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Fixed poles in the model matching problem for systems over semirings[☆]

Ying Shang [a,*], Michael K. Sain [b,1]

[a] Department of Electrical and Computer Engineering, Southern Illinois University Edwardsville, Campus Box 1801, Edwardsville, IL 62026, USA

[b] Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA

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ABSTRACT

In this paper, solution existence conditions for the model matching problem are studied for systems over semirings, which are used in many applications, such as queueing systems, communication networks, and manufacturing systems. The main contribution is the discovery of fixed pole structure in solutions to the model matching problem. This fixed pole structure provides essential information contained in all the solutions to the model matching problem. For a discrete-event dynamic system example, a common Petri net component in the solutions of the model matching problem can be discovered from the fixed pole structure.

1. Introduction

A semiring can be understood as a set of objects not all of which have inverses with respect to the corresponding operators. There are many examples of this special algebraic structure, such as the max-plus algebra [3], the min-plus algebra [5], and the Boolean semiring [11]. Systems over semirings are systems evolving with variables taking values in semimodules over a semiring. Intuitively, such systems are not equipped with "additive inverses" and are used in many applications. For instance, systems over the max-plus algebra model queueing systems [6], systems over the min-plus algebra

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* Corresponding author. Tel.: +1 618 650 5193; fax: +1 618 650 3374.
E-mail addresses: yshang@siue.edu (Y. Shang), avemaria@nd.edu (M.K. Sain).

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model communication networks [5], and systems over the Boolean semiring model hysteretic discrete structural systems [21].

This paper focuses on the model matching problem (MMP) for systems over semirings. The goal of the MMP is to design a compensator for a given plant, or to design a plant for a given compensator, so that the response of the closed-loop system matches a given reference model. The MMP is indeed a well-studied problem for systems over a field. In 1951, just after the mid-century, Aaron [1] presented the MMP for single input and single output systems. Two decades later, work on the MMP began to intensify. Wolovich [30] precisely defined the MMP via linear feedback as the problem of finding a feedback pair such that the transfer function matrix of the closed-loop system is identical to a prescribed transfer function matrix. The solution existence condition was obtained using coordinate transformations and a factorization of the transfer function matrix. At about the same time, the MMP with static and dynamic state feedback controllers was studied using various methods, for instance, by solving linear algebraic equations [29], by geometric methods [17], and by solving rational matrix equations [28]. Continuing on into the next decade, MMP was addressed by solving matrix Diophantine equations [27], by utilizing the theory of inverses and state space algorithms, and so forth. Later, in the 1990s, the MMP was studied for more general classes of systems, such as linear systems with delays [20], periodic discrete-time systems [7], and nonlinear systems [14,15]. With this type of study proceeding into the new century, recently, the MMP for systems over a field has been applied to various application areas, for example, adaptive flight control systems [18], maneuver vehicles [19], and propulsion control aircraft [12].

The MMP for linear systems over a semiring, however, has not been investigated as well as for traditional linear systems. The MMP for systems over a special semiring, the max-plus algebra, has been studied in [9,16], in which residuation theory is used to characterize solutions. This paper studies systems over an arbitrary semiring and presents the fixed pole structures to characterize essential information in solutions to the MMP. It is a generalization of the study of the MMP for traditional linear systems over a field in the paper [8] by Conte et al. to systems over a semiring. In [8], the fixed pole modules are introduced and the relationship is established between the fixed pole modules and the pole modules of the solutions to the MMP. Moreover, the fixed pole modules characterize the "essential solutions" that are contained in any solutions to the MMP. The advantage of the coordinate free and module-theoretic definitions of poles and zeros [13,31] is that point poles and point zeros become pole and zero spaces. The pole and zero spaces are independent of whether the operators in the state space have inverses or not. Therefore, they can be generalized to the semimodule case, while the point poles and point zeros cannot be generalized due to lack of the subtraction operator. In this paper, two fixed pole structures are generalized to systems over a semiring, and relationships are established between these fixed poles and the pole semimodules of the solutions to MMP. These fixed pole structures characterize common components in the solutions to MMP. This observation is illustrated by a discrete-event dynamic system modeled as a linear system over the max-plus algebra. A common Petri net component can be generated from the fixed pole semimodule, and it is contained in the Petri net realizations of any solutions to the MMP.

The paper is organized as follows. Section 2 introduces some mathematical preliminaries. Section 3 defines systems over a semiring and its application to discrete-event systems. Section 4 studies solution existence conditions for the MMP. Section 5 defines pole semimodules, zero semimodules, and two fixed pole semimodules. Relationships are established between the fixed pole semimodules and the pole semimodules of the solutions to the MMP. These fixed pole semimodules provide essential information about the realizations of solutions. In Section 6, a discrete-event dynamic system is used to illustrate that essential information in the solutions to the MMP can be obtained by studying the fixed pole semimodules, and is characterized by a Petri net. Section 7 concludes this paper with future research.

2. Mathematical preliminaries

2.1. Semirings and semimodules

A monoid R is a semigroup $(R, \boxplus)$ with an identity element e_R with respect to the binary operation $\boxplus$. The term *semiring* means a set, $R = (R, \boxplus, e_R, \boxtimes, 1_R)$ with two binary associative operations, $\boxplus$ and $\boxtimes$, such that $(R, \boxplus, e_R)$ is a commutative monoid and $(R, \boxtimes, 1_R)$ is a monoid, which are connected by a two-sided distributive law of $\boxtimes$ over $\boxplus$. Moreover, $e_R \boxtimes r = r \boxtimes e_R = e_R$, for all r in R. Let $(R, \boxplus, e_R, \boxtimes, 1_R)$ be a semiring, and $(M, \square_M, e_M)$ be a commutative monoid. M is called a *left R-semimodule* if there exists a map $\mu : R \times M \to M$, denoted by $\mu(r, m) = rm$, for all $r \in R$ and $m \in M$, such that the following conditions are satisfied:

(1) $r(m_1 \square_M m_2) = rm_1 \square_M rm_2$;
(2) $(r_1 \boxplus r_2)m = r_1 m \square_M r_2 m$;
(3) $r_1(r_2 m) = (r_1 \boxtimes r_2)m$;
(4) $1_R\, m = m$;
(5) $r\, e_M = e_M = e_R\, m$;

for any $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. In this paper, e denotes the unit semimodule. A *sub-semimodule* K of M is a submonoid of M with $rk \in K$, for all $r \in R$ with $k \in K$. A sub-semimodule K of M is called *subtractive* if $k \in K$ and $k \square_M m \in K$ imply $m \in K$, for $m \in M$. An R-morphism between two semimodules $(M, \square_M, e_M)$ and $(N, \square_N, e_N)$ is a map $f : M \to N$ satisfying

(1) $f(m_1 \square_M m_2) = f(m_1) \square_N f(m_2)$;
(2) $f(rm) = rf(m)$;

for all m, m_1, $m_2 \in M$ and $r \in R$. The kernel of an R-semimodule morphism $f : M \to N$ is defined as $\operatorname{Ker} f = \{x \in M | f(x) = e_N\}$.

Let N be a subset of a R-semimodule $(M, \square_M, e_M)$. We denote N_0 as the set of all elements of the form $\square_{Mi}\lambda_i n_i$ where the n_i are elements in N, the λ_i are elements in R, and the i are elements in an index set I. The sub-semimodule N_0 is said to be generated by N, and N is called a system of generators of N_0. The subset N of an R-semimodule M is called *linearly independent* if $\square_{Mi}\lambda_i n_i = \square_{Mi}\beta_i n_i$ implies $\lambda_i = \beta_i$ for all $i \in I$. An R-semimodule M is called a free R-semimodule if it has a linearly independent subset N which generates M; and then N is called a basis of M. If N has a finite number of elements, M is called a finitely generated R-semimodule.

2.2. Bourne relation, image and proper image

The Bourne relation is introduced in [11, p. 164] for an R-semimodule. If K is a sub-semimodule of an R-semimodule M, then the *Bourne relation* is defined by setting $m \equiv_K m'$ if and only if there exist two elements k and k' of K such that $m \square_M k = m' \square_M k'$. The factor semimodule $M/\equiv_K$ induced by $\equiv_K$ is also written as M/K.

If K is equal to the kernel of an R-semimodule morphism $f : M \to N$, then $m \equiv_{\operatorname{Ker} f} m'$ if and only if there exist two elements k, k' of $\operatorname{Ker} f$, such that $m \square_M k = m' \square_M k'$. Applying f on both sides, we obtain that $f(m) = f(m')$, i.e. $m \equiv_f m'$. Hence this special Bourne relation and the relation induced by the morphism f satisfy the partial order $\leqslant$, i.e. $\equiv_{\operatorname{Ker} f} \leqslant \equiv_f$. In general, we do not have $\equiv_f \leqslant \equiv_{\operatorname{Ker} f}$ for an R-semimodule morphism f. If an R-semimodule morphism $f : M \to N$ satisfies $\equiv_f \leqslant \equiv_{\operatorname{Ker} f}$, then f is called a *steady* or *k-regular* R-semimodule morphism.

There are two different kinds of images for R-semimodule morphisms [26]. Given an R-semimodule morphism $f : M \to N$, one image is defined to be the set of all values $f(m)$, $m \in M$, i.e.

$$f(M) = \{n \in N | n = f(m), m \in M\}. \tag{1}$$

It is called the *proper image* of an R-semimodule morphism f. The other image of f is defined as

$$\operatorname{Im} f = \{n \in N | n \square_N f(m) = f(m') \text{ for some } m, m' \in M\}. \tag{2}$$

It is called the *image* of f to distinguish from the proper image. It is easy to see that the two images coincide for the module case. For the semimodule case, if the two images are the same, i.e. $f(M) = \operatorname{Im} f$, the R-morphism of semimodules $f : M \to N$ is called *i-regular*.

