv_{\mathrm{Ker}\, f} m'$ if and only if there exist two elements k, k' of $\mathrm{Ker}\, f$, such that $m \ \Box_M \ k = m' \ \Box_M \ k'$. Applying f on both sides, we obtain that $f(m) = f(m')$, i.e., $m \equiv_f m'$. Hence this special Bourne relation and the relation induced by the morphism f satisfy the partial order $\leqslant$, i.e., $\equiv_{\mathrm{Ker}\, f} \leqslant \equiv_f$. In general, we do not have $\equiv_f \leqslant \equiv_{\mathrm{Ker}\, f}$ for an R-semimodule morphism f. If an R-semimodule morphism $f : M \to N$ satisfies $\equiv_f \leqslant \equiv_{\mathrm{Ker}\, f}$, then f is called a *steady* or *k-regular* R-semimodule morphism.

Given an R-semimodule morphism $f : M \to N$, one image is defined to be the set of all values $f(m)$, $m \in M$, i.e.,

$$f(M) = \{n \in N | n = f(m), \ m \in M\}. \tag{1}$$

It is called the *proper image* of an R-semimodule morphism f. The other image of f is defined as

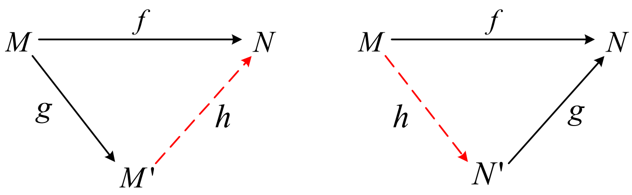
$$\mathrm{Im}\, f = \{n \in N | n \ \Box_N \ f(m) = f(m') \text{ for some } m, m' \in M\}. \tag{2}$$

It is called the *image* of f to distinguish from the proper image. It is easy to see that the two images coincide for the module case. For the semimodule case, if the two images are the same, i.e., $f(M) = \mathrm{Im}\, f$, the R-morphism of semimodules $f : M \to N$ is called *i-regular*. The morphism $f : M \to N$ is called *surjective*, if for any $n \in N$ there exists an element $m \in M$ such that $f(m) = n$.

Given two R-semimodule morphisms $f : (A, \Box_A, e_A) \to (B, \Box_B, e_B)$ and $g : (B, \Box_B, e_B) \to (C, \Box_C, e_C)$, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{3}$$

is called an *exact sequence* if $\mathrm{Im}\, f = \mathrm{Ker} g$. Since $\mathrm{Im}\, f$ is not the same as the proper image $f(A)$ for the R-semimodule morphism f, the sequence is said to be a *proper exact sequence* if $f(A) = \mathrm{Ker}\, g$. For the module case, the image of f, $\mathrm{Im}\, f$, is the same as the proper image $f(A)$, so every exact sequence of modules is also proper exact.



(1) (2)

Fig. 1. Triangle diagrams in the Factor Theorem 1.

Lemma 1 [20]**.** *Given an R-semimodule B and its sub-semimodule A, the following sequence*

$$e \to A \xrightarrow{i} B \xrightarrow{p} B/A \to e \tag{4}$$

is a short exact sequence, i.e., it is exact at each semimodule, where i is an insertion and p is a natural projection.

Theorem 1 [2, pp. 50, Factor Theorem]**.** *Let M, M', N and N' be left R-semimodules and let $f : M \to N$ be an R-semimodule morphism.*

(1) If $g : M \to M'$ is a surjective k-regular R-semimodule morphism with $\mathrm{Ker}\ g \subset \mathrm{Ker}\ f$, then there exists a unique R-semimodule morphism $h : M' \to N$ such that $f = h \circ g$. Moreover, if f is injective then h is also injective. Also, $\mathrm{Ker}\ h = g(\mathrm{Ker}\ f)$, $\mathrm{Im}\ h = \mathrm{Im}\ f$ and $f(M) = h(M')$, so that h has a unit kernel, i.e. $\mathrm{Ker}\ h = e_{M'}$, if and only if $\mathrm{Ker}\ g = \mathrm{Ker}\ f$. h is surjective if and only if f is surjective (see (1) in Fig. 1).

(2) If $g : N' \to N$ is an i-regular injective R-semimodule morphism with $\mathrm{Im}\ f \subset \mathrm{Im}\ g$, then there exists a unique R-semimodule morphism $h : M \to N'$ such that $f = g \circ h$. Moreover, $\mathrm{Ker}\ h = \mathrm{Ker}\ f$ and $\mathrm{Im}\ h = g^{-1}(\mathrm{Im}\ f)$, h is injective if and only if f is injective, h is surjective if and only if $g(N') = f(M)$ (see (2) in Fig. 1).

3. Model matching problem for systems over a semiring

3.1. Systems over a semiring

Systems over a semiring R are described by the following equations:

$$\begin{aligned} x(k+1) &= A\, x(k) \,\square_X\, B\, u(k), \\ y(k) &= C\, x(k) \,\square_Y\, D\, u(k), \end{aligned} \tag{5}$$

where x is in the state semimodule X, y is in the output semimodule Y, and u is in the input semimodule U, which are all assumed to be free. $A : X \to X$, $B : U \to X$, $C : X \to Y$ and $D : U \to Y$ are four R-semimodule morphisms. $R[z]$ denotes the polynomial semiring with coefficients in R and ΩX, ΩU, and ΩY denote the polynomial $R[z]$-semimodules of states, inputs, and outputs, respectively. ΩX is an alternative notation for the polynomial $R[z]$-semimodules $X[z]$ of states. For instance, given a sequence of states

$$\cdots, x(-2), x(-1), x(0), x(1), x(2), \cdots,$$

$\Omega X = x(0) \,\square_X\, x(1)z \,\square_X\, x(2)z^2 \,\square_X \cdots \square_X\, x(n)z^n$, $n < \infty$, is isomorphic to a finite sequence of states starting from the time instant 0 to the future. Let $R(z)$ denote the set of formal laurent series in z^{-1} with coefficients in R. In like manner, let $X(z)$ denote the set of formal laurent series in z^{-1}, with coefficients in X. We define $U(z)$ and $Y(z)$ similarly to $X(z)$. The transfer function $G(z) : U(z) \to Y(z)$ of this system (5) is

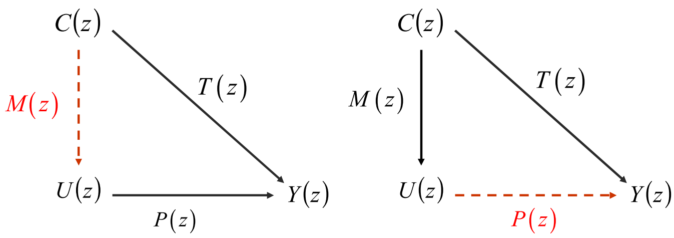


Fig. 2. The MMP commutative diagrams.

$$
\begin{aligned}
G(z) &= C(z^{-1}A)^* B z^{-1} \square_Y D \\
&= CBz^{-1} \; \square_Y \; CABz^{-2} \; \square_Y \; \cdots \; \square_Y \; CA^{n-1}Bz^{-n} \; \square_Y \; \cdots \; \square_Y \; D.
\end{aligned} \tag{6}
$$

The star operator A^* for an $n \times n$ matrix mapping $A : X \rightarrow X$ is defined as

$$
A^* = I_{n\times n} \; \square_X \; A \; \square_X \; \cdots \; \square_X \; A^n \; \square_X \; \cdots, \tag{7}
$$

where the operator $\square_X$ is induced from the state semimodule X, and $I_{n\times n}$ denotes the identity matrix mapping from X to X. The transfer function $G(z)$, as defined in Eq. (6), is of course in a natural way an $R(z)$-morphism from the $R(z)$-semimodule $U(z)$ to the $R(z)$-semimodule $Y(z)$. Transfer functions are considered as $R(z)$-morphisms in the definitions of the MMP and the solution existence conditions. In other parts of the paper, however, the mappings may be taken as $R[z]$-“linear” instead of $R(z)$-“linear”.

3.2. Model matching problem

The model matching problem has two cases as shown in Fig. 2. Given two transfer functions $T(z) : C(z) \rightarrow Y(z)$ and $P(z) : U(z) \rightarrow Y(z)$, one is to find $M(z) : C(z) \rightarrow U(z)$, or given two transfer functions $T(z) : C(z) \rightarrow Y(z)$ and $M(z) : C(z) \rightarrow U(z)$, one is to find $P(z) : U(z) \rightarrow Y(z)$, such that the model matching equation

$$
T(z) = P(z)\, M(z), \tag{8}
$$

is satisfied, which is illustrated in the commutative diagrams of Fig. 2.

4. Zero semimodules and extended zero semimodules

4.1. Zero semimodules of input and output type

This section extends the concepts of the zero modules and the extended zero modules introduced by Wyman and Sain in [24] to the semimodule case.

Definition 1. The zero semimodule of input type for the transfer function $T(z) : C(z) \rightarrow Y(z)$ is defined as

$$
Z_I(T(z)) = \frac{\Omega Y \cap T(C(z))}{\Omega Y \cap T(\Omega C)}. \tag{9}
$$

The quotient structure of the $R[z]$-semimodule $Z_I(T(z))$ is obtained by means of the *Bourne relation*. Intuitively, the zero semimodule of input type consists of the polynomial outputs produced by the inputs with poles. The poles of the inputs are canceled by the zeros of the plant, hence, the zero semimodule $Z_I(T(z))$ leads to the discovery of the plant's zeros. In the zero semimodule $Z_I(T(z))$, the polynomial outputs produced by the inputs without poles are removed, because they cannot discover the plant's zeros.

Definition 2. The zero semimodule of output type for a given transfer function $T(z) : C(z) \to Y(z)$ is defined as

$$Z_O(T(z)) = \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap (\text{Ker } T \;\square_{C(z)}\; \Omega C)}. \tag{10}$$

The operator $\square_{C(z)}$ is the binary operator in $C(z)$ induced by the operator $\square_C$ in C. Intuitively, the zero semimodule of output type consists of the inputs with poles, which produce outputs without poles. The poles in the inputs are canceled by the zeros of the plant, therefore, the zero semimodule $Z_O(T(z))$ leads to the discoveries of the plant's zeros. The zero semimodule of output type removes the kernel of the transfer function $T(z)$ and the polynomial inputs because they will not help to discover the zeros of the plant.

In the module case, the zero modules of input and output type are isomorphic forms of each other; however, that is not the case for the zero semimodules of input and output type. There exists an $R[z]$-semimodule epimorphism, that is, a surjective $R[z]$-morphism, instead of an $R[z]$-isomorphism from the zero semimodule of output type to the zero semimodule of input type.

Lemma 2. *Given a transfer function $T(z) : C(z) \to Y(z)$ and the zero semimodules of input and output type as shown in Eq. (9) and Eq. (10), there exists an $R[z]$-semimodule epimorphism from the zero semimodule of output type to the zero semimodule of input type.*

Proof of Lemma 2. We need to show that there exists an $R[z]$-semimodule epimorphism $\overline{T}(z)$ such that the following diagram is commutative.

$$\begin{array}{ccccc}
T^{-1}(\Omega Y) & \xrightarrow{T(z)} & \Omega Y \cap T(C(z)) & & \\
\downarrow p_1 & & \downarrow p_2 & & \\
\frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y)\cap(\text{Ker } T \;\square_{C(z)}\; \Omega C)} & \xrightarrow{\overline{T}(z)} & \frac{\Omega Y \cap T(C(z))}{\Omega Y \cap T(\Omega C)} & \longrightarrow e &
\end{array}$$

Factor Theorem 1 [2, pp. 50] states that for a surjective k-regular morphism satisfying the kernel inclusion condition, there exists a morphism to complete the commutative diagram. We know that the projection p_1 is k-regular, so we only need to show that Ker $p_1 \subset$ Ker $(p_2 \circ T)$. By the definitions of Ker p_1 and Ker $(p_2 \circ T)$, they are

$$\begin{aligned}
\text{Ker } p_1 = \{ & c(z) \in T^{-1}(\Omega Y) | c(z) \;\square_{C(z)}\; c_1(z) = c_2(z), \\
& \exists c_1(z),\; c_2(z) \in T^{-1}(\Omega Y) \cap (\text{Ker } T \;\square_{C(z)}\; \Omega C)\},
\end{aligned}$$

and

$$\begin{aligned}
\text{Ker } (p_2 \circ T) = \{ & c(z) \in T^{-1}(\Omega Y) | T(z)c(z) \;\square_{Y(z)}\; y_1(z) = y_2(z), \\
& \exists\; y_1(z),\; y_2(z) \in \Omega Y \cap T(\Omega C)\}.
\end{aligned}$$

It can be seen that any element in Ker p_1 is also in Ker $(p_2 \circ T)$. Using the Factor Theorem, there exists a unique $R[z]$-semimodule morphism $\overline{T}(z) : Z_O(T(z)) \to Z_I(T(z))$ such that the diagram is commutative and it is defined by the action

$$\overline{T}(z) : \frac{c(z)}{T^{-1}(\Omega Y) \cap (\text{Ker } T \;\square_{C(z)}\; \Omega C)} \mapsto \frac{T(z)c(z)}{\Omega Y \cap T(\Omega C)},$$

for $c(z) \in T^{-1}(\Omega Y)$. $\overline{T}(z)$ is surjective because $p_2 \circ T(z)$ is surjective. Hence, there exists an $R[z]$-semimodule epimorphism from the zero semimodule of output type to the zero semimodule of input type. $\square$

4.2. The Γ-zero semimodule

Given a transfer function $T(z) : C(z) \to Y(z)$, the *Γ-zero semimodule* is defined as

$$Z_\Gamma(T(z)) = \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap \Omega C}. \tag{11}$$

When Ker T is unit, $Z_\Gamma(T(z))$ is identical to $Z_O(T(z))$. When Ker T is not unit, the nature of the Γ-zero semimodule can be characterized by the following short exact sequence:

$$e \longrightarrow \Gamma(T(z)) \xrightarrow{\alpha} Z_\Gamma(T(z)) \xrightarrow{\beta} Z_O(T(z)) \longrightarrow e, \tag{12}$$

where $\Gamma(T(z)) = \text{Ker } T/\{\text{Ker } T \cap \Omega C\}$. We define $R[z]$-semimodule morphisms α and β by the actions

$$\alpha : \frac{c_1(z)}{\text{Ker } T \cap \Omega C} \mapsto \frac{c_1(z)}{T^{-1}(\Omega Y) \cap \Omega C}, \quad c_1(z) \in \text{Ker } T,$$
$$\beta : \frac{c_2(z)}{T^{-1}(\Omega Y) \cap \Omega C} \mapsto \frac{c_2(z)}{T^{-1}(\Omega Y) \cap (\text{Ker } T \ \Box_{C(z)} \ \Omega C)}, \quad c_2(z) \in T^{-1}(\Omega Y).$$

The morphism α is induced by the insertion from Ker T to $T^{-1}(\Omega Y)$. The morphism β is induced by the identity map from $T^{-1}(\Omega Y)$ to itself, therefore, β is an $R[z]$-epimorphism. A sketch of the proof for the exact sequence in Eq. (12) is shown as follows. By the definition of Ker α,

$$\begin{aligned}
\text{Ker } \alpha &= \left\{ \frac{c(z)}{\text{Ker } T \cap \Omega C}, c(z) \in \text{Ker } T \;\middle|\; \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} = e_{Z_\Gamma(T(z))} \right\} \\
&= \left\{ \frac{c(z)}{\text{Ker } T \cap \Omega C}, c(z) \in \text{Ker } T \;\middle|\; c(z) \ \Box_{C(z)} \ c_1(z) = c_2(z), \right. \\
&\qquad \left. c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \right\} \\
&= \left\{ \frac{c(z)}{\text{Ker } T \cap \Omega C}, c(z) \in \text{Ker } T \;\middle|\; c(z) \in \Omega C \right\} = \text{Ker } T \cap \Omega C.
\end{aligned}$$

Hence, α has a unit kernel. To prove that the sequence in Eq. (12) is exact at $Z_\Gamma(T(z))$, we only need to show that Im α = Ker β. By the definition, we have

$$\begin{aligned}
\text{Im } \alpha &= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \;\middle|\; \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \ \Box_{C(z)} \ \frac{c_1(z)}{T^{-1}(\Omega Y) \cap \Omega C} \right. \\
&\qquad \left. = \frac{c_2(z)}{T^{-1}(\Omega Y) \cap \Omega C},\ c(z) \in T^{-1}(\Omega Y),\ c_1(z), c_2(z) \in \text{Ker } T \right\} \\
&= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \;\middle|\; c(z) \ \Box_{C(z)} \ c_1(z) \ \Box_{C(z)} \ k_1(z) = c_2(z) \ \Box_{C(z)} \ k_2(z), \right. \\
&\qquad \left. c_1(z),\ c_2(z) \in \text{Ker } T,\ k_1(z), \in k_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \right\}
\end{aligned}$$

and

$$\begin{aligned}
\text{Ker } \beta &= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \;\middle|\; \frac{c(z)}{T^{-1}(\Omega Y) \cap (\text{Ker } \ \Box_{C(z)} \ \Omega C)} = e_{Z_I(T(z))}, \right. \\
&\qquad \left. \text{where } c(z) \in T^{-1}(\Omega Y) \right\}
\end{aligned}$$

$$= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \middle| c(z) \,\square_{C(z)}\, c_1(z) \,\square_{C(z)}\, k_1(z) = c_2(z) \,\square_{C(z)}\, k_2(z), \right.$$
$$c_i(z) \,\square_{C(z)}\, k_i(z) \in T^{-1}(\Omega Y) \cap (\text{Ker } T \,\square_{C(z)}\, \Omega C), i = 1, 2,$$
$$\left. \text{where } c(z) \in T^{-1}(\Omega Y) \right\}.$$

Because $T^{-1}(\Omega Y) \cap (\text{Ker } T \,\square_{C(z)}\, \Omega C) = (T^{-1}(\Omega Y) \cap \Omega C) \,\square_{C(z)}\, \text{Ker } T$, which can be proved by Lemma 3, we have the exactness at $Z_\Gamma(T(z))$, namely Im α = Ker β. Therefore, the sequence in Eq. (12) is exact.

Lemma 3. *Given A, B, and C, three sub-semimodules of an R-semimodule X with an operator $\square$, and $A \subset C$, the following equality*

$$(A \,\square\, B) \cap C = A \,\square\, (B \cap C) \tag{13}$$

holds if C is subtractive.

Proof. $(A \,\square\, B) \cap C \supset A \,\square\, (B \cap C)$ is obvious, we only need to show that $(A \,\square\, B) \cap C \subset A \,\square\, (B \cap C)$. For any $x \in (A \,\square\, B) \cap C$, we have $x \in C$ and $x = a \,\square\, b$ where $a \in A$, $b \in B$. If C is subtractive, then $b \in B \cap C$. Therefore, the equality holds. $\square$

In the module case, the short exact sequence in Eq. (12) means that the Γ-zero semimodule $Z_\Gamma(T(z))$ is a direct summand of $\Gamma(T(z))$and the zero semimodule of output type $Z_O(T(z))$. The reason is the module $\Gamma(T(z))$ is both torsion and divisible. An element u in a module M over a commutative integral domain R is called a torsion element if $uc = 0$ for some c in $R\backslash\{0\}$. All torsion elements in M is called the torsion submodule of M. If all elements of M is torsion, then M is called a *torsion module*. M is called *divisible*, if the equation $u = ax$, where $u \in M$ and $a \in R\backslash\{0\}$, always has a solution x in M. The module $\Gamma(T(z))$ is torsion because every element $k(z)$ in Ker T can find a polynomial $r(z)$ in the polynomial ring $R[z]$, which is a principal ideal domain, such that $r(z)k(z)$ in the zero element Ker $T \cap \Omega C$ in $\Gamma(T(z))$. To prove that $\Gamma(T(z))$ is divisible, we can verify that scalar multiplication $p(z)$: Ker $T(z) \to kerT(z)$ is an epimorphism of $R[z]$-modules whenever $p(z)$ is nonzero in $R[z]$. Then, $\Gamma(T(z))$ inherits this same property. As a divisible module over $R[z]$, $\Gamma(T(z))$ is injective; and so its image under injection into $\Gamma(T(z))$ is a direct summand. This conclusion follows from Theorem 3.20 and Proposition 4.24 in [7]. In the semimodule case, this short exact sequence does not split, so we can not understand the Γ-zero semimodule as the direct sum of two components $\Gamma(T(z))$ and $Z_O(T(z))$.