2.3. Exact and proper exact sequences

Given two R-semimodule morphisms $f:(A,\Box_A,e_A)\to(B,\Box_B,e_B)$ and $g:(B,\Box_B,e_B)\to(C,\Box_C,e_C)$, the sequence

$$A\xrightarrow{f}B\xrightarrow{g}C \tag{3}$$

is called an *exact sequence* if $\operatorname{Im} f=\operatorname{Ker} g$. Since $\operatorname{Im} f$ is not the same as the proper image $f(A)$ for the R-semimodule morphism f, the sequence is said to be a *proper exact sequence* if $f(A)=\operatorname{Ker} g$. For the module case, the image of f, $\operatorname{Im} f$, is the same as the proper image $f(A)$, so every exact sequence of modules is also proper exact.

Lemma 1. *Given an R-semimodule B and its sub-semimodule A, the following sequence*:

$$e\to A\xrightarrow{i}B\xrightarrow{p}B/A\to e \tag{4}$$

is a short exact sequence, i.e. it is exact at each semimodule, where i is an insertion and p is a natural projection.

Proof. Because i is an insertion and p is a projection, the sequence is exact at A and B/A. We only need to show $\operatorname{Im} i=\operatorname{Ker} p$. By definition, $\operatorname{Ker} p=\{b\in B|b\;\Box_B\;a_1=a_2,\ \text{where}\ a_1,\ a_2\in A\}=\operatorname{Im} i$. Therefore, the sequence is exact. □

Given two R-semimodules $(B,\Box_B,e_B)$ and $(C,\Box_C,e_C)$, a sequence of the form

$$B\xrightarrow{\beta}C\to e$$

is said to *split* if there is an R-semimodule morphism $\gamma:C\to B$ such that $\beta\circ\gamma=I_C$, where I_C denotes the identity map on C. Given R-semimodules $(A,\Box_A,e_A)$ and $(B,\Box_B,e_B)$, a sequence of the form

$$e\to A\xrightarrow{\alpha}B$$

is said to *split* if there is an R-semimodule morphism $\xi:B\to A$ such that $\xi\circ\alpha=I_A$, where I_A denotes the identity map on A. A short exact sequence of R-semimodule morphisms of the form:

$$e\to A\xrightarrow{\alpha}B\xrightarrow{\beta}C\to e \tag{5}$$

is said to *split* on the right (resp. on the left) if $B\to C\to e$ (resp. $e\to A\to B$) splits. Based on the splitting lemma for modules [4], that the above sequence is splitting on the left (resp. on the right) is equivalent to $B=A\oplus C$. However, for the semimodule case, even if the sequence splits on the left (or right), it does not mean that B is a direct sum of A and C without further assumptions on the morphisms. The reason is due to the differences between modules and semimodules; for instance, for the module case, there are inverses for the binary operators, there are no distinctions between exact and proper exact sequences, and no differences between image and proper image. Moreover, every R-module morphism is both i-regular and k-regular.

3. Systems over a semiring and its applications

3.1. Systems over a semiring

Systems over a semiring R are described by the following equations:

$$\begin{aligned} x(k+1)&=Ax(k)\;\Box_X\;Bu(k),\\ y(k)&=Cx(k)\;\Box_Y\;Du(k), \end{aligned} \tag{6}$$

where x is in the state semimodule X, y is in the output semimodule Y, and u is in the input semimodule U, which are all assumed to be free. $A:X\to X$, $B:U\to X$, $C:X\to Y$ and $D:U\to Y$ are four

R-semimodule morphisms. $R[z]$ denotes the polynomial semiring with coefficients in R and ΩX, ΩU, and ΩY denote the polynomial $R[z]$-semimodules of states, inputs, and outputs, respectively. ΩX is an alternative notation for the polynomial $R[z]$-semimodules $X[z]$ of states. For instance, given a sequence of states

$$\ldots, x(-2), x(-1), x(0), x(1), x(2), \ldots,$$

$\Omega X = x(0) \square_X x(1)z \square_X x(2)z^2 \square_X \cdots$ is isomorphic to a finite sequence of states starting from the time instant 0 to the future. Let $R(z)$ denote the set of formal laurent series in z^{-1}, with coefficients in R, with finite left support, and having the property that, for any element $a(z) \in R(z)$, there exists an element $r(z)$ in $R[z]$, such that $r(z)a(z) \in R[z]$. In like manner, let $X(z)$ denote the set of formal laurent series in z^{-1}, with coefficients in X, with finite left support, and having the property that, for any element $x(z)$ in $X(z)$, there exists an element $r(z)$ in $R[z]$ such that $r(z)x(z) \in \Omega X$. We define $U(z)$ and $Y(z)$ similarly to $X(z)$. The transfer function $G(z) : U(z) \to Y(z)$ of this system (6) is

$$\begin{aligned} G(z) &= C(z^{-1}A)^* B z^{-1} \square_Y D \\ &= CBz^{-1} \square_Y CABz^{-2} \square_Y \cdots \square_Y CA^{n-1}Bz^{-n} \square_Y \cdots \square_Y D. \end{aligned} \tag{7}$$

The star operator A^* for an $n \times n$ matrix mapping $A : X \to X$ is defined as

$$A^* = I_{n\times n} \square_X A \square_X \cdots \square_X A^n \square_X \cdots, \tag{8}$$

where the operator $\square_X$ is induced from the state semimodule X, and $I_{n\times n}$ denotes the identity matrix mapping from X to X. The transfer function $G(z)$, as defined in Eq. (7), is of course in a natural way an $R(z)$-morphism from the $R(z)$-semimodule $U(z)$ to the $R(z)$-semimodule $Y(z)$. Transfer functions are considered as $R(z)$-morphisms in the definitions of the MMP and the solution existence conditions. In other parts of the paper, however, the mappings may be taken as $R[z]$-“linear” instead of $R(z)$-“linear”.

3.2. Discrete-event system application: queueing system example

A special semiring, the max-plus algebra, has been used to model a class of discrete-event systems evolving with time, such as queueing systems, transportation networks, and communication systems. The max-plus algebra is a set of real numbers, where the traditional addition and multiplication are replaced by the max operation and plus operation, i.e.

$$\begin{aligned} &\text{Addition:} \quad a \oplus b \equiv \max\{a, b\}, \\ &\text{Multiplication:} \quad a \otimes b \equiv a + b. \end{aligned}$$

The max-plus algebra is usually denoted by $\mathbb{R}_{\text{Max}} = (\mathbb{R} \cup \{\epsilon\}, \oplus, \epsilon, \otimes, e)$, where $\mathbb{R}$ denotes the set of real numbers, $\epsilon = -\infty$, and $e = 0$.

Consider a queueing system with one server [6] and its Petri net model as shown in Fig. 1. Petri nets are often used to model discrete-event systems. A *Petri net* is a four-tuple (P, T, A, w) where P is a finite set of *places*; T is a finite set of *transitions*; A is a set of *arcs*, a subset of the set $(P \times T) \cap (T \times P)$, and w is a *weight function*, $w : A \to \{1, 2, 3, \ldots\}$. We use $I(t_j)$ to represent the set of input places to transition t_j and $O(t_j)$ to represent the set of output places from transition t_j and

$$I(t_j) = \{p_i : (p_i, t_j) \in A\} \quad \text{and} \quad O(t_j) = \{p_i : (t_j, p_i) \in A\}.$$

A marking x of a Petri net is a function $x : P \to \{0, 1, 2, \ldots\}$. The number represents how many tokens are in a place. A marked Petri net is a five-tuple (P, T, A, w, x_0) where (P, T, A, w) is a Petri net and x_0 is the initial marking. For a timed Petri net, when the transition t_j is enabled for the kth time, it does not fire immediately, but it has a firing delay, $v_{j,k}$, during which the tokens are kept in the input places of t_j. The clock structure associated with the set of timed transitions, $T_D \subseteq T$, of a marked Petri net (P, T, A, w, x) is a set $\mathbf{V} = \{\mathbf{v}_j : t_j \in T_D\}$ of lifetime sequences $\mathbf{v}_j = \{v_{j,1}, v_{j,2}, \ldots\}$, $t_j \in T_D$, $v_{j,k} \in \mathbb{R}^+$, $k = 1, 2, \ldots$. A *timed Petri net* is a six-tuple $(P, T, A, w, x, \mathbf{V})$ where (P, T, A, w, x) is a marked Petri net and $\mathbf{V} = \{\mathbf{v}_j : t_j \in T_D\}$ is a clock structure.