4.3. The Ω-zero semimodule

The *Ω-zero semimodule* $Z_\Omega(T(z))$ of the transfer function $T(z) : C(z) \to Y(z)$ is given by

$$Z_\Omega(T(z)) = \frac{\Omega Y}{\Omega Y \cap T(\Omega C)}. \tag{14}$$

This $R[z]$-semimodule is finitely generated, because ΩY is finitely generated. If the proper image $T(C(z))$ of $T(z)$ is equal to $Y(z)$, then the zero semimodule of input type $Z_I(T(z))$ is equal to the Ω-zero semimodule of $T(z)$. The Ω-zero semimodule differs from the zero semimodule of input type when the cokernel of $T(z)$ is not a unit semimodule. In this case, $Z_I(T(z))$ is a sub-semimodule of $Z_\Omega(T(z))$, and there exists a natural inclusion from $Z_I(T(z))$ to $Z_\Omega(T(z))$, with the cokernel as $\Omega(T(z)) = \Omega Y/\{\Omega Y \cap T(C(z))\}$. Thus, there is a short exact sequence of $R[z]$-semimodules and $R[z]$-semimodule morphisms:

$$e \longrightarrow Z_I(T(z)) \xrightarrow{i} Z_\Omega(T(z)) \xrightarrow{p} \Omega(T(z)) \longrightarrow e. \tag{15}$$

This sequence is exact because of the exact sequence of Eq. (4) in Lemma 1. In the module case, the splitting lemma implies that the short exact sequence in Eq. (15) splits from both sides, i.e., there exists a morphism $t : Z_\Omega(T(z)) \to Z_I(T(z))$, such that $t \circ i$ is the identify map on $Z_I(T(z))$, and on the other hand, there exists a morphism $u : \Omega(T(z)) \to Z_{\Omega}(T(z))$, such that $p \circ u$ is the identify map on $\Omega(T(z))$. Moreover, $Z_{\Omega}(T(z))$ is isomorphic to the direct summand of the $Z_I(T(z))$ and $\Omega(T(z))$. The reason is that the module $\Omega(T(z))$ is torsion-free. A finitely generated, torsion-free module over the principal ideal domain $R[z]$, $\Omega(T(z))$ is a free module from Lemma 3.19 in [7]. From Theorem 3.20, the sequence splits. In the semimodule case, however, the sequence in Eq. (15) does not split, so $Z_{\Omega}(T(z))$ is not isomorphic to the direct summand of the $Z_I(T(z))$ and $\Omega(T(z))$.

4.4. Relation between Γ-zeros and Ω-zeros

In the module case, the zero semimodules of input and output type are isomorphic to each other, so there exists an exact sequence connecting the Γ-zero semimodule and the Ω-zero semimodule. However, in the semimodule case, there is no direct connection between them without further assumptions on the transfer function $T(z)$. If the transfer function $T(z)$ is steady, then we can establish a similar relation between Γ-zeros and Ω-zeros as in the module case.

Theorem 2. *Given a steady or k-regular transfer function $T(z) : C(z) \to Y(z)$, the Γ-zero semimodule $Z_{\Gamma}(T(z))$, and the Ω-zero semimodule $Z_{\Omega}(T(z))$. There exists an exact sequence*

$$e \to \Gamma(T(z)) \xrightarrow{\alpha} Z_{\Gamma}(T(z)) \xrightarrow{\phi} Z_{\Omega}(T(z)) \xrightarrow{p} \Omega(T(z)) \to e \tag{16}$$

of $R[z]$-semimodules and $R[z]$-semimodule morphisms.

Proof. From Eqs. (12) and (15), we have established the exact sequences of

$$e \longrightarrow \Gamma(T(z)) \xrightarrow{\alpha} Z_{\Gamma}(T(z)),$$

and

$$Z_{\Omega}(T(z)) \xrightarrow{p} \Omega(T(z)) \longrightarrow e.$$

Therefore, we only need to prove the existence of an $R[z]$-semimodule morphism ϕ and the exactness at $Z_{\Gamma}(T(z))$ and $Z_{\Omega}(T(z))$. The existence of an $R[z]$-semimodule morphism ϕ is a consequence of the following commutative diagram:

$$\begin{array}{ccc} T^{-1}(\Omega Y) & \xrightarrow{T(z)} & \Omega Y \\ \downarrow p_1 & & \downarrow p_2 \\ \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap \Omega C} & \xrightarrow{\phi} & \frac{\Omega Y}{\Omega Y \cap T(\Omega C)}. \end{array}$$

Factor Theorem 1 [2, pp. 50] states that for a surjective k-regular morphism satisfying the kernel inclusion condition, there exists a morphism to complete the commutative diagram. To verify this inclusion condition $\mathrm{Ker}\ p_1 \subset \mathrm{Ker}\ (p_2 \circ T)$, we use the kernel definition to obtain

$$\begin{aligned} \mathrm{Ker}\ p_1 &= \left\{ c(z) \in T^{-1}(\Omega Y) \,\middle|\, \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} = e_{Z_{\Gamma}(T(z))} \right\} \\ &= \Big\{ c(z) \in T^{-1}(\Omega Y) \Big| \, c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\ &\qquad c_1(z),\ c(z) \in T^{-1}(\Omega Y) \cap \Omega C \Big\} \\ &= T^{-1}(\Omega Y) \cap \Omega C, \\ \mathrm{Ker}\ p_2 \circ T &= \left\{ c(z) \in T^{-1}(\Omega Y) \,\middle|\, \frac{T(z)c(z)}{\Omega Y \cap T(\Omega C)} = e_{Z_{\Omega}(T)} \right\} \\ &= \Big\{ c(z) \in T^{-1}(\Omega Y) \Big| \, T(z)c(z) \,\square_{Y(z)}\, y_1(z) = y_2(z), \\ &\qquad y_1(z), y_2(z) \in \Omega Y \cap T(\Omega C) \Big\}. \end{aligned}$$

Therefore, we have $\mathrm{Ker}\ p_1 \subset \mathrm{Ker}\ p_2 \circ T$, which guarantees that a unique $R[z]$-morphism ϕ exists and is defined by the action

$$\phi : \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \mapsto \frac{T(z)c(z)}{\Omega Y \cap T(\Omega C)},$$

for $c(z) \in T^{-1}(\Omega Y)$.

The next question is whether or not the sequence is exact at $Z_\Gamma(T(z))$ and $Z_\Omega(T(z))$, namely whether $\mathrm{Ker}\ \phi = \mathrm{Im}\ \alpha$ and $\mathrm{Ker}\ p = \mathrm{Im}\ \phi$. By the kernel definition, it can be seen that $\mathrm{Ker}\ \phi = \mathrm{Im}\ \alpha$, which means the exactness at $Z_\Gamma(T(z))$. To prove the exactness at $Z_\Omega(T(z))$, we use the kernel and the image definitions to obtain

$$\begin{aligned}
\mathrm{Ker}\ p &= \left\{ \frac{y(z)}{\Omega Y \cap T(\Omega C)}, y(z) \in \Omega Y \,\middle|\, \frac{y(z)}{\Omega Y \cap T(C(z))} = e_{\Omega(T(z))} \right\} \\
&= \left\{ \frac{y(z)}{\Omega Y \cap T(\Omega C)}, y(z) \in \Omega Y \,\middle|\, y(z) \,\square_{Y(z)}\, y_1(z) = y_2(z), \text{ where} \right. \\
&\qquad \left. y_1(z), y_2(z) \in \Omega Y \cap T(C(z)) \right\};
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Im}\ \phi &= \left\{ \frac{y(z)}{\Omega Y \cap T(\Omega C)}, y(z) \in \Omega Y \,\middle|\, \frac{y(z)}{\Omega Y \cap T(\Omega C)} \,\square_{Y(z)}\, \frac{y_1(z)}{\Omega Y \cap T(\Omega C)} \right. \\
&\qquad \left. = \frac{y_2(z)}{\Omega Y \cap T(\Omega C)},\ y_1(z), y_2(z) \in \Omega Y \cap T(C(z)) \right\} \\
&= \left\{ \frac{y(z)}{\Omega Y \cap T(\Omega C)}, y(z) \in \Omega Y \,\middle|\, y(z) \,\square_{Y(z)}\, y_1(z) \,\square_{Y(z)}\, y_p^1(z) \right. \\
&\qquad = y_2(z) \,\square_{Y(z)}\, y_p^2(z), y_1(z), y_2(z) \in \Omega Y \cap T(C(z)), \\
&\qquad \left. y_p^1(z),\ y_p^2(z) \in \Omega Y \cap T(\Omega C) \right\}.
\end{aligned}$$

The inclusion $\mathrm{Ker}\ p \subset \mathrm{Im}\ \phi$ is trivial, by viewing $y_p^1(z)$ and $y_p^2(z)$ as $e_{Y(z)}$. The inclusion $\mathrm{Im}\ \phi \subset \mathrm{Ker}\ p$ is true because $y_1(z) \,\square_{Y(z)}\, y_p^1(z)$ and $y_2(z) \,\square_{Y(z)}\, y_p^2(z)$ are in $\Omega Y \cap T(C(z))$. Therefore, we have $\mathrm{Ker}\ p = \mathrm{Im}\ \phi$, which means the exactness at $Z_\Omega(T(z))$. Hence we proved the sequence in Eq. (16) is exact under the steadiness assumption on the transfer function $T(z)$. □

In summary, this section studies a relation between the Γ-zero and Ω-zero semimodules, which play a crucial role in the study of the fixed zeros of the solutions to the MMP.

5. Fixed zeros structure for the model matching problem

This section generalizes the study of fixed zeros for the solutions to the MMP by Sain et al. [18] to systems over a semiring R.

5.1. Fixed zeros of the solution $M(z)$ to $T(z) = P(z)M(z)$

Define an $R[z]$-semimodule $Z(T, P)$ as follows:

$$Z(T, P) = \frac{T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)}{T(\Omega C)}. \tag{17}$$

This form, though a Bourne-type semimodule, is otherwise identical to the form in [18], for modules. The $R[z]$-semimodule $Z(T, P)$ is called the fixed zero semimodule for the solution $M(z)$ to the model matching equation $T(z) = P(z)M(z)$. In the next theorem, we will establish a relation between the fixed zero semimodule $Z(T, P)$ and the Ω-zero semimodule of the solution $M(z)$ to the MMP.

Proposition 1. *If we are given two transfer functions $T(z) : C(z) \to Y(z), P(z) : U(z) \to Y(z)$, and the solution $M(z) : C(z) \to U(z)$ satisfying the model matching equation $T(z) = P(z)M(z)$, there exists an $R[z]$-epimorphism $\beta_\Omega(z)$ between the Ω-zero semimodule of $M(z)$, $Z_\Omega(M(z))$, and the fixed zero semimodule, $Z(T, P)$:*

$$Z_\Omega(M(z)) \xrightarrow{\beta_\Omega(z)} Z(T, P) \longrightarrow e. \tag{18}$$

Proof. Recall that the Ω-zero semimodule of $M(z)$ is

$$Z_\Omega(M(z)) = \frac{\Omega U}{\Omega U \cap M(\Omega C)}.$$

The $R[z]$-semimodule morphism $\beta_\Omega(z)$ can be established by the following commutative diagram:

$$\begin{array}{ccccc} \Omega U & \xrightarrow{P(z)} & P(\Omega U) \,\square_{Y(z)}\, T(\Omega C) & & \\ \downarrow p_1 & & \downarrow p_2 & & \\ \frac{\Omega U}{\Omega U \cap M(\Omega C)} & \xrightarrow{\beta_\Omega(z)} & \frac{P(\Omega U) \,\square_{Y(z)}\, T(\Omega C)}{T(\Omega C)} & \longrightarrow e \end{array}$$