The Petri net model of the queuing system in Fig. 1 has three places: Q (queue), I (idle), and B (busy), two timed transitions: a (customer arrives) and d (service completes and customer departs),

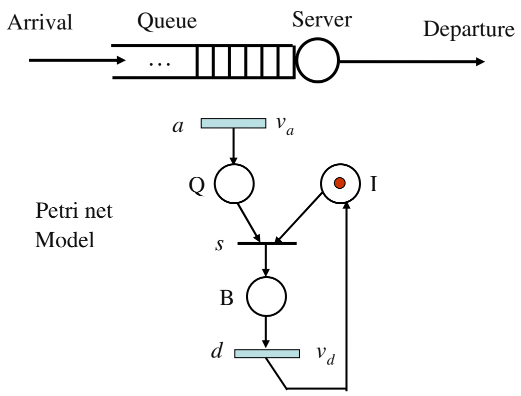

Fig. 1. A queueing system with one server [6].

and one transition s (service starts) without time delays. The clock structure of this model has constant sequences $\mathbf{v}_a = \{C_a, C_a, \ldots\}$ and $\mathbf{v}_d = \{C_d, C_d, \ldots\}$. The rectangles present the timed transitions. The initial marking is $x_0 = \{0, 1, 0\}$ in the order of (Q, I, B). Using the max-plus algebra, the $k + 1$th arrival time a_{k+1} and the kth departure time d_k, $k = 0, 1, 2, \ldots$, can be described by

$$\begin{bmatrix} a_{k+1} \\ d_k \end{bmatrix} = \begin{bmatrix} C_a + a_k, \\ \max\{a_k, d_{k-1}\} + C_d \end{bmatrix}, \quad \text{where } \begin{bmatrix} a_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} C_a \\ 0 \end{bmatrix}.$$

Defining

$$x_k = \begin{bmatrix} a_{k+1} \\ d_k \end{bmatrix}, \quad A = \begin{bmatrix} C_a & \epsilon \\ C_d & C_d \end{bmatrix} \Longrightarrow x_{k+1} = A \otimes x_k,$$

which is a linear system over the max-plus algebra with no inputs. The queuing system example is used to illustrate the connection between the semiring system theory and discrete-event systems. In discrete-event systems, the basic phenomenon is the nonlinear and non-smooth synchronization. The advantage of the semiring theory in discrete-event systems is that it brings back the linearity, so nonlinear models can be studied by generalizing the traditional linear system theory to systems over a semiring. Existing concepts, such as controllability, observability, stabilization, and feedback synthesis, can be revisited to study discrete-event systems.

4. Solution existence conditions

The MMPs for systems over a semiring R can be described as follows. Given two transfer functions, $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$, one is to find another transfer function, $M(z) : C(z) \to U(z)$; or given $T(z)$ and $M(z)$, another is to find another transfer function $P(z)$, such that the following model matching equation is satisfied:

$$T(z) = P(z)\, M(z), \tag{9}$$

which is illustrated in the commutative diagrams of Fig. 2.

If F is a field and $F(z)$ denotes the induced quotient field of the polynomial ring $F[z]$, then, for traditional systems over a field, the existence conditions for the MMP are stated as follows. Given two transfer functions $T(z)$ and $P(z)$, then there exists an $F(z)$-linear map $M(z)$ such that $P(z)M(z) = T(z)$ if and only if the image $T(z)C(z)$ is a subspace of the image $P(z)U(z)$, namely $T(z)C(z) \subset P(z)U(z)$. Given two transfer functions $T(z)$ and $M(z)$, then there exists an $F(z)$-linear map $P(z)$ such that $P(z)M(z) = T(z)$ if and only if the kernel of $M(z)$ is contained in the kernel of $T(z)$, namely $\operatorname{Ker} M(z) \subset \operatorname{Ker} T(z)$. For systems over a semiring, the image and the kernel inclusion conditions are necessary but not sufficient.

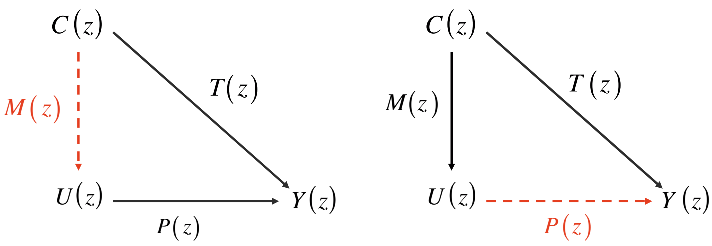

Fig. 2. The MMP commutative diagrams.

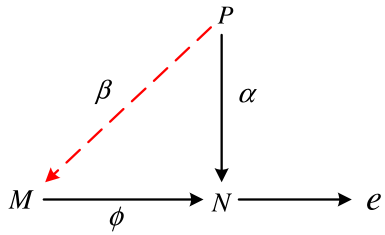

Fig. 3. The projective semimodule definition.

4.1. Model matching with unknown controller

If a reference transfer function $T(z) : C(z) \to Y(z)$ and a plant $P(z) : U(z) \to Y(z)$ are given, a controller $M(z) : C(z) \to U(z)$ is sought to satisfy the model matching equation $P(z)M(z) = T(z)$. The image inclusion condition $T(z)C(z) \subset P(z)U(z)$, though necessary, is not sufficient for the existence of an $R(z)$-semimodule morphism $M(z)$. With the help of the projective assumption on the domain of $T(z)$, an $R(z)$-semimodule morphism $M(z)$ to the MMP exists.

A left R-semimodule P is *projective* if and only if the following condition holds: if $\phi : M \to N$ is a surjective R-semimodule morphism, that is, an R-semimodule epimorphism, and $\alpha : P \to N$ is an R-semimodule morphism, then there exists an R-semimodule morphism $\beta : P \to M$ such that $\phi \circ \beta = \alpha$ as shown in Fig. 3.

Proposition 1. *Given two $R(z)$-semimodule morphisms $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$, suppose that $C(z)$ is projective; then there exists an $R(z)$-semimodule morphism $M(z) : C(z) \to U(z)$ such that $P(z)M(z) = T(z)$ if and only if $T(z)C(z) \subset P(z)U(z)$.*

Proof. If there exists an $R(z)$-semimodule morphism $M(z)$ such that $P(z)M(z) = T(z)$, then $T(z)C(z) \subset P(z)U(z)$. This statement is directly obtained from Theorem 3.3 in [4, p. 21]. On the other hand, if $T(z)C(z) \subset P(z)U(z)$ and we assume the projectivity of $C(z)$, there exists an $R(z)$-morphism $M(z) : C(z) \to U(z)$ such that $\overline{P}(z)M(z) = \overline{T}(z)$, where $\overline{P}(z) : U(z) \to P(z)U(z)$ and $\overline{T}(z) : C(z) \to P(z)U(z)$ are defined as $u(z) \mapsto P(z)u(z)$ and $c(z) \mapsto T(z)c(z)$, respectively, for arbitrary elements $u(z) \in U(z)$ and $c(z) \in C(z)$. Since $P(z) = i \circ \overline{P}(z)$ and $T(z) = i \circ \overline{T}(z)$, we obtain that $i \circ \overline{P}(z)M(z) = i \circ \overline{T}(z)$, which implies $P(z)M(z) = T(z)$. □

Proposition 1 states the existence conditions for an $R(z)$-semimodule morphism $M(z)$ to the MMP. Because every free semimodule is projective, which is stated in Proposition 17.14 in [11, p. 195], the result can be restated as the following corollary.

Corollary 1. *Given two $R(z)$-semimodule morphisms $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$, suppose that $C(z)$ is a free $R(z)$-semimodule, then there exists an $R(z)$-semimodule morphism $M(z) : C(z) \to U(z)$ such that the model matching equation $P(z)M(z) = T(z)$ holds if and only if $T(z)C(z) \subset P(z)U(z)$.*

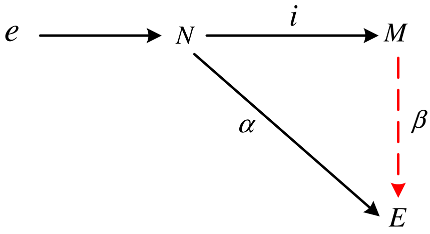

Fig. 4. The injective semimodule definition.

Proposition 1 and Corollary 1 give us motivation to add the freedom assumption on the $R(z)$-semimodule $C(z)$. In other words, if we assume that the domain of the transfer function $T(z)$ is free, then the existence condition for a controller $M(z)$ to the MMP is the image inclusion condition, that is, $T(z)C(z) \subset P(z)U(z)$, which is the same as for the traditional systems over a field. In the definition of systems over a semiring in Eq. (6), the state semimodule X, the input semimodule U, and the output semimodule Y are free. Therefore, $X(z)$, $U(z)$, and $Y(z)$ are $R(z)$-free, which means any element in $X(z)$, $U(z)$, and $Y(z)$ can be uniquely expressed as a linear combination of the basis of X, U, and Y with coefficients in $R(z)$, respectively. Of course that does not mean that $C(z)$ is free, but it seems quite appropriate to treat C as free, like X, U, and Y, in the context of MMP. For these systems over a semiring, the solution existence condition for MMP with unknown controller is the same as that for traditional systems over a field in [8].

4.2. Model matching with unknown plant

If a reference transfer function $T(z)$ and a controller $M(z)$ are given, we wish to find a plant $P(z)$ satisfying the model matching equation $P(z)M(z) = T(z)$. Unlike the results in [8], the kernel inclusion condition, $\operatorname{Ker} M(z) \subset \operatorname{Ker} T(z)$, is not sufficient to guarantee the existence of an $R(z)$-semimodule morphism $P(z)$. However, with an injective assumption on $Y(z)$, the codomain of $T(z)$, the necessary and sufficient condition exists for the existence of a transfer function $P(z)$.

A left R-semimodule E is called *injective* [11] if and only if, given a left R-semimodule M and its sub-semimodule N, any R-semimodule morphism α from N to E can be extended to an R-semimodule morphism β from M to E as shown in Fig. 4. Proposition 2 states the existence conditions for an $R(z)$-semimodule solution morphism $P(z)$ to MMP.

Proposition 2. *Given two $R(z)$-semimodule morphisms $T(z) : C(z) \to Y(z)$ and $M(z) : C(z) \to U(z)$, suppose that $Y(z)$ is injective. Then there exists an $R(z)$-morphism $P(z) : U(z) \to Y(z)$ such that $P(z)M(z) = T(z)$ if and only if the following condition is satisfied*:

$$\forall x, y \in C(z), \quad M(z)x = M(z)y \Longrightarrow T(z)x = T(z)y. \tag{10}$$

Moreover, if $Y(z)$ is injective and $M(z)$ is steady, namely k-regular, then there exists such an $R(z)$-semimodule morphism $P(z)$ if and only if $\operatorname{Ker} M(z) \subset \operatorname{Ker} T(z)$.

Proof. If there exists an $R(z)$-semimodule morphism $P(z) : U(z) \to Y(z)$ such that $P(z)M(z) = T(z)$, then the condition in Eq. (10) is satisfied because of Theorem 3.3 in [4, p. 21]. On the other hand, if the condition in Eq. (10) is satisfied, by Theorem 3.3 in [4, p. 21], there exists a mapping $\overline{P}(z)$ such that $\overline{P}(z)\overline{M}(z) = T(z)$, where $\overline{M}(z) : C(z) \to M(z)C(z)$ is an $R(z)$-semimodule morphism defined by the action $\overline{M}(z) : c(z) \mapsto M(z)c(z)$, $c(z) \in C(z)$ and $\overline{P}(z) : M(z)C(z) \to Y(z)$ is defined by the action $\overline{P}(z) : M(z)c(z) \mapsto T(z)c(z)$, $c(z) \in C(z)$. The mapping $\overline{P}(z)$ is an $R(z)$-semimodule morphism because, for any $M(z)c(z)$, $M(z)c_1(z)$, $M(z)c_2(z)$ in $M(z)C(z)$, and $r(z) \in R(z)$, the following statements hold:

$$\begin{aligned}\overline{P}(z)(M(z)c_1(z) \,\square_{U(z)}\, M(z)c_2(z)) &= \overline{P}(z)(\overline{M}(z)c_1(z) \,\square_{Y(z)}\, \overline{M}(z)c_2(z))\\ &= \overline{P}(z)\overline{M}(z)(c_1(z) \,\square_{C(z)}\, c_2(z))\\ &= T(z)(c_1(z) \,\square_{C(z)}\, c_2(z))\\ &= T(z)c_1(z) \,\square_{Y(z)}\, T(z)c_2(z)\\ &= \overline{P}(z)\overline{M}(z)c_1(z) \,\square_{Y(z)}\, \overline{P}(z)\overline{M}(z)c_2(z)\\ &= \overline{P}(z)M(z)c_1(z) \,\square_{Y(z)}\, \overline{P}(z)M(z)c_2(z),\end{aligned}$$

$$\begin{aligned}\overline{P}(z)(r(z)M(z)c(z)) &= \overline{P}(z)(r(z)\overline{M}(z)c(z))\\ &= \overline{P}(z)\overline{M}(z)(r(z)c(z))\\ &= T(z)(r(z)c(z)) = r(z)T(z)c(z)\\ &= r(z)\overline{P}(z)\overline{M}(z)c(z)\\ &= r(z)\overline{P}(z)(M(z)c(z)),\end{aligned}$$

where $\square_{C(z)}$, $\square_{U(z)}$, and $\square_{Y(z)}$ are operations on $C(z)$, $U(z)$, and $Y(z)$. With the injective assumption on $Y(z)$, the $R(z)$-semimodule morphism $\overline{P}(z) : M(z)C(z) \to Y(z)$ can be extended to $P(z) : U(z) \to Y(z)$.

In the module case, the kernel inclusion condition $\operatorname{Ker} M(z) \subset \operatorname{Ker} T(z)$ is equivalent to the condition in Eq. (10). Thus, the kernel inclusion condition in the module case guarantees the existence of a mapping $P(z)$. This is not the case for semimodules; the kernel condition does not imply the condition in Eq. (10) due to lack of an inverse for the $\square_{C(z)}$ operation. However, if $\operatorname{Ker} M(z) \subset \operatorname{Ker} T(z)$ and $M(z)$ is steady, then $M(z)(x) = M(z)(y) \iff x \,\square_{C(z)}\, k_1 = y \,\square_{C(z)}\, k_2$, where k_1, k_2 are in $\operatorname{Ker} M(z)$. Apply $T(z)$ on both sides of this equality to obtain, $T(z)(x) \,\square_{Y(z)}\, T(z)(k_1) = T(z)(y) \,\square_{Y(z)}\, T(z)(k_2) \iff T(z)(x) = T(z)(y)$, because k_1 and k_2 are in $\operatorname{Ker} T$. Hence, the kernel inclusion condition implies Eq. (10). Therefore, there exists an $R(z)$-semimodule morphism $\overline{P}(z) : M(z)C(z) \to Y(z)$ such that the model matching equation is satisfied. The injectivity assumption on the codomain of $T(z)$ guarantees that, for a sub-semimodule $M(z)C(z)$ of $U(z)$, an $R(z)$-semimodule morphism $\overline{P}(z) : M(z)C(z) \to Y(z)$ can be extended to an $R(z)$-semimodule morphism $P(z) : U(z) \to Y(z)$. With the injective and steady assumptions, the existence condition for the MMP is the same as that for the module case. □

In summary, in the case of the MMP with an unknown controller, the projective assumption is consistent with the freedom assumption on the state semimodule, the input semimodule, and the output semimodule for the system over a semiring. Moreover, the freedom assumption is consistent with the freedom assumption in traditional systems over a field. For systems so defined in this paper, the existence condition of an $R(z)$-semimodule morphism $M(z)$ is then essentially the same as in [8]. In the MMP with an unknown plant, we add a less intuitive assumption, namely injectivity on the codomain of $T(z)$, to obtain the existence condition. However, the two cases of the MMP can be viewed as dual problems of each other. The less intuitive injective assumption can be replaced by the freedom assumption for its dual problem.

5. Fixed pole structures

This section presents suitable algebraic notions of “fixed poles” for the MMP for both cases in Fig. 2. These fixed poles allow us to describe certain essential structure of the solutions to the MMP. The main results are generalizations of the study of the MMP for traditional linear systems over a field in the paper [8] by Conte et al. to systems over a semiring. In [8], the fixed pole modules are introduced and the relationship is established between the fixed pole modules and the pole modules of the solutions to the MMP. Moreover, the fixed pole modules characterize the “essential solutions” that will be contained in any solutions to the MMP.

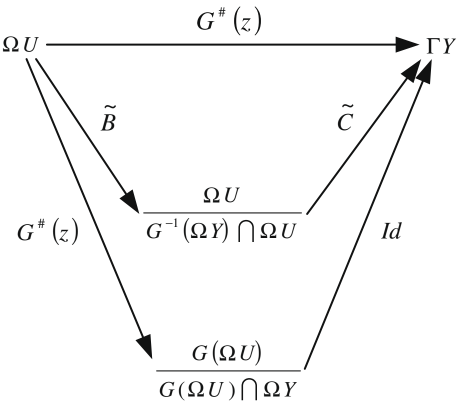

Fig. 5. A generalized Kalman realization diagram.

5.1. Pole semimodules of input and output type

This section reviews the definitions of pole semimodules of input and output type in [22,24]. Using the standard construction in automata theory, the states of a system can be viewed as equivalence classes in the input semimodule induced by the transfer function. If we consider the Kalman input/output map $G^{\#}(z) : \Omega U \to \Gamma Y$, where $\Gamma Y = Y(z)/\Omega Y$ is a factor semimodule induced by ΩY, then we can obtain the commutative diagram shown in Fig. 5. In this diagram, the pole semimodule of output type is defined as

$$X_O(G) = \frac{G(\Omega U)}{G(\Omega U) \cap \Omega Y}. \tag{11}$$

The pole semimodule of input type is defined as

$$X_I(G) = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}. \tag{12}$$

$X_I(G)$ is actually $\Omega U/\mathrm{Ker}\, G^{\#}$, because $\mathrm{Ker}\, G^{\#} = G^{-1}(\Omega Y) \cap \Omega U$. The mappings $\widetilde{B} : \Omega U \to X$ and $\widetilde{C} : X \to \Gamma Y$ are defined by

$$\widetilde{B}(z \cdot u) = ABu;$$
$$\widetilde{C}(x) = Cxz^{-1} \,\square_{Y(z)}\, C(Ax)z^{-2} \,\square_{Y(z)}\, C(A^2x)z^{-3} \,\square_{Y(z)}\, \cdots \bmod \Omega Y.$$

The mapping *Id* is an identity map and $\square_{Y(z)}$ is the operator in the $R(z)$-semimodule $Y(z)$ induced by the output semimodule Y. The pole semimodules $X_I(G)$ and $X_O(G)$ can have $R[z]$-semimodule structure if, for any polynomial $r(z) \in R[z]$, the scalar multiplication is defined by the action $r(z)x = r(A)x$. In the module case, each formula has been used by the preference of the researchers, because $X_I(G)$ is isomorphic to $X_O(G)$. However, for the semimodule case, there exists an $R[z]$-semiisomorphism between $X_I(G)$ and $X_O(G)$ instead, i.e. an unit kernel $R[z]$-semimodule epimorphism.