Using the Factor Theorem, there exists a unique $R[z]$-semimodule morphism $\beta_\Omega(z)$ if Ker $p_1 \subset$ Ker $(p_2 \circ P(z))$ is satisfied. By the kernel definition, we obtain the kernels of the natural projection p_1 and the morphism $p_2 \circ P(z)$ shown in the following equations

$$\begin{aligned} \text{Ker } p_1 = \{ & u(z) \in \Omega U | u(z) \,\square_{U(z)}\, M(z)c_1(z) = M(z)c_2(z), \\ & M(z)c_1(z), M(z)c_2(z) \in \Omega U,\ c_1(z), c_2(z) \in \Omega C\}, \end{aligned}$$

and

$$\begin{aligned} \text{Ker } p_2 \circ P(z) = \{ & u(z) \in \Omega U | P(z)u(z) \,\square_{Y(z)}\, T(z)c_1(z) = T(z)c_2(z), \\ & \text{where } c_1(z), c_2(z) \in \Omega C\}. \end{aligned}$$

If we apply $P(z)$ to the element $u(z) \in$ Ker p_1, we obtain

$$\begin{aligned} & P(z)u(z) \,\square_{Y(z)}\, P(z)M(z)c_1(z) = P(z)M(z)c_2(z) \quad \text{implies} \\ & P(z)u(z) \,\square_{Y(z)}\, T(z)c_1(z) = T(z)c_2(z). \end{aligned}$$

Therefore, any element $u(z)$ in Ker p_1 is also in Ker $p_2 \circ P(z)$. The mapping $\beta_\Omega(z)$ is an epimorphism because $p_2 \circ P(z)$ is surjective. Therefore, there exists an $R[z]$-epimorphism between the Ω-zero semimodule of $M(z)$, $Z_\Omega(M(z))$, and the fixed zero semimodule, $Z(T, P)$. $\square$

In the remainder of the section, we will give a description of the fixed zero semimodule $Z(T, P)$ in terms of the Ω-zero semimodule and the pole semimodules of the transfer functions $T(z)$ and $P(z)$. Consider an $R(z)$-semimodule morphism $[T(z)\ P(z)] : C(z) \times U(z) \to Y(z)$ with the action

$$[T(z)\ P(z)] : (c(z), u(z)) \mapsto T(z)c(z) \,\square_{Y(z)}\, P(z)u(z).$$

The pole semimodules of output type for the transfer functions $T(z)$ and $[T(z)\ P(z)]$, respectively, are

$$X_O(T(z)) = \frac{T(\Omega C)}{T(\Omega C) \cap \Omega Y}, \tag{19}$$

$$X_O([T(z)\ P(z)]) = \frac{T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}. \tag{20}$$

Recall that the Ω-zero semimodules of $T(z)$ and $[T(z)\ P(z)]$, respectively, are

$$Z_\Omega(T(z)) = \frac{\Omega Y}{T(\Omega C) \cap \Omega Y}, \tag{21}$$

$$Z_\Omega([T(z)\ P(z)]) = \frac{\Omega Y}{(T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)) \cap \Omega Y}. \tag{22}$$

The following theorem characterizes the fixed zero semimodule $Z(T, P)$ using short exact sequences of the pole semimodules and the Ω-zero semimodules.

Theorem 3. *If we are given two transfer functions $T(z) : C(z) \to Y(z), P(z) : U(z) \to Y(z)$, and the solution $M(z) : C(z) \to U(z)$ to the model matching equation $T(z) = P(z)M(z)$, there exist $R[z]$-semimodules Z_1 and P_1, which are defined below, such that the following three sequences are exact*:

$$e \longrightarrow Z_1 \xrightarrow{i} Z_\Omega(T(z)) \xrightarrow{p} Z_\Omega([T(z)\ P(z)]) \longrightarrow e; \tag{23}$$

$$e \longrightarrow X_O(T(z)) \xrightarrow{\alpha} X_O([T(z)\ P(z)]) \xrightarrow{\beta} P_1 \longrightarrow e; \tag{24}$$

$$Z_1 \xrightarrow{\phi} Z(T, P) \xrightarrow{\psi} P_1 \longrightarrow e, \tag{25}$$

where

$$Z_1 = \frac{(T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)) \cap \Omega Y}{T(\Omega C) \cap \Omega Y},$$

$$P_1 = \frac{T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)}{(T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)) \cap \Omega Y\ \square_{Y(z)}\ T(\Omega C)}.$$

The morphism ϕ has a unit kernel if $T(\Omega C)$ is subtractive, in which the last sequence becomes the following short exact sequence,

$$e \longrightarrow Z_1 \xrightarrow{\phi} Z(T, P) \xrightarrow{\psi} P_1 \longrightarrow e. \tag{26}$$

Proof. The first exact sequence in Eq. (23) is proven by using the exact sequence of Eq. (4) in Lemma 1. The existence of a unit kernel $R[z]$-semimodule morphism α and an $R[z]$-epimorphism β in Eq. (24) can be proved by the following commutative diagrams:

$$\begin{array}{ccccc} & & T(\Omega C) & \xrightarrow{j} & T(\Omega C)\ \square_{Y(z)}\ P(\Omega U) \\ & & \downarrow p_1 & & \downarrow p_2 \\ e & \longrightarrow & \frac{T(\Omega C)}{T(\Omega C)\cap\Omega Y} & \xrightarrow{\alpha} & \frac{T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)}{(T(\Omega C)\ \square_{Y(z)}\ P(\Omega U))\cap\Omega Y} \end{array}$$

and

$$\begin{array}{ccccc} T(\Omega C)\ \square_{Y(z)}\ P(\Omega U) & \xrightarrow{\mathrm{Id}} & T(\Omega C)\ \square_{Y(z)}\ P(\Omega U) & & \\ \downarrow p_2 & & \downarrow p_3 & & \\ \frac{T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)}{(T(\Omega C)\ \square_{Y(z)}\ P(\Omega U))\cap\Omega Y} & \xrightarrow{\beta} & \frac{T(\Omega C)\ \square_{Y(z)}\ P(\Omega U)}{(T(\Omega C)\ \square_{Y(z)}\ P(\Omega U))\cap\Omega Y\ \square_{Y(z)}\ T(\Omega C)} & \to & e \end{array}$$

where j is an inclusion and Id is an identity map. Using the Factor Theorem, a unit kernel $R[z]$-semimodule morphism α exists because the kernel equality condition Ker p_1 = Ker $p_2 \circ j$ is satisfied. An $R[z]$-semimodule epimorphism β exists because the kernel inclusion condition Ker $p_2 \subset$ Ker $p_3 \circ$ Id is satisfied and the morphism $p_3 \circ$ Id is surjective. To see that, we prove Ker p_1 = Ker $p_2 \circ j$ by the kernel definition, that is,

$$\begin{aligned}\text{Ker } p_1 = \{&y(z) \in T(\Omega C) | y(z) \,\square_{Y(z)}\, y_1(z) = y_2(z),\\ &y_1(z),\ y_2(z) \in T(\Omega C) \cap \Omega Y\} = T(\Omega C) \cap \Omega Y\end{aligned}$$

and

$$\begin{aligned}\text{Ker } p_2 \circ j = \{&y(z) \in T(\Omega C) | y(z) \,\square_{Y(z)}\, y_1(z) = y_2(z),\\ &y_1(z), y_2(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y\} = T(\Omega C) \cap \Omega Y.\end{aligned}$$

In order to prove Ker $p_2 \subset$ Ker $p_3 \circ$ Id, we obtain that the kernels of the natural projection p_2 and of the morphism $p_3 \circ$ Id, respectively, are

$$\begin{aligned}\text{Ker } p_2 = \{&y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) | y(z) \,\square_{Y(z)}\, y_1(z) = y_2(z),\\ &y_1(z),\ y_2(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y\}\\ = &(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y\end{aligned}$$

and

$$\begin{aligned}\text{Ker } p_3 \circ \text{Id} = \{&y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) | y(z) \,\square_{Y(z)}\, y_1(z) = y_2(z),\\ &y_1(z),\ y_2(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C)\}.\end{aligned}$$

Therefore, the morphisms α and β are defined by the actions

$$\begin{aligned}\alpha &: \frac{y_1(z)}{T(\Omega C) \cap \Omega Y} \mapsto \frac{y_1(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y},\\ \beta &: \frac{y_2(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y} \mapsto \frac{y_2(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C)},\end{aligned}$$

where $y_1(z) \in T(\Omega C)$ and $y_2(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)$.

The last step is to show that the second sequence in Eq. (24) is exact at $X_O([T(z)\ P(z)])$, namely Ker $\beta =$ Im α. This conclusion can be proved by the definition, that is,

$$\begin{aligned}\text{Ker } \beta &= \left\{ \frac{y(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) \right|\\ &\qquad \left. \frac{y(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C)} = e_{P_1} \right\}\\ &= \left\{ \frac{y(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) \right|\\ &\qquad y(z) \,\square_{Y(z)}\, \bar{y}_1(z) = \bar{y}_2(z), \bar{y}_1(z), \bar{y}_2(z) \in\\ &\qquad \left. (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C) \right\}\end{aligned}$$

and

$$\begin{aligned}\text{Im } \alpha = &\left\{ \frac{y(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) \right|\\ &\quad \frac{y(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y} \,\square_{Y(z)}\, \frac{y_1(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}\\ &\quad \left. = \frac{y_2(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y},\ y_1(z),\ y_2(z) \in T(\Omega C) \right\}\end{aligned}$$

$$= \left\{ \frac{y(z)}{(T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y}, y(z) \in T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) \right| \\ y(z) \,\Box_{Y(z)}\, y_1(z) \,\Box_{Y(z)}\, l_1(z) = y_2(z) \,\Box_{Y(z)}\, l_2(z), \\ \left. y_1(z),\ y_2(z) \in T(\Omega C),\ l_1(z), l_2(z) \in (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y \right\}.$$

Because $y_1(z) \,\Box_{Y(z)}\, l_1(z)$ and $y_2(z) \,\Box_{Y(z)}\, l_2(z)$ are in $(T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\Box_{Y(z)}\, T(\Omega C)$, they can be understood as $\bar{y}_1(z)$ and $\bar{y}_2(z)$ in the definition of Ker β. Therefore, we have Ker $\beta = \mathrm{Im}\ \alpha$. At this point, we have shown that the second sequence in Eq. (24) is exact.