Lemma 2 [24]. *Given a transfer function $G(z) : U(z) \to Y(z)$ and the pole semimodules of input and output type, there exists an $R[z]$-semimodule semiisomorphism, that is an unit kernel $R[z]$-semimodule epimorphism, from $X_I(G)$ to $X_O(G)$.*

Example 1. This example is to illustrate the relationship between the pole semimodule and the matrix A in the state space realization. Consider a transfer function $G(z) : U(z) \to Y(z)$ over the max-plus algebra as

$$G(z) = \begin{bmatrix} z^{-2} & \epsilon \\ \epsilon & \epsilon \end{bmatrix}.$$

Consider an operation by z upon the columns of this transfer function: the first column is

$$z\begin{bmatrix} z^{-2} \\ \epsilon \end{bmatrix} = \begin{bmatrix} z^{-1} \\ \epsilon \end{bmatrix}, \quad \text{then } z\begin{bmatrix} z^{-1} \\ \epsilon \end{bmatrix} = \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix}.$$

The procedure stops at this point because linear dependency occurs, and the basis of the state space X is

$$e_1 = \begin{bmatrix} z^{-2} \\ \epsilon \end{bmatrix}, \quad e_2 = \begin{bmatrix} z^{-1} \\ \epsilon \end{bmatrix}.$$

This is because $z : X(z) \to X(z)$ is equivalent to $A : X \to X$. The state matrix A can be obtained as

$$A = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

5.2. Fixed poles of the solution $M(z)$ to $T(z) = P(z)M(z)$

We are given two transfer functions $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$. Assume that they satisfy the existence conditions in Proposition 1. Then there exists an $R(z)$-semimodule morphism $M(z)$ such that the model matching equation $T(z) = P(z)M(z)$ is satisfied. To study the pole structure of this solution $M(z)$, an $R[z]$-semimodule $P(T,P)$ is defined as follows:

$$P(T,P) = \frac{T(\Omega C) \;\Box_{Y(z)}\; P(\Omega U)}{P(\Omega U)}. \tag{13}$$

The quotient structure in the $R[z]$-semimodule $P(T,P)$ is obtained by means of the *Bourne relation*.

Definition 1. The semimodule $P(T,P)$ in Eq. (13) is called the fixed pole semimodule for the solution $M(z)$ to the model matching equation $T(z) = P(z)M(z)$.

The same terminology, fixed poles, is adopted from systems over a field in [8], in which the fixed pole module behaves as a factor module. The reason is that, in the classical case, there exists a natural projection from the pole semimodule of output type of a solution to the MMP to the fixed pole structure $P(T,P)$. Therefore, a splitting short exact sequence can be constructed using the kernel of the natural projection. Splitting lemma guarantees that the pole semimodule of output type of a solution will contain the fixed pole structure, which behaves as a factor module. However, the splitting lemma does not hold for the semimodule case, therefore, the fixed pole semimodule does not behave as a factor semimodule of the pole semimodule for the solution $M(z)$. But there are nonetheless connections between the structures of the fixed poles and the pole semimodule of the solution $M(z)$. The fixed pole semimodule helps us discover essential components in the solutions to MMP, which will be illustrated later with a discrete-event system example. The $R[z]$-semimodule $P(T,P)$ is a fixed pole structure because it has a direct relation with the pole semimodule of output type for the solution $M(z)$.

Proposition 3. *For the MMP with two known transfer functions $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$ satisfying the conditions in Corollary 1, there exists an $R[z]$-epimorphism, that is a surjective $R[z]$-semimodule morphism, from the pole semimodule of output type $X_O(M)$ of the solution $M(z)$ to the $R[z]$-semimodule $P(T,P)$.*

Proof. We need to prove that there exists an $R[z]$-semimodule epimorphism $\overline{P}(z) : X_O(M) \to P(T,P)$ such that the following diagram:

$$\begin{array}{ccccc}
M(\Omega C) & \xrightarrow{P(z)} & T(\Omega C) \;\Box_{Y(z)}\; P(\Omega U) & & \\
\Big\downarrow p_1 & & \Big\downarrow p_2 & & \\
\dfrac{M(\Omega C)}{M(\Omega C)\cap \Omega U} & \xrightarrow{\overline{P}(z)} & \dfrac{T(\Omega C) \;\Box_{Y(z)}\; P(\Omega U)}{P(\Omega U)} & \longrightarrow & e
\end{array}$$

is commutative. By definition, we obtain the kernel of the natural projection p_1 as

$$\operatorname{Ker} p_1 = \{u \in M(\Omega C) | u \,\square_{U(z)}\, u_1 = u_2, u_1, u_2 \in M(\Omega C) \cap \Omega U\}.$$

Since ΩU is subtractive, the element u in $\operatorname{Ker} p_1$ is also in ΩU, that is, $\operatorname{Ker} p_1 = M(\Omega C) \cap \Omega U$. The kernel of $p_2 \circ P(z)$ is given by

$$\operatorname{Ker} p_2 \circ P(z) = \{u \in M(\Omega C) | P(u) \,\square_{Y(z)}\, P(u_1) = P(u_2), u_1, u_2 \in \Omega U\}.$$

It is easy to see that $\operatorname{Ker} p_1 \subset \operatorname{Ker} p_2 \circ P(z)$. By the Factor Theorem in [2, p. 50] and Theorem 2 in [22], there exists an $R[z]$-semimodule morphism $\overline{P}(z)$ such that the previous diagram is commutative. This morphism is defined by the action

$$\overline{P}(z) : \frac{M(z)c}{\Omega U} \mapsto \frac{T(z)c}{\Omega U}, \quad c \in \Omega C.$$

The morphism $\overline{P}(z)$ is surjective because $p_2 \circ P(z)$ is surjective. Hence, $\overline{P}(z)$ is an $R[z]$-semimodule epimorphism. □

There is an alternative form of the pole semimodule of output type, which is semiisomorphic to $X_O(M)$, for a given transfer function $M(z) : C(z) \to U(z)$. The following proposition states that there exists also an $R[z]$-semimodule epimorphism from the alternative pole semimodule of output type to $P(T, P)$. The proof is a direct generalization of Proposition 3, so it is omitted here.

Proposition 4. *For the MMP with two known transfer functions $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$ satisfying the conditions in Corollary 1, there exists a solution $M(z) : C(z) \to U(z)$ satisfying the model matching equation $T(z) = P(z)\, M(z)$. Moreover, there exists an $R[z]$-semimodule epimorphism from the alternative form*

$$X'_O(M) = \frac{M(\Omega C) \,\square_{U(z)}\, \Omega U}{\Omega U}$$

of the pole semimodule of output type for $M(z)$ to the $R[z]$-semimodule $P(T, P)$.

In the remainder of the subsection, a description is given for the fixed pole semimodule $P(T, P)$ in terms of pole and zero features of the transfer functions $T(z)$ and $P(z)$. The pole semimodule of output type for $P(z)$ is

$$X_O(P) = \frac{P(\Omega U)}{P(\Omega U) \cap \Omega Y}$$

and the zero semimodule of input type for $P(z)$ [23] may be defined by

$$Z_I(P) = \frac{P(U(z)) \cap \Omega Y}{P(\Omega U) \cap \Omega Y}.$$

Define a new $R(z)$-semimodule morphism $[T(z)\ P(z)] : C(z) \times U(z) \to Y(z)$ by

$$[T(z)\ P(z)] : (c(z), u(z)) \mapsto T(z)c(z) \,\square_{Y(z)}\, P(z)u(z).$$

The pole semimodule of output type for $[T(z)\ P(z)]$ is

$$X_O([T(z)\ P(z)]) = \frac{T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}.$$

The zero semimodule of input type for this new $R(z)$-morphism is

$$Z_I([T\ P]) = \frac{(T(C(z)) \,\square_{Y(z)}\, P(U(z))) \cap \Omega Y}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y} = \frac{P(U(z)) \cap \Omega Y}{(T(\Omega C) \,\square_{Y(z)}\, P(\Omega U)) \cap \Omega Y}.$$

The second equality is true because $T(z)C(z) \subset P(z)U(z)$.

Using Lemma 1 the fixed pole semimodule $P(T,P)$ can be characterized by zero semimodules and pole semimodules of the transfer functions $T(z)$ and $P(z)$.

Theorem 1. *Consider two transfer functions $T(z) : C(z) \to Y(z)$ and $P(z) : U(z) \to Y(z)$ satisfying the conditions in Corollary 1. Then there exist $R[z]$-semimodules Z_1 and P_1, and $R[z]$-semimodule morphisms α, β, ϕ, and ψ such that the following three sequences are exact:*

$$e \to Z_1 \xrightarrow{i} Z_I(P) \xrightarrow{p} Z_I([T\ P]) \to e; \tag{14}$$

$$e \to X_O(P) \xrightarrow{\alpha} X_O([T\ P]) \xrightarrow{\beta} P_1 \to e; \tag{15}$$

$$Z_1 \xrightarrow{\phi} P(T,P) \xrightarrow{\psi} P_1 \to e, \tag{16}$$

where Z_1 and P_1 are defined as

$$Z_1 = \frac{(T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y}{P(\Omega U) \cap \Omega Y}, \tag{17}$$

$$P_1 = \frac{T(\Omega C) \Box_{Y(z)} P(\Omega U)}{(T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y \Box_{Y(z)} P(\Omega U)}. \tag{18}$$

The map ϕ has a unit kernel if $P(\Omega U)$ is subtractive, and then the last sequence becomes

$$e \to Z_1 \xrightarrow{\phi} P(T,P) \xrightarrow{\psi} P_1 \to e, \tag{19}$$

which is a short exact sequence.