The existence of the $R[z]$-semimodule morphisms ϕ and ψ in the third sequence Eq. (25) can be proved by the following commutative diagrams.

$$\begin{array}{ccc} e \to (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y & \xrightarrow{j} & T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) \\ \downarrow p_1 & & \downarrow p_2 \\ \frac{(T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y}{T(\Omega C) \cap \Omega Y} & \xrightarrow{\phi} & \frac{T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)}{T(\Omega C)} \end{array}$$

and

$$\begin{array}{ccc} T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) & \xrightarrow{\mathrm{Id}} & T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) \\ \downarrow p_2 & & \downarrow p_3 \\ \frac{T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)}{T(\Omega C)} & \xrightarrow{\psi} & \frac{T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)}{(T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\Box_{Y(z)}\, T(\Omega C)} \to e \end{array}$$

where j is an inclusion and Id is an identity map. Using the Factor Theorem, the morphisms ϕ and ψ exist because the kernel inclusion conditions Ker $p_1 \subset$ Ker $p_2 \circ j$ and Ker $p_2 \subset$ Ker $p_3 \circ \mathrm{Id}$ are satisfied. To prove that, the kernels of p_1 and $p_2 \circ j$, respectively, are

$$\begin{aligned} \mathrm{Ker}\ p_1 &= \{y(z) \in (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y) | y(z) \,\Box_{Y(z)}\, y_1(z) = y_2(z), \\ &\quad y_1(z),\ y_2(z) \in T(\Omega C) \cap \Omega Y\} \\ \mathrm{Ker}\ p_2 \circ j &= \{y(z) \in (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y | \\ &\quad y(z) \,\Box_{Y(z)}\, y_1(z) = y_2(z),\ y_1(z),\ y_2(z) \in T(\Omega C)\} \end{aligned}$$

Therefore, the inclusion Ker $p_1 \subset$ Ker $p_2 \circ j$ holds. Similarly, Ker$p_2 \subset$ Ker $p_3 \circ \mathrm{Id}$ is proved by the kernel definition, that is,

$$\begin{aligned} \mathrm{Ker}\ p_2 &= \{y(z) \in T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) | y(z) \,\Box_{Y(z)}\, y_1(z) = y_2(z), \\ &\quad y_1(z),\ y_2(z) \in T(\Omega C)\} \\ \mathrm{Ker}\ p_3 \circ \mathrm{Id} &= \{y(z) \in T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) | y(z) \,\Box_{Y(z)}\, y_1(z) = y_2(z), \\ &\quad y_1(z),\ y_2(z) \in (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\Box_{Y(z)}\, T(\Omega C)\}. \end{aligned}$$

Therefore, the morphisms ϕ and ψ are defined by the actions described in the following equations:

$$\begin{aligned} \phi &: \frac{y_1(z)}{T(\Omega C) \cap \Omega Y} \mapsto \frac{y_1(z)}{T(\Omega C)}, \\ \psi &: \frac{y_2(z)}{T(\Omega C)} \mapsto \frac{y_2(z)}{(T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\Box_{Y(z)}\, T(\Omega C)}, \end{aligned}$$

where $y_1(z) \in (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y$ and $y_2(z) \in T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)$. The morphism ψ is an epimorphism because $p_3 \circ \mathrm{Id}$ is surjective.

To prove the sequence in Eq. (25) is exact, we only need to show the exactness at $Z(T, P)$, namely Ker $\psi = \mathrm{Im}\ \phi$. By the definitions, we have

$$\begin{aligned}\text{Ker }\psi &= \left\{\frac{y(z)}{T(\Omega C)}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) \middle| \right.\\ &\qquad \left. \frac{y(z)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C)} = e_{P_1}\right\}\\ &= \left\{\frac{y(z)}{T(\Omega C)}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) | y(z) \,\square_{Y(z)}\, \bar{y}_1(z) = \bar{y}_2(z),\right.\\ &\qquad \left.\bar{y}_1(z), \bar{y}_2(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C)\right\};\end{aligned}$$

and

$$\begin{aligned}\text{Im }\phi &= \left\{\frac{y(z)}{T(\Omega C)}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) \middle| \frac{y(z)}{T(\Omega C)} \,\square_{Y(z)}\, \frac{y_1(z)}{T(\Omega C)}\right.\\ &\qquad \left. = \frac{y_2(z)}{T(\Omega C)}, y_1(z), y_2(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y\right\}\\ &= \left\{\frac{y(z)}{T(\Omega C)}, y(z) \in T(\Omega C) \,\square_{Y(z)}\, P(\Omega U) | y(z) \,\square_{Y(z)}\, y_1(z) \,\square_{Y(z)}\, l_1(z)\right.\\ &\qquad = y_2(z) \,\square_{Y(z)}\, l_2(z),\ y_1(z),\ y_2(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y,\\ &\qquad \left. l_1(z),\ l_2(z) \in T(\Omega C)\right\}\end{aligned}$$

Because $y_1(z) \,\square_{Y(z)}\, l_1(z)$ and $y_2(z) \,\square_{Y(z)}\, l_2(z)$ are in $(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \,\square_{Y(z)}\, T(\Omega C)$, they can be understood as $\bar{y}_1(z)$ and $\bar{y}_2(z)$ in the definition of Ker ψ. Therefore, the equality Ker $\psi =$ Im ϕ holds, and the sequence in Eq. (25) is exact.

The remaining question is that whether or not the kernel of ϕ is a unit semimodule. By the definition,

$$\begin{aligned}\text{Ker }\phi = &\left\{\frac{y(z)}{T(\Omega C) \cap \Omega Y}, y(z) \in (T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y \middle| \right.\\ &\left. y(z) \,\square_{Y(z)}\, T(z)c_1(z) = T(z)c_2(z),\ c_1(z), c_2(z) \in \Omega C\right\}.\end{aligned}$$

If $T(\Omega C)$ is subtractive, then this kernel is equal to the unit element in Z_1. Hence, we obtain the short exact sequence in Eq. (26). $\square$

In the module case, the sequences in Eqs. (23)–(26) in Theorem 3 are always proper exact. Moreover, the morphism ϕ has a unit kernel, because $T(\Omega C)$ is subtractive. In the semimodule case, the sequences are in general exact but not proper exact. Moreover, unlike the module case, the fixed zero semimodule $Z(T, P)$ can not be understood as a direct summand of Z_1 and P_1.

5.2. Fixed zeros of the solution $P(z)$ to $T(z) = P(z)M(z)$

We define an $R[z]$-semimodule $Z(T, M)$ as follows:

$$Z(T, M) = \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}. \tag{27}$$

This form, though a Bourne-type semimodule, is otherwise identical to the form in [18] for modules. This $R[z]$-semimodule $Z(T, M)$ is called the fixed zero semimodule of the solution $P(z)$ to the model matching equation $T(z) = P(z)M(z)$. In the next theorem, we will establish the relation between the fixed zero semimodule $Z(T, M)$ and the Γ-zero semimodule of the solution $P(z)$ to the model matching equation $T(z) = P(z)M(z)$.

Proposition 2. *If we are given two transfer functions $T(z) : C(z) \to Y(z), M(z) : C(z) \to U(z)$, and the solution $P(z) : U(z) \to Y(z)$ to the model matching equation $T(z) = P(z)M(z)$, there exists a unit kernel $R[z]$-semimodule morphism $\beta_\Gamma(z)$ from the fixed zero semimodule $Z(T, M)$ to the Γ-zero semimodule $Z_\Gamma(P(z))$ of $P(z)$, that is,*

$$e \longrightarrow Z(T, M) \xrightarrow{\beta_\Gamma(z)} Z_\Gamma(P(z)). \tag{28}$$

Proof. Recall that the Γ-zero semimodule of $P(z)$ is

$$Z_\Gamma(P(z)) = \frac{P^{-1}(\Omega Y)}{P^{-1}(\Omega Y) \cap \Omega U}. \tag{29}$$

A unit kernel $R[z]$-semimodule morphism $\beta_\Gamma(z)$ is constructed by the commutative diagram

$$\begin{array}{ccccc} & & T^{-1}(\Omega Y) & \xrightarrow{M(z)} & P^{-1}(\Omega Y) \\ & & \downarrow p_1 & & \downarrow p_2 \\ e & \longrightarrow & \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} & \xrightarrow{\beta_\Gamma(z)} & \frac{P^{-1}(\Omega Y)}{P^{-1}(\Omega Y) \cap \Omega U}. \end{array}$$

Using the Factor Theorem, such a morphism $\beta_\Gamma(z)$ exists and has a unit kernel because Ker $p_1 =$ Ker $(p_2 \circ M(z))$. This equality can be directly proved by the definition: Ker $p_1 = T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)$ and

$$\begin{aligned} \text{Ker } (p_2 \circ M(z)) &= \Big\{ c(z) \in T^{-1}(\Omega Y) | M(z)c(z) \;\; \Box_{C(z)} \;\; c_1(z) = c_2(z), \\ &\qquad c_1(z), c_2(z) \in P^{-1}(\Omega Y) \cap \Omega U \Big\} \\ &= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U). \end{aligned}$$

The last equality is true because ΩU is subtractive. The morphism $\beta_\Gamma(z)$ is defined by the action

$$\beta_\Gamma(z) : \frac{c(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} \mapsto \frac{M(z)c(z)}{P^{-1}(\Omega Y) \cap \Omega U},$$

for $c(z)$ in $T^{-1}(\Omega Y)$. □

In the remainder of the section, we will give a description of the fixed zero semimodule $Z(T, M)$ in terms of the Γ-zero semimodules and the pole semimodules of input type for the transfer functions $T(z)$ and $M(z)$. We can obtain similar results in [18] without further assumptions on the given transfer functions. We consider an $R(z)$-morphism $\begin{bmatrix} T(z) \\ M(z) \end{bmatrix} : C(z) \to U(z) \times Y(z)$ defined by the action

$$\begin{bmatrix} T(z) \\ M(z) \end{bmatrix} : c(z) \mapsto (T(z)c(z), M(z)c(z)).$$

The following theorem characterizes the fixed zero semimodule $Z(T, M)$ using the pole semimodules of input type and the Γ-zero semimodules of transfer functions $T(z)$ and $\begin{bmatrix} T(z) \\ M(z) \end{bmatrix}$.