Proof. The first sequence is exact directly from Lemma 1. We only need to prove the second and the third sequences to be exact. The $R[z]$-semimodule morphisms $\alpha : X_O(P) \to X_O([T\ P])$ and $\phi : Z_1 \to P(T,P)$ are induced by insertions $i_1 : P(\Omega U) \to T(\Omega C) \Box_{Y(z)} P(\Omega U)$ and $i_2 : (T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y \to T(\Omega C) \Box_{Y(z)} P(\Omega U)$. The kernel of α is a unit $R[z]$-semimodule; but the kernel of ϕ is not a unit $R[z]$-semimodule, because $(T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y$ is subtractive but $P(\Omega U)$ is not subtractive. The $R[z]$-semimodule morphisms $\beta : X_O([T\ P]) \to P_1$ and $\psi : P(T,P) \to P_1$ are induced by the identity morphism. The morphisms ψ and β are surjective.

Next we prove the exactness at $X_O([T\ P])$ in the sequence Eq. (15), namely $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$. By definition, the kernel $\mathrm{Ker}\,\beta$ and the image $\mathrm{Im}\,\alpha$, respectively are

$$\begin{aligned}\mathrm{Ker}\,\beta = \Big\{ &\frac{y}{(T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y},\ y \in T(\Omega C) \Box_{Y(z)} P(\Omega U) |\\ &y \Box_{Y(z)} \bar{y}_1 = \bar{y}_2,\\ &\bar{y}_1, \bar{y}_2 \in (T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y \Box_{Y(z)} P(\Omega U) \Big\},\\ \mathrm{Im}\,\alpha = \Big\{ &\frac{y}{(T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y},\ y \in T(\Omega C) \Box_{Y(z)} P(\Omega U) |\\ &y \Box_{Y(z)} y_1 \Box_{Y(z)} l_1 = y_2 \Box_{Y(z)} l_2,\ y_1, y_2 \in P(\Omega U),\\ &l_1, l_2 \in (T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y \Big\}.\end{aligned}$$

Notice that, in the definition of $\mathrm{Im}\,\alpha$, $y_1 \Box_{Y(z)} l_1$ and $y_2 \Box_{Y(z)} l_2$ are elements in $(T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y \Box_{Y(z)} P(\Omega U)$ and they can be viewed as $\bar{y}_1$ and $\bar{y}_2$ in the definition of $\mathrm{Ker}\,\beta$. Therefore, we have $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$. Thus, the sequence in Eq. (15) is exact.

We still need to show the exactness at $P(T,P)$ of the last sequence in Eq. (16), i.e. $\mathrm{Im}\,\phi = \mathrm{Ker}\,\psi$. By definition, the kernel $\mathrm{Ker}\,\psi$ and the image $\mathrm{Im}\,\phi$, respectively are

$$\begin{aligned}\mathrm{Ker}\,\psi = \Big\{ &\frac{y}{P(\Omega U)},\ y \in T(\Omega C) \Box_{Y(z)} P(\Omega U) | y \Box_{Y(z)} y_1 = y_2,\\ &y_1, y_2 \in (T(\Omega C) \Box_{Y(z)} P(\Omega U)) \cap \Omega Y \Box_{Y(z)} P(\Omega U) \Big\},\end{aligned}$$

$$\operatorname{Im}\phi = \left\{ \frac{y}{P(\Omega U)},\ y \in T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U) \mid \right.$$
$$y \,\Box_{Y(z)}\, y_1 \,\Box_{Y(z)}\, l_1 = y_2 \,\Box_{Y(z)}\, l_2,$$
$$\left. y_1, y_2 \in (T(\Omega C) \,\Box_{Y(z)}\, P(\Omega U)) \cap \Omega Y,\ l_1, l_2 \in P(\Omega U) \right\}.$$

This completes the proof for the sequence in Eq. (16) to be exact. If $P(\Omega U)$ is subtractive, the kernel of ϕ is the unit $R[z]$-semimodule, so the sequence Eq. (16) is a short exact sequence. □

In summary, this subsection focuses on the MMP with an unknown controller $M(z)$ and studies the fixed pole structure of the solutions $M(z)$ to the model matching equation $T(z) = P(z)M(z)$. The three sequences in Eqs. (14)–(16) (or Eq. (19)) permit one to give insight into the nature of $P(T, P)$. In the results for traditional systems over a field [8], the sequence in Eq. (19) is always true without further assumptions on $P(\Omega U)$. For systems over a semiring, we can make the subtractive assumption to obtain the short exact sequence in Eq. (19). For the traditional case, the sequence in Eq. (19) is not only exact but also splits, therefore, the fixed pole structure $P(T, P)$ is a direct sum of Z_1 and P_1 using the splitting lemma [4]. However, for the semimodule case, splitting lemma does not hold without adding further assumptions on the morphisms. The main reason is due to the differences between module and semimodule. For example, subtraction operation is used in the proof for the module case, there is no difference between proper exact and exact sequences of modules, and every R-module morphism is both i-regular and k-regular. All of above reasons result in the conclusion of the fixed pole structure is not a necessarily a direct sum of Z_1 and P_1. For systems over a semiring, however, the fixed pole semimodule $P(T, P)$ still contains information in Z_1 and P_1.

5.3. Fixed poles of the solution $P(z)$ to $T(z) = P(z)M(z)$

We are given two transfer functions, $T(z) : C(z) \to Y(z)$ and $M(z) : C(z) \to U(z)$, and we assume that they satisfy the existence condition in Proposition 2 with $Y(z)$ injective. Therefore, there exists a solution $P(z)$ to the model matching equation $T(z) = P(z)M(z)$. To study the pole structure of this solution $P(z)$, we define an $R[z]$-semimodule $P(T, M)$ as follows:

$$P(T, M) = \frac{M^{-1}(\Omega U)}{M^{-1}(\Omega U) \cap T^{-1}(\Omega Y)}. \tag{20}$$

Definition 2. The semimodule $P(T, M)$ in Eq. (20) is called the fixed pole semimodule for the solution $P(z)$ to the model matching equation $T(z) = P(z)M(z)$.

Proposition 5. *Given two transfer functions $T(z) : C(z) \to Y(z)$ and $M(z) : C(z) \to U(z)$ satisfying the condition in Proposition 2 with $Y(z)$ injective, there exists a unit kernel $R[z]$-semimodule morphism from $P(T, M)$ to the pole semimodule of the input type $X_I(P)$,*

$$X_I(P) = \frac{\Omega U}{\Omega U \cap P^{-1}(\Omega Y)} \tag{21}$$

of this solution $P(z)$.

In traditional systems, the fixed pole module is a submodule of the pole module of a solution $P(z)$. In systems over a semiring, however, the morphism between them has a unit kernel but is not a monomorphism, or in other words, an injective $R[z]$-semimodule morphism. Although the same terminology, fixed poles, is adopted from [8], unlike the traditional case, the fixed pole semimodule is not a sub-semimodule of the pole semimodule for a solution $P(z)$. However, there are connections between the fixed pole semimodule $P(T, M)$ and the pole semimodule of the solution $P(z)$.

In the remainder of the subsection, a description is given for the fixed pole semimodule $P(T, M)$ in terms of Γ-zero semimodules and the pole semimodules of the known transfer functions $T(z)$ and $M(z)$. The Γ-zero semimodule of a transfer function $M(z) : C(z) \to U(z)$ is defined as

$$Z_\Gamma(M) = \frac{M^{-1}(\Omega U)}{M^{-1}(\Omega U) \cap \Omega C}.$$

We are able to obtain a result similar to that in [8] by changing the zero semimodule into the Γ-zero semimodule. Define a new $R(z)$-semimodule morphism

$$\begin{bmatrix} T(z) \\ M(z) \end{bmatrix} : C(z) \to Y(z) \times U(z).$$

The action is defined by

$$\begin{bmatrix} T(z) \\ M(z) \end{bmatrix} (c(z)) = (T(z)c(z), M(z)c(z)).$$

The Γ-zero semimodule of this transfer function is

$$Z_\Gamma\left(\begin{bmatrix} T \\ M \end{bmatrix}\right) = \frac{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C} \tag{22}$$

and the pole semimodule of input type is

$$X_I\left(\begin{bmatrix} T \\ M \end{bmatrix}\right) = \frac{\Omega C}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C}. \tag{23}$$

A natural projection $p : X_I\left(\begin{bmatrix} T \\ M \end{bmatrix}\right) \to X_I(M)$ exists between these two pole semimodules, where

$$X_I(M) = \frac{\Omega C}{M^{-1}(\Omega U) \cap \Omega C}.$$

Theorem 2. *Given two transfer functions $T(z) : C(z) \to Y(z)$ and $M(z) : C(z) \to U(z)$ satisfying the conditions in Proposition 2 with $Y(z)$ injective, there exist two $R[z]$-semimodules Z_2 and P_2 and $R[z]$-semimodule morphisms $\alpha, \beta, \phi, \psi$ such that the following sequences are exact:*

$$e \to P_2 \xrightarrow{i} X_I\left(\begin{bmatrix} T \\ M \end{bmatrix}\right) \xrightarrow{p} X_I(M) \to e, \tag{24}$$

$$e \to Z_\Gamma\left(\begin{bmatrix} T \\ M \end{bmatrix}\right) \xrightarrow{\alpha} Z_\Gamma(M) \xrightarrow{\beta} Z_2 \to e, \tag{25}$$

$$e \to P_2 \xrightarrow{\phi} P(T, M) \xrightarrow{\psi} Z_2 \to e, \tag{26}$$

where Z_2 and P_2 are defined as

$$Z_2 = \frac{M^{-1}(\Omega U)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \,\square_{C(z)}\, M^{-1}(\Omega U) \cap \Omega C}, \tag{27}$$

$$P_2 = \frac{M^{-1}(\Omega U) \cap \Omega C}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C}, \tag{28}$$

respectively.

This subsection studies the fixed pole structure for solutions $P(z)$ to the model matching equation $T(z) = P(z)M(z)$. For systems over a field, the sequence in Eq. (25) uses zero modules of output type [8]. However, for systems over a semiring, we choose the Γ-zero semimodules instead to obtain the exact sequence in Eq. (26). Zero semimodules of output type [23] for $M(z)$ and $[T(z) \quad M(z)]^T$ are defined as

$$Z_O(M) = \frac{M^{-1}(\Omega U)}{M^{-1}(\Omega U) \cap (\operatorname{Ker} M \,\square_{C(z)}\, \Omega C)}, \tag{29}$$

$$Z_O\left(\begin{bmatrix} T \\ M \end{bmatrix}\right) = \frac{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U)}{T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap ((\operatorname{Ker} M \cap \operatorname{Ker} T) \,\Box_{C(z)}\, \Omega C)}. \tag{30}$$

Therefore, if the transfer function $M(z)$ has a unit kernel, then the zero semimodule of output type coincides with its Γ-zero semimodule. In that case, the Γ-zero semimodules in Theorem 2 can be replaced by the zero semimodule of output type, which will be the same as the results for traditional linear systems in [8].

Another difference from the module case is that all these sequence are exact, not proper exact. Notice that, unlike the previous subsection, we can obtain the exact sequence in Eq. (26) without the subtractive assumption, because $T^{-1}(\Omega Y) \cap M^{-1}(\Omega U) \cap \Omega C$ is already subtractive. Moreover, for traditional systems over a field, the fixed pole module $P(T, M)$ can be interpreted as a direct sum of Z_2 and P_2. For systems over a semiring, however, the fixed pole semimodule $P(T, M)$ contains information in Z_2 and P_2, but not necessarily a direct sum of Z_2 and P_2.

5.4. Essential and inessential poles

Propositions 3–5 state the relationship between the fixed pole semimodules and solutions to MMP. In particular, for the MMP with unknown controller $M(z)$, an $R[z]$-epimorphism $\overline{P}(z) : X_O(M) \to P(T, P)$ is established in the proof of Proposition 3. Therefore, we have the following short exact sequence:

$$e \to C(M) \overset{\alpha}{\to} X_O(M) \overset{\overline{P}(z)}{\to} P(T, P) \to e \tag{31}$$

with $C(M) = \operatorname{Ker} \overline{P}(z)$, which is called the *inessential pole semimodule* of the solutions to the MMP. For an arbitrary transfer function solution $P(z)$, the inessential pole semimodule $C(M)$ cannot be easily expressed as a semimodule form. However, for a steady transfer function $P(z)$, the inessential pole semimodule $C(M)$ can be expressed explicitly.

Corollary 2. *For the MMP with given $T(z) : C(z) \to Y(z)$ and a steady $P(z) : U(z) \to Y(z)$, the sequence in Eq. (31) with*

$$C(M) = \frac{\operatorname{Ker} P \cap (M(\Omega C) \,\Box_{U(z)}\, \Omega U)}{\Omega U}$$

is a short exact sequence.

Proof. Proposition 4 stated the existence of an $R[z]$-epimorphism $\overline{P}(z) : X_O(M) \to P(T, P)$ in the sequence (31). We only need to show that there exists a unit kernel $R[z]$-morphism $\alpha : C(M) \to X_O(M)$ such that $\operatorname{Ker} \overline{P}(z) = \operatorname{Im} \alpha$. That $R[z]$-morphism α exists can be proved by the following commutative diagram:

$$\begin{array}{ccccc} & & \operatorname{Ker} P \cap (M(\Omega C) \,\Box_{U(z)}\, \Omega U) & \xrightarrow{\ i\ } & M(\Omega C) \,\Box_{U(z)}\, \Omega U \\ & & \downarrow p_1 & & \downarrow p_2 \\ e & \longrightarrow & \dfrac{\operatorname{Ker} P \cap (M(\Omega C) \,\Box_{U(z)}\, \Omega U)}{\Omega U} & \xrightarrow{\ \alpha\ } & \dfrac{M(\Omega C) \,\Box_{U(z)}\, \Omega U}{\Omega U}, \end{array}$$

where i is an insertion. Because ΩU is subtractive, the morphism α has a unit kernel. The image of α is

$$\begin{aligned} \operatorname{Im} \alpha &= \left\{ \frac{u}{\Omega U}, u \in M(\Omega C) \,\Box_{U(z)}\, \Omega U \middle| \frac{u}{\Omega U} \,\Box_{U(z)}\, \frac{k_1}{\Omega U} = \frac{k_2}{\Omega U}, \right. \\ &\quad \left. k_1, k_2 \in \operatorname{Ker} P \cap (M(\Omega C) \,\Box_{U(z)}\, \Omega U) \right\}, \\ &= \left\{ \frac{u}{\Omega U}, u \in M(\Omega C) \,\Box_{U(z)}\, \Omega U | u \,\Box_{U(z)}\, k_1 \,\Box_{U(z)}\, u_p^1 = k_2 \,\Box_{U(z)}\, u_p^2, \right. \\ &\quad \left. u_1, u_2 \in \Omega U \right\}. \end{aligned}$$

The kernel of a steady transfer function $\overline{P}(z)$ is

$$\begin{aligned}\operatorname{Ker}\overline{P}(z) &= \{u \in M(\Omega C) \Box_{U(z)} \Omega U | Pu\Box_{Y(z)} Pu_p^1 = Pu_p^2, u_p^1, u_p^2 \in \Omega U\} \\ &= \{u \in M(\Omega C) \Box_{U(z)} \Omega U | u \Box_{U(z)} u_p^1 \Box_{U(z)} k_1 = u_p^2 \Box_{U(z)} k_2, \\ &\qquad u_p^1, u_p^2 \in \Omega U, k_1, k_2 \in \operatorname{Ker} P\}.\end{aligned}$$

Therefore, $\operatorname{Ker}\overline{P}(z) = \operatorname{Im}\alpha$, i.e. the sequence in Eq. (31) is exact. □

For the MMP with unknown plant $P(z)$, a unit kernel $R[z]$-morphism $\overline{M}(z) : P(T,M) \to X_I(P)$ is established in the proof of Proposition 5. Therefore, using Lemma 1, we obtain the short exact sequence:

$$e \to P(T,M) \overset{\overline{M}(z)}{\to} X_I(P) \overset{\beta}{\to} \widetilde{C}(P) \to e \tag{32}$$

with $\widetilde{C}(M) = X_I(P)/P(T,M)$, which is the *inessential pole semimodule* of the solutions to the model matching problem in this case. In the next section, a discrete-event dynamic system is used to illustrate that the fixed pole semimodules provide important information for the pole semimodule of solutions to the MMP. The inessential pole semimodules may not be contained in any solution to the MMP.

6. A discrete-event system application

This section uses a discrete-event dynamical system over the max-plus algebra to illustrate the relationship between the fixed pole semimodule and the pole semimodule of solutions to the MMP. The fixed pole semimodule characterizes a common component in the solutions to MMP. Suppose given a reference transfer function $T(z) : C(z) \to Y(z)$ and a plant $P(z) : U(z) \to Y(z)$ over the max-plus algebra $\mathbb{R}_{\text{Max}}$:

$$T(z) = \begin{bmatrix} z^{-3} \oplus z^{-4} & \epsilon \\ z^{-3} & \epsilon \end{bmatrix}, \quad P(z) = \begin{bmatrix} z^{-1} \oplus z^{-2} & \epsilon \\ z^{-1} & \epsilon \end{bmatrix}.$$

Recall that the fixed pole semimodule $P(T,P)$ is

$$P(T,P) = \frac{T(\Omega C) \oplus P(\Omega U)}{P(\Omega U)}.$$

If we define $P(\Omega U)$ and $T(\Omega C)$ by means of the usual power series equipped with the customary polynomial multiplication, and if $u_p = [u_p^1 \quad u_p^2]^T$ and $c_p = [c_p^1 \quad c_p^2]^T$, where $u_p^1, u_p^2 \in \Omega U$, and $c_p^1, c_p^2 \in \Omega C$ are represented by

$$\begin{aligned} u_p^1 &= a_0 \oplus a_1 z \oplus a_2 z^2 \oplus \cdots \oplus a_{n_1} z^{n_1}, \\ u_p^2 &= b_0 \oplus b_1 z \oplus b_2 z^2 \oplus \cdots \oplus b_{n_2} z^{n_2}, \\ c_p^1 &= p_0 \oplus p_1 z \oplus p_2 z^2 \oplus \cdots \oplus p_{m_1} z^{m_1}, \\ c_p^2 &= q_0 \oplus q_1 z \oplus q_2 z^2 \oplus \cdots \oplus q_{m_2} z^{m_2}, \end{aligned}$$

we can obtain that

$$\begin{aligned} P(\Omega U) &= \{P(z)u_p | \forall u_p \in \Omega U\} \\ &= \left\{ \begin{bmatrix} z^{-1} \oplus z^{-2} & \epsilon \\ z^{-1} & \epsilon \end{bmatrix} \begin{bmatrix} u_p^1 \\ u_p^2 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} a_0 z^{-2} \oplus \bar{a}_1 z^{-1} \oplus y_p^1 \\ a_0 z^{-1} \oplus y_p^2 \end{bmatrix}, \ y_p^1, \ y_p^2 \in \Omega Y \right\}, \end{aligned}$$