Theorem 4. *If we are given two transfer functions $T(z) : C(z) \to Y(z), M(z) : C(z) \to U(z)$, and the solution $P(z) : U(z) \to Y(z)$ to the model matching equation $T(z) = P(z)M(z)$, then the following three sequences*

$$e \longrightarrow P_2 \xrightarrow{i} X_I\left(\begin{bmatrix} T(z) \\ M(z) \end{bmatrix}\right) \xrightarrow{p} X_I(T(z)) \longrightarrow e; \tag{30}$$

$$e \longrightarrow Z_\Gamma\left(\begin{bmatrix} T(z) \\ M(z) \end{bmatrix}\right) \xrightarrow{\alpha} Z_\Gamma(T(z)) \xrightarrow{\beta} Z_2 \longrightarrow e; \tag{31}$$

$$e \longrightarrow P_2 \xrightarrow{\phi} Z(T, M) \xrightarrow{\psi} Z_2 \longrightarrow e, \tag{32}$$

are exact, where the two $R[z]$-semimodules Z_2 and P_2 are

$$P_2 = \frac{T^{-1}(\Omega Y) \cap \Omega C}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C},$$
$$Z_2 = \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega Y)}.$$

Proof. The first sequence in Eq. (30) is exact because the mapping i in this sequence is a natural inclusion and the mapping p is a natural projection using Lemma 1. The existence of a unit kernel $R[z]$-semimodule morphism α and an $R[z]$-epimorphism β in Eq. (31) can be proved by the following commutative diagrams:

$$\begin{array}{cccc}
 & T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) & \xrightarrow{j} & T^{-1}(\Omega Y) \\
 & \downarrow p_1 & & \downarrow p_2 \\
e \longrightarrow & \frac{T^{-1}(\Omega Y)\cap M^{-1}(\Omega U)}{T^{-1}(\Omega Y)\cap M^{-1}(\Omega U)\cap \Omega C} & \xrightarrow{\alpha} & \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y)\cap \Omega C}
\end{array}$$

and

$$\begin{array}{cccc}
T^{-1}(\Omega Y) & \xrightarrow{\mathrm{Id}} & T^{-1}(\Omega Y) & \\
\downarrow p_2 & & \downarrow p_3 & \\
\frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y)\cap \Omega C} & \xrightarrow{\beta} & \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y)\cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y)\cap M^{-1}(\Omega U)} & \longrightarrow e
\end{array}$$

where j is an inclusion and Id is an identity map. Using the Factor Theorem, the morphism α exists and has a unit kernel because Ker $p_1 =$ Ker $p_2 \circ j$. The morphism β exists because Ker $p_2 \subset$ Ker $p_3 \circ \mathrm{Id}$ are satisfied. To prove Ker $p_1 =$ Ker $p_2 \circ j$, we use the kernel definition to obtain

$$\begin{aligned}
\text{Ker } p_1 &= \{c(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\qquad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C\} \\
&= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C, \\
\text{Ker } p_2 \circ j &= \{c(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\qquad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap \Omega C\} \\
&= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C.
\end{aligned}$$

We can also prove Ker $p_2 \subset$ Ker $p_3 \circ \mathrm{Id}$ using the kernel definition, that is,

$$\begin{aligned}
\text{Ker } p_2 &= \{c(z) \in T^{-1}(\Omega Y) | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\qquad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap \Omega C\} = T^{-1}(\Omega Y) \cap \Omega C, \\
\text{Ker } p_3 \circ \mathrm{Id} &= \{c(z) \in T^{-1}(\Omega Y) | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \exists c_1(z), c_2(z) \\
&\qquad \in T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)\}.
\end{aligned}$$

Because $p_3 \circ \mathrm{Id}$ is surjective, the morphism β is an $R[z]$-epimorphism. Hence, the morphism β has a unit cokernel. The two morphisms α and β exist and are defined by the action

$$\alpha : \frac{c_1(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C} \mapsto \frac{c_1(z)}{T^{-1}(\Omega Y) \cap \Omega C},$$

$$\beta : \frac{c_2(z)}{T^{-1}(\Omega Y) \cap \Omega C} \mapsto \frac{c_2(z)}{T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)},$$

where $c_1(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)$ and $c_2(z) \in T^{-1}(\Omega Y)$.

The last step is to show that the second sequence in Eq. (31) is exact at $Z_\Gamma(T(z))$, namely Ker $\beta =$ Im α. This conclusion can be proved by the kernel and the image definitions:

$$\begin{aligned}
\text{Ker } \beta &= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C}, c(z) \in T^{-1}(\Omega Y) \middle| \right. \\
&\qquad \left. \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} = e_{Z_\Gamma(T(z))} \right\} \\
&= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C}, c(z) \in T^{-1}(\Omega Y) \middle| c(z) \,\square_{C(z)}\, \bar{c}_1(z) = \bar{c}_2(z), \right. \\
&\qquad \left. \bar{c}_1(z), \bar{c}_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \right\}
\end{aligned}$$

and

$$\begin{aligned}
\text{Im } \alpha &= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C}, c(z) \in T^{-1}(\Omega Y) \middle| \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C} \,\square_{C(z)} \right. \\
&\qquad \frac{c_1(z)}{T^{-1}(\Omega Y) \cap \Omega C} = \frac{c_2(z)}{T^{-1}(\Omega Y) \cap \Omega C}, \\
&\qquad \left. c_1(z), c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \right\} \\
&= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C}, c(z) \in T^{-1}(\Omega Y) \middle| c(z) \,\square_{C(z)}\, c_1(z) \,\square_{C(z)}\, l_1(z) = \right. \\
&\qquad c_2(z) \,\square_{C(z)}\, l_2(z),\ \ c_1(z), c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U), \\
&\qquad \left. l_1(z), l_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \right\}.
\end{aligned}$$

The elements $c_1(z) \,\square_{C(z)}\, l_1(z)$ and $c_2(z) \,\square_{C(z)}\, l_2(z)$ in the definition of Im α can be viewed as $\bar{c}_1(z)$ and $\bar{c}_2(z)$ in the definition of Ker β. Therefore, we have Ker $\beta =$ Im α. We prove that the second sequence in Eq. (31) is exact.

That the unit kernel $R[z]$-semimodule morphism ϕ and the $R[z]$-semimodule epimorphism ψ in the third sequence Eq. (32) exist can be proved similarly by the following commutative diagrams:

$$\begin{array}{ccc}
T^{-1}(\Omega Y) \cap \Omega C & \xrightarrow{j} & T^{-1}(\Omega Y) \\
\downarrow p_1 & & \downarrow p_2 \\
e \longrightarrow \frac{T^{-1}(\Omega Y) \cap \Omega C}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C} & \xrightarrow{\phi} & \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}
\end{array}$$

$$\begin{array}{ccc}
T^{-1}(\Omega Y) & \xrightarrow{\mathrm{Id}} & T^{-1}(\Omega Y) \\
\downarrow p_2 & & \downarrow p_3 \\
\frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} & \xrightarrow{\psi} & \frac{T^{-1}(\Omega Y)}{T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} \longrightarrow e
\end{array}$$

where j is an inclusion and Id is an identity map. Using the Factor Theorem, the morphism α exists and has a unit kernel because Ker $p_1 =$ Ker $p_2 \circ j$. The morphism β exists because Ker $p_2 \subset$ Ker $p_3 \circ$ Id are satisfied. To prove Ker $p_1 =$ Ker $p_2 \circ j$, we use the kernel definition to obtain

$$\begin{aligned}
\text{Ker } p_1 &= \{c(z) \in T^{-1}(\Omega Y) \cap \Omega C | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\quad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)\} \\
&= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C, \\
\text{Ker } p_2 \circ j &= \{c(z) \in T^{-1}(\Omega Y) \cap \Omega C | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\quad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)\} \\
&= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C.
\end{aligned}$$

We can also prove Ker $p_2 \subset$ Ker $p_3 \circ \mathrm{Id}$ using the kernel definition, that is,

$$\begin{aligned}
\text{Ker } p_2 &= \{c(z) \in T^{-1}(\Omega Y) | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\quad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)\} \\
&= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U), \\
\text{Ker } p_3 \circ \mathrm{Id} &= \{c(z) \in T^{-1}(\Omega Y) | c(z) \,\square_{C(z)}\, c_1(z) = c_2(z), \\
&\quad c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)\}.
\end{aligned}$$

Therefore, the morphisms ϕ and ψ exist and are defined by the action described in the following equations:

$$\begin{aligned}
\phi &: \frac{c_1(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C} \mapsto \frac{c_1(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}, \\
\psi &: \frac{c_2(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} \mapsto \frac{c_2(z)}{T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)},
\end{aligned}$$

where $c_1(z) \in T^{-1}(\Omega Y) \cap \Omega C$ and $c_2(z) \in T^{-1}(\Omega Y)$. The morphism ψ is an $R[z]$-epimorphism because $p_3 \circ \mathrm{Id}$ is surjective.

To prove the sequence in Eq. (32) is exact, we only need to show the exactness at $Z(T, M)$, that is Ker ψ = Im ϕ. This conclusion can be proved by the kernel and the image definitions:

$$\begin{aligned}
\text{Ker } \psi &= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}, c(z) \in T^{-1}(\Omega Y) \right| \\
&\quad \left. \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} = e_{Z(T,M)} \right\} \\
&= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap \Omega C}, c(z) \in T^{-1}(\Omega Y) | c(z) \,\square_{C(z)}\, \bar{c}_1(z) = \bar{c}_2(z), \right. \\
&\quad \left. \bar{c}_1(z), \bar{c}_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \,\square_{C(z)}\, T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\text{Im } \phi &= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}, c(z) \in T^{-1}(\Omega Y) \right| \frac{c(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} \\
&\quad \square_{C(z)}\, \frac{c_1(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} \\
&\quad \left. = \frac{c_2(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)},\ c_1(z), c_2(z) \in T^{-1}(\Omega Y) \cap \Omega C \right\}
\end{aligned}$$

$$= \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)} \middle| c(z) \square_{C(z)} c_1(z) \square_{C(z)} l_1(z) = \right.$$
$$c_2(z) \square_{C(z)} l_2(z),\ c(z) \in T^{-1}(\Omega Y),\ c_1(z),\ c_2(z) \in T^{-1}(\Omega Y) \cap \Omega C,$$
$$\left. l_1(z),\ l_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \right\}.$$

Because the elements $c_1(z) \square_{C(z)} l_1(z)$ and $c_2(z) \square_{C(z)} l_2(z)$ are in $T^{-1}(\Omega Y) \cap \Omega C \square_{C(z)} T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)$, we can view them as $\bar{c}_1(z)$ and $\bar{c}_2(z)$ in the definition of Ker β. Therefore, we have Ker $\phi =$ Im ψ.

The remaining question is whether or not the kernel of ϕ is the unit semimodule. By the kernel definition,

$$\text{Ker } \phi = \left\{ \frac{c(z)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}, c(z) \in T^{-1}(\Omega Y) \cap \Omega U \middle| \right.$$
$$\left. c(z) \square_{C(z)} c_1(z) = c_2(z), c_1(z), c_2(z) \in T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \right\}$$
$$= T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C.$$

The last step is true because $T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)$ is subtractive. So the kernel of ϕ is equal to the unit semimodule in P_2. Hence, we can obtain the short exact sequence in Eq. (32). □

Notice that, unlike the previous section, we can obtain the exact sequence in Eq. (32) without the subtractive assumption, because $T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C$ is already subtractive. In the module case, the sequences in Theorem 4 are always proper exact. In the semimodule case, the sequences are exact but not proper exact. Moreover, unlike the module case, the fixed zero semimodule $Z(T, M)$ can not be understood as a direct sum Z_2 and P_2, because the sequence in Eq. (32) does not split.

5.3. Essential and inessential zeros

Propositions 1 and 2 in the previous section state the relationships between the fixed zero semimodules and the extended zero semimodules of the solutions to the model matching problem. We call the fixed zeros $Z(T, P)$ and $Z(T, M)$ the *essential zero semimodules* of the solutions to the MMP. In particular, for the model matching problem with an unknown controller $M(z)$, we have the exact sequence:

$$e \longrightarrow C(M) \longrightarrow Z_\Omega(M(z)) \xrightarrow{\beta_\Omega(z)} Z(T, P) \longrightarrow e, \tag{33}$$

where $C(M) =$ Ker $\beta_\Omega(z)$, which is called the *inessential zero semimodule* of the solutions to the MMP. For an arbitrary transfer function $P(z)$, the inessential zero semimodule $C(M)$ cannot be easily expressed as a concrete form. However, with proper assumptions, the inessential zero semimodule can be expressed explicitly, shown in the following Corollary.