where $\bar{a}_1 = a_0 \oplus a_1$; and

$$\begin{aligned} T(\Omega C) &= \{T(z)c_p | \forall c_p \in \Omega C\} \\ &= \left\{ \begin{bmatrix} z^{-3} \oplus z^{-4} & \epsilon \\ z^{-3} & \epsilon \end{bmatrix} \begin{bmatrix} c_p^1 \\ c_p^2 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} p_0 z^{-4} \oplus \bigoplus_{i=1}^{3} \bar{p}_i z^{i-4} \oplus \bar{y}_p^1 \\ p_0 z^{-3} \oplus p_1 z^{-2} \oplus p_2 z^{-1} \oplus \bar{y}_p^2 \end{bmatrix} \right\}, \end{aligned}$$

where $\bar{p}_1 = p_0 \oplus p_1, \bar{p}_2 = p_1 \oplus p_2, \bar{p}_3 = p_2 \oplus p_3$, and $\bar{y}_p^i$, $i = 1, 2$, are polynomial outputs in ΩY. Therefore, we can compute $P(\Omega U) \oplus T(\Omega C)$ and obtain

$$T(\Omega C) \oplus P(\Omega U) = T(\Omega C).$$

Now discard elements in $P(\Omega U)$ from $T(\Omega C) \oplus P(\Omega U)$, which means to remove vectors with the first element which has a degree greater than or equal to -2 and with the second element which has a degree greater than or equal to -1. In the module case, the construction defined above provides the fixed pole module $P(T,P)$. In the semimodule case, however, the quotient structure cannot be understood as dividing out the denominator component, so this construction may be a larger version of the fixed pole semimodule $P(T,P)$. However, there is still a relation between this construction and the actual fixed pole semimodule, which will be demonstrated later. A new $R(z)$-morphism $[T(z) \quad P(z)] : C(z) \oplus U(z) \to Y(z)$ is defined by

$$[T(z)\ P(z)] = \begin{bmatrix} z^{-3} \oplus z^{-4} & \epsilon & z^{-1} \oplus z^{-2} & \epsilon \\ z^{-3} & \epsilon & z^{-1} & \epsilon \end{bmatrix}.$$

Consider an operation by z upon the columns of this new transfer function: the first column is

$$z \begin{bmatrix} z^{-3} \oplus z^{-4} \\ z^{-3} \end{bmatrix} = \begin{bmatrix} z^{-3} \\ z^{-2} \end{bmatrix},$$
$$z \begin{bmatrix} z^{-3} \\ z^{-2} \end{bmatrix} = \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix}$$

and the third column is

$$z \begin{bmatrix} z^{-1} \oplus z^{-2} \\ z^{-1} \end{bmatrix} = \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix}.$$

Notice that the last two equalities hold because any vector with the first element which has a degree greater than or equal to -2 and with the second element which has a degree greater than or equal to -1 is considered a unit element. The procedure stops at this point because linear dependency occurs, and the basis of the state space X is

$$x_1 = \begin{bmatrix} z^{-3} \oplus z^{-4} \\ z^{-3} \end{bmatrix},$$
$$x_2 = \begin{bmatrix} z^{-3} \\ z^{-2} \end{bmatrix},$$
$$x_3 = \begin{bmatrix} z^{-1} \oplus z^{-2} \\ z^{-1} \end{bmatrix}.$$

This is because $z : X(z) \to X(z)$ is equivalent to $A : X \to X$. Therefore, the realization matrix A_f for the fixed pole semimodule $P(T,P)$ is

$$A_f = \begin{bmatrix} \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{bmatrix}.$$

The next step is to pick two arbitrary solutions $M_1(z)$ and $M_2(z)$ to the model matching equation $P(z)M(z) = T(z)$ and construct the realization matrices A_1 and A_2 for them. For instance, $M_1(z)$ and $M_2(z)$ are given as

$$M_1(z) = \begin{bmatrix} z^{-2} & \epsilon \\ \epsilon & \epsilon \end{bmatrix}, \quad M_2(z) = \begin{bmatrix} z^{-2} & \epsilon \\ z^{-2} & z^{-2} \end{bmatrix}.$$

Recall that the pole semimodules of output type for $M_1(z)$ and $M_2(z)$ are

$$X_O(M_1) = \frac{M_1(\Omega C)}{(M_1(\Omega C) \cap \Omega U)}, \quad X_O(M_2) = \frac{M_2(\Omega C)}{(M_2(\Omega C) \cap \Omega U)}.$$

By using a construction similar to that for A_f, two realizations for these two controllers are obtained as

$$A_1 = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}, \quad B_1 = \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \end{bmatrix}, \quad C_1 = \begin{bmatrix} \epsilon & e \\ \epsilon & \epsilon \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix}, \quad B_2 = \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & e \\ \epsilon & \epsilon \end{bmatrix}, \quad C_2 = \begin{bmatrix} \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e \end{bmatrix}.$$

Notice that these two matrices A_1 and A_2 both contain the essential matrix A_{fixed}, which is defined by

$$A_{\text{fixed}} = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

Because the transfer function $P(z)$ is steady, and the kernel of $P(z)$ is

$$\operatorname{Ker} P(z) = \left\{ \begin{bmatrix} \epsilon \\ u \end{bmatrix}, u \in U(z) \right\},$$

the inessential pole semimodule of $M_1(z)$ is a unit semimodule, i.e. $C(M_1) = \frac{\operatorname{Ker} P \cap (M_1(\Omega C) \oplus \Omega U)}{\Omega U} = e$. Therefore, all poles of $M_1(z)$ are essential poles. This is true because the matrix A_1 is contained in A_f, which is obtained from the fixed pole semimodule $P(T, P)$. The inessential pole semimodule of $M_2(z)$ is

$$C(M_2) = \frac{\operatorname{Ker} P \cap (M_2(\Omega C) \oplus \Omega U)}{\Omega U} = \begin{bmatrix} \epsilon \\ u_{-2}z^{-2} \oplus u_{-1}z^{-1} \end{bmatrix},$$

where u_{-1} and u_{-2} are in U. The realization matrix $A_{\text{inessential}}$ for the inessential pole semimodule $C(M_2)$ is

$$A_{\text{inessential}} = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix}.$$

This is true because the inessential matrix is contained in the lower right diagonal block of the matrix A_2. This fact can also be observed from the Petri net realization for the two solutions $M_1(z)$ and $M_2(z)$. The state semimodule representation for $M_1(z)$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{k+1} = \begin{bmatrix} \epsilon & \epsilon \\ e & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_k \oplus \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_k = \begin{bmatrix} u_1 \\ x_1 \end{bmatrix}_k,$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_k = \begin{bmatrix} \epsilon & e \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_k = \begin{bmatrix} x_2 \\ \epsilon \end{bmatrix}_k$$

and the state semimodule representation for $M_2(z)$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{k+1} = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k \oplus \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \\ e & e \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_k = \begin{bmatrix} u_1 \\ x_1 \\ u_1 \oplus u_2 \\ x_3 \end{bmatrix}_k,$$

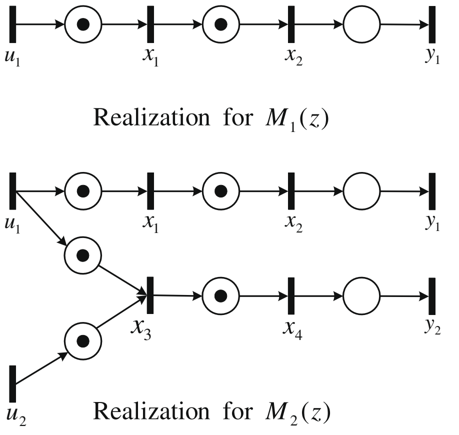

Fig. 6. The Petri net realizations for the solutions $M_1(z)$ and $M_2(z)$.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_k = \begin{bmatrix} \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}_k .$$

These two systems can be modeled by Petri nets shown in Fig. 6. These Petri net realizations for the two controllers $M_1(z)$ and $M_2(z)$ each contain the same set of components, which are generated by the essential matrix A_{fixed}.

We know that there exists a surjective morphism between the pole semimodule of output type of a solution to the fixed pole structure $P(T, P)$ by Proposition 3. For the module case, this implies that the fixed pole module is a contained in the pole semimodule of the solution due to the splitting lemma. For the semimodule case, the fixed pole semimodule contains common components as the pole semimodule of the solution to the MMP. In this discrete-event example, the fixed pole semimodule can generate a common Petri net component in the solutions to the MMP. Although the fixed pole semimodule is not a sub-semimodule of the pole semimodule, it contains essential information about the pole semimodule of output type for a solution to the MMP.

The purpose of the example is to illustrate that the fixed pole semimodule can be used to find a common Petri net component in the solutions of the MMP. Especially in controller synthesis problems, if we need to design a controller for a given plant in order to match the reference model, a common Petri net model can be built using the fixed pole semimodule. This common Petri net component is contained in any controllers satisfying the design requirements. The additional components in the controllers can be added based on the specific design requirements of the users in order to save time and cost.

7. Conclusion

In this paper, the MMP is studied for systems over semirings, which are used to model a class of discrete-event dynamic systems, such as queueing systems, communication networks, and manufacturing systems. The main contribution of this paper is the discovery of fixed pole structure for solutions to MMP. The fixed pole structure provides essential information contained in solutions to MMP. For systems over a semiring, the fixed poles cannot all be viewed as components in the pole structures of the solutions. However, there are parts of the fixed pole structures which will appear in the pole semimodules of the solutions to MMP. For a discrete-event system, a common Petri net component in the solutions to MMP can be discovered from the fixed pole structure. In future research, fixed zeros for systems over a semiring will be explored, along with the further interpretation of the fixed poles and fixed zeros.

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