Corollary 1. *For the MMP with given $T(z) : C(z) \to Y(z)$ and a steady $P(z) : U(z) \to Y(z)$, if $M(\Omega C) \square_{U(z)}$ Ker P is subtractive, then the sequence in Eq. (33) with*

$$C(M) = \frac{\Omega U \cap (M(\Omega C) \square_{U(z)} \text{Ker } P)}{M(\Omega C) \cap \Omega U}$$

is a short exact sequence.

Proof. The morphism from $C(M)$ to $Z_\Omega(M)$ is an insertion i, therefore, in order to prove the short exact sequence, we only need to show Im $i =$ Ker β_Ω. By the kernel and image definitions, we have

$$\begin{aligned}
\text{Ker } \beta_{\Omega} &= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U \,\middle|\, \frac{P(u_p)}{T(\Omega C)} = e_{Z(T,P)} \right\} \\
&= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U \,\middle|\, Pu_p \,\square_{Y(z)}\, Tc_p^1 = Tc_p^2,\ c_p^1, c_p^2 \in \Omega C \right\} \\
&= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U \,\middle|\, Pu_p \square_{Y(z)}\, PMc_p^1 = PMc_p^2,\ c_p^1, c_p^2 \in \Omega C \right\} \\
&= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U \,\middle|\, u_p \square_{U(z)} Mc_p^1 \,\square_{U(z)}\, k_1 = Mc_p^2 \,\square_{U(z)}\, k_2, \right. \\
&\qquad \left. c_p^1, c_p^2 \in \Omega C, k_1, k_2 \in \text{Ker } P \right\}, \text{because } P(z) \text{ is steady} \\
&= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U | u_p \in M(\Omega C) \square_{U(z)} \text{Ker } P \right\}, \text{because} \\
&\qquad (M(\Omega C) \,\square_{U(z)}\, \text{Ker } P) \text{ is subtractive} \\
&= \frac{\Omega U \cap (M(\Omega C) \,\square_{U(z)}\, \text{Ker } P)}{M(\Omega C) \cap \Omega U},
\end{aligned}$$

and

$$\begin{aligned}
\text{Im } i &= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U \,\middle|\, \frac{u_p}{M(\Omega C) \cap \Omega U} \square_{U(z)} \frac{u_p^1}{M(\Omega C) \cap \Omega U} \right. \\
&= \left. \frac{u_p^2}{M(\Omega C) \cap \Omega U}, u_p^1, u_p^2 \in \Omega U \cap (M(\Omega C) \,\square_{U(z)}\, \text{Ker } P) \right\} \\
&= \left\{ \frac{u_p}{M(\Omega C) \cap \Omega U}, u_p \in \Omega U \,\middle|\, u_p \square_{U(z)} u_p^1 \,\square_{U(z)}\, v_p^1 = u_p^2 \,\square_{U(z)}\, v_p^2 \right. \\
&\qquad \left. v_p^1, v_p^2 \in \Omega U \cap M(\Omega C), u_p^1, u_p^2 \in \Omega U \cap (M(\Omega C) \,\square_{U(z)}\, \text{Ker } P) \right\} \\
&= \frac{\Omega U \cap (M(\Omega C) \,\square_{U(z)}\, \text{Ker } P)}{M(\Omega C) \cap \Omega U}, \\
&\qquad \text{because } M(\Omega C) \,\square_{U(z)}\, \text{Ker } P \text{ issubtractive}.
\end{aligned}$$

Therefore, the sequence in Eq. (33) is a short exact sequence. □

For the model matching problem with an unknown plant $P(z)$, we have

$$e \longrightarrow Z(T, M) \xrightarrow{\beta_{\Gamma}(z)} Z_{\Gamma}(P(z)) \longrightarrow \widetilde{C}(P) \longrightarrow e, \tag{34}$$

where $\widetilde{C}(M) = \text{coker } \beta_{\Gamma}(z) = Z_{\Gamma}(P(z))/Z(T, M)$. We call $\widetilde{C}(P)$ the *inessential zero semimodule* of the solutions to the MMP. In the semimodule case, the sequence is exact but not proper exact, so the fixed zeros can not necessarily be viewed as components in the extended zeros of the solutions.

6. A discrete event system application

The Petri net models for the four-machine manufacturing system (top) and the two-machine manufacturing system (bottom) are given in Fig. 3. The control problem is to design a compensator such that the manufacturing system with four machines has the same processing time as the manufacturing system with two machines. The discrete event systems can be modeled as a system over the max-plus

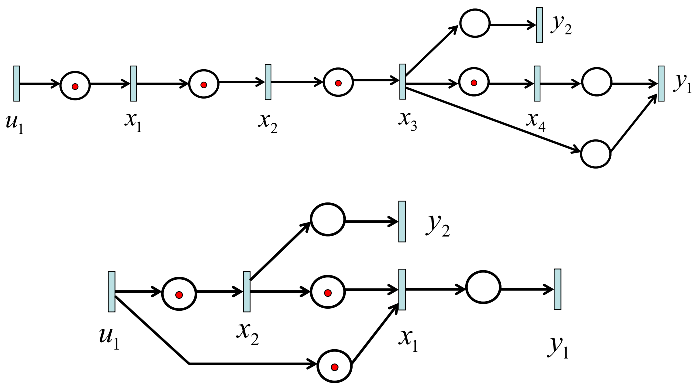


Fig. 3. The Petri net realizations for the two manufacturing systems.

algebra. The max-plus algebra is a set $\mathbb{R}_{\mathrm{Max}} = (R \cup \{-\infty\}, \oplus, -\infty, \otimes, 0)$, where $\oplus$ denotes max and $\otimes$ denotes addition. The unit elements are denoted as $\epsilon = -\infty$ and $e = 0$. The state space representation for the manufacturing system with four machines, denoted as the plant, is given as the following form:

$$x(k+1) = A_P x(k) \oplus B_P u(k), \quad y(k) = C_P x(k),$$
$$\text{where } A_P = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix}, \; B_P = \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ \epsilon & \epsilon \\ \epsilon & \epsilon \end{bmatrix}, \text{ and } C_P = \begin{bmatrix} \epsilon & \epsilon & e & e \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix}.$$

For the plant, the state semimodule is $X = \mathbb{R}^4_{\mathrm{Max}}$, the input semimodule $U = \mathbb{R}^2_{\mathrm{Max}}$, and the output semimodule is $Y = \mathbb{R}^2_{\mathrm{Max}}$. The plant transfer function $P(z) : U(z) \to Y(z)$ is obtained as the following form:

$$P(z) = \begin{bmatrix} z^{-3} \oplus z^{-4} & \epsilon \\ z^{-3} & \epsilon \end{bmatrix},$$

where $U(z)$ and $Y(z)$ are the set of formal Laurent series in z^{-1} with coefficients in U and Y, respectively.

The state space representation for the manufacturing system with two machines, denoted as the reference system, is given below:

$$x(k+1) = A_T x(k) \oplus B_T c(k), \quad y(k) = C_T x(k),$$
$$\text{where } A_T = \begin{bmatrix} \epsilon & e \\ \epsilon & \epsilon \end{bmatrix}, \; B_T = \begin{bmatrix} e & \epsilon \\ e & \epsilon \end{bmatrix}, \text{ and } C_T = \begin{bmatrix} e & \epsilon \\ \epsilon & e \end{bmatrix}.$$

For the reference system, the state semimodule is $X = \mathbb{R}^2_{\mathrm{Max}}$, the input semimodule $C = \mathbb{R}^2_{\mathrm{Max}}$, and the output semimodule is $Y = \mathbb{R}^2_{\mathrm{Max}}$. The reference transfer function $T(z) : C(z) \to Y(z)$ is obtained as the following form:

$$T(z) = \begin{bmatrix} z^{-1} \oplus z^{-2} & \epsilon \\ z^{-1} & \epsilon \end{bmatrix},$$

where $C(z)$ and $Y(z)$ are the set of formal Laurent series in z^{-1} with coefficients in C and Y, respectively.

In the fixed zero semimodule $Z(T, P)$ of Eq. (17), $P(\Omega U)$ and $T(\Omega C)$ are defined by the usual power series multiplication, for any $u_p = \begin{bmatrix} u_p^1 & u_p^2 \end{bmatrix}^T$ and $c_p = \begin{bmatrix} c_p^1 & c_p^2 \end{bmatrix}^T$, where $u_p^1, u_p^2 \in \Omega\mathbb{R}_{\mathrm{Max}}$, c_p^1 and $c_p^2 \in \Omega\mathbb{R}_{\mathrm{Max}}$ are represented by

$$u_p^1 = a_0 \oplus a_1 z \oplus a_2 z^2 \cdots \oplus a_n z^n,$$
$$u_p^2 = b_0 \oplus b_1 z \oplus b_2 z^2 \cdots \oplus b_n z^n,$$
$$c_p^1 = p_0 \oplus p_1 z \oplus p_2 z^2 \cdots \oplus p_n z^n,$$
$$c_p^2 = q_0 \oplus q_1 z \oplus q_2 z^2 \cdots \oplus q_n z^n,$$

where the coefficients a_i, b_i, p_i and q_i are from $\mathbb{R}_{\text{Max}}$ for $i = \{0, \ldots, n\}$. We can obtain that

$$\begin{aligned} T(\Omega C) &= \{T(z)c_p | \forall c_p \in \Omega C\} \\ &= \left\{ \begin{bmatrix} p_0 z^{-2} \oplus (p_0 \oplus p_1) z^{-1} \oplus (p_1 \oplus p_2) \oplus (p_2 \oplus p_3) z \\ \oplus \cdots \oplus (p_{n-1} \oplus p_n) z^{n-2} \oplus p_n z^{n-1} \\ ------------------- \\ p_0 z^{-1} \oplus p_1 \oplus p_2 z \oplus \cdots \oplus p_{n-1} z^{n-2} \oplus p_n z^{n-1} \end{bmatrix} \right\}, \end{aligned}$$

and

$$\begin{aligned} P(\Omega U) &= \{P(z)u_p | \forall u_p \in \Omega U\} \\ &= \left\{ \begin{bmatrix} a_0 z^{-4} \oplus (a_0 \oplus a_1) z^{-3} \oplus (a_1 \oplus a_2) z^{-2} \oplus (a_2 \oplus a_3) z^{-1} \oplus \\ (a_3 \oplus a_4) \oplus (a_4 \oplus a_5) z \oplus \cdots \oplus (a_{n-1} \oplus a_n) z^{n-4} \oplus a_n z^{n-3} \\ ------------------- \\ a_0 z^{-3} \oplus a_1 z^{-2} \oplus a_2 z^{-1} \oplus a_3 \oplus a_4 z \oplus \cdots \oplus a_n z^{n-3} \end{bmatrix} \right\}. \end{aligned}$$

In the module case of real numbers, the fixed zero structure $Z(T, P) = \frac{T(\Omega C)+P(\Omega U)}{T(\Omega C)}$ denotes the set of equivalent classes in which y_1 is equivalent with y_2, where $y_1, y_2 \in T(\Omega C) + P(\Omega U)$, if and only if they satisfy $y_1 = y_2 + y$, where $y \in T(\Omega C)$. The representative for the equivalent classes can be understood as the element in $T(\Omega C) + P(\Omega U)$ by removing the components in $T(\Omega C)$. In the semimodule case of the fixed pole structure, the equivalent classes cannot be constructed by simply removing $T(\Omega C)$ from $T(\Omega C) \oplus P(\Omega U)$ because of the Bourne equivalent relation. Instead, two elements $y_1 = T(z)c_p \oplus P(z)u_p$ and $y_2 = T(z)\tilde{c}_p \oplus P(z)\tilde{u}_p$ in $T(\Omega C) \oplus P(\Omega U)$ are equivalent, if and only if $y_1 \oplus l_1 = y_2 \oplus l_2$, where l_1 and l_2 in $T(\Omega C)$. By tedious mathematical manipulations, one can show that y_1 and y_2 are equivalent if and only if the first rows in u_p^1 and $\tilde{u}_p^1$ have the same coefficients for z^0 and z^1. Due to limited space, the detail derivations are omitted here. In other words, the representative of the equivalent classes in the fixed zero semimodule is

$$P(z)u_p = P(z) \begin{bmatrix} a_0 \oplus a_1 z \oplus w_p^1 \\ w_p^2 \end{bmatrix},$$

where w_p^1 is an arbitrary polynomial element in $\Omega\mathbb{R}_{\text{Max}}$ with the order of z higher than 1 and w_p^2 is an arbitrary polynomial element in $\Omega\mathbb{R}_{\text{Max}}$. Based on the proceeding discussion and the calculations for $T(\Omega C)$and $P(\Omega U)$ in the previous paragraph, we can observe the equivalent classes in the fixed zero semimodule have the same coefficients for z^{-4} and z^{-3} in the first row and have the same coefficients for z^{-3} and z^{-2} in the second row. It can be equivalently understood as the representative in the quotient equivalence classes are $P(z)u_p$ by removing vectors with the first element which has the order of z greater than -2 and with the second element which has the order of z greater than -1.

Consider the operation of z upon the columns of the transfer function $P(z)$, the first column is

$$z \begin{bmatrix} z^{-3} \oplus z^{-4} \\ z^{-3} \end{bmatrix} = \begin{bmatrix} z^{-3} \\ z^{-2} \end{bmatrix}, \quad z \begin{bmatrix} z^{-3} \\ z^{-2} \end{bmatrix} = \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix}.$$

We can stop at this point because linear dependency occurs, and identify the basis as

$$x_1 = \begin{bmatrix} z^{-3} \oplus z^{-4} \\ z^{-3} \end{bmatrix}, \quad x_2 = \begin{bmatrix} z^{-3} \\ z^{-2} \end{bmatrix}.$$

Therefore, we can find the matrix A_f from the fixed zero semimodule $Z(T, P)$ as

$$A_f = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

We will show later that this matrix is connected to the state space realizations of the solutions to the model matching problem. Next, we take any two solutions $M_1(z)$ and $M_2(z)$ to the model matching equation $P(z)M(z) = T(z)$ and construct the realization matrices A_1 and A_2 for them. For instance, two solutions $M_1(z)$ and $M_2(z)$ are given as follows:

$$M_1(z) = \begin{bmatrix} z^2 & \epsilon \\ \epsilon & z \end{bmatrix}, \quad M_2(z) = \begin{bmatrix} z^2 & \epsilon \\ z^2 & z^2 \end{bmatrix}.$$

Because

$$M_1(\Omega C) = \left\{ \begin{bmatrix} u_p^1 \\ u_p^2 \end{bmatrix}, \text{ order of } z \text{ in } u_p^1 \geqslant 2, \quad \text{order of } z \text{ in } u_p^2 \geqslant 1, \right\},$$

$$M_2(\Omega C) = \left\{ \begin{bmatrix} u_p^1 \\ u_p^2 \end{bmatrix}, \text{ order of } z \text{ in } u_p^1 \geqslant 2, \quad \text{order of } z \text{ in } u_p^2 \geqslant 2, \right\},$$

where $u_p^1, u_p^2 \in \Omega \mathbb{R}_{\text{Max}}$. The Ω-zero semimodules of $M_1(z)$ and $M_2(z)$ are

$$Z_\Omega(M_1) = \frac{\Omega U}{M_1(\Omega C) \cap \Omega U} = \left\{ \begin{bmatrix} a_0 \oplus a_1 z \\ b_0 \end{bmatrix}, a_0, a_1, b_0 \in \mathbb{R}_{\text{Max}} \right\},$$

$$Z_\Omega(M_2) = \frac{\Omega U}{M_2(\Omega C) \cap \Omega \mathbb{R}_{\text{Max}}} = \left\{ \begin{bmatrix} a_0 \oplus a_1 z \\ b_0 \oplus b_1 z \end{bmatrix}, a_0, a_1, b_0, b_1 \in \mathbb{R}_{\text{Max}} \right\}.$$

Therefore, the Ω-zero semimodules of $M_1(z)$ and $M_2(z)$ generate two A matrices for these two controllers:

$$A_1 = \begin{bmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon \end{bmatrix}, A_2 = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix}.$$

Notice that these two matrices A_1 and A_2 both contain the essential matrix A_{fixed}, which is defined by

$$A_{\text{fixed}} = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

This essential matrix A_{fixed} is obtained in the fixed zero structure. The kernel of $P(z)$ is

$$\text{Ker } P(z) = \left\{ \begin{bmatrix} \epsilon \\ u \end{bmatrix}, u \in U(z) \right\}.$$

Because the transfer function $P(z)$ is steady, and $M_i(\Omega C) \oplus \text{ Ker } P$ is subtractive for $i = 1, 2$, the inessential zero semimodules of $M_1(z)$ and $M_2(z)$ are

$$C(M_1) = \frac{\Omega U \cap (M_1(\Omega C) \oplus \text{Ker } P)}{M_1(\Omega C) \cap \Omega U} = \left\{ \begin{bmatrix} \epsilon \\ b_0 \end{bmatrix}, b_0 \in U \right\},$$

$$C(M_2) = \frac{\Omega U \cap (M_2(\Omega C) \oplus \text{Ker } P)}{M_1(\Omega C) \cap \Omega U} = \left\{ \begin{bmatrix} \epsilon \\ b_0 \oplus b_1 z \end{bmatrix}, b_0, b_1 \in U \right\}.$$

The generated two state space realization matrices, respectively, are

$$A^1_{\text{inessential}} = \epsilon, \; A^2_{\text{inessential}} = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

These matrices will appear in the state space representations of the solutions to the model matching problem as the inessential components.

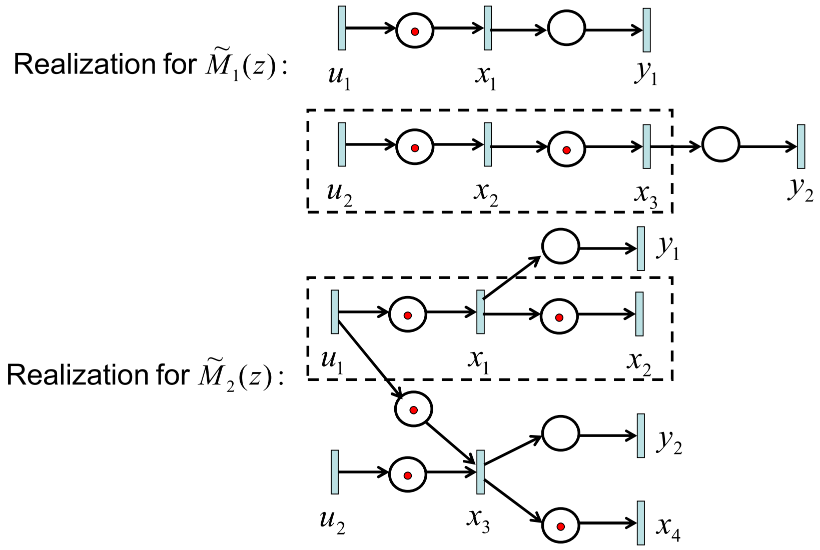


Fig. 4. The Petri net realizations for $\widetilde{M}_1(z)$ and $\widetilde{M}_2(z)$.

Notice that the two controllers are not the standard transfer function forms as in Eq. (6), but we can obtain the realization matrices for the following two causal transfer functions related to these two controllers:

$$\widetilde{M}_1(z) = z^{-3}M_1(z) = \begin{bmatrix} z^{-1} & \epsilon \\ \epsilon & z^{-2} \end{bmatrix}, \ \widetilde{M}_2(z) = z^{-3}M_2(z) = \begin{bmatrix} z^{-1} & \epsilon \\ z^{-1} & z^{-1} \end{bmatrix}.$$

The state space representations for $\widetilde{M}_1(z)$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k \oplus \begin{bmatrix} e & \epsilon \\ \epsilon & e \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_k = \begin{bmatrix} u_1 \\ u_2 \\ x_2 \end{bmatrix}_k$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_k = \begin{bmatrix} e & \epsilon & \epsilon \\ \epsilon & \epsilon & e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}_k$$

and the state space representation for $\widetilde{M}_2(z)$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{k+1} = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k \oplus \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & e \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_k = \begin{bmatrix} u_1 \\ x_1 \\ u_1 \oplus u_2 \\ x_3 \end{bmatrix}_k,$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_k = \begin{bmatrix} e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}_k.$$

Moreover, we can model these two systems by Petri nets shown in Fig. 4. We find out that two Petri net realizations for $\widetilde{M}_1(z)$ and $\widetilde{M}_2(z)$, which are obtained from the two controllers $M_1(z)$ and $M_2(z)$, respectively, each contain the same set of components marked in the dashed boxes, which are generated by the essential matrix A_{fixed}.

7. Conclusion

In this paper, the MMP is studied for systems over semirings, which are used to model a class of discrete-event dynamic systems, such as queueing systems, communication networks, and manufacturing systems. The main contribution of this paper is the discovery of fixed zero structure for solutions to MMP. The fixed zero semimodules provide essential information contained in solutions to MMP. For systems over a semiring, the fixed zeros cannot all be viewed as components in the extended zero structures of the solutions. However, we can still find that parts of the fixed zeros will appear in the extended zero semimodules of the solutions to MMP. For a discrete event system, a common Petri net component obtained from the solutions to MMP can be discovered from the fixed zero semimodules.

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