

# Observer-based Controllers for Max-plus Linear Systems

Laurent Hardouin, Ying Shang, Carlos Andrey Maia, Bertrand Cottenceau

## Abstract

Max-plus algebra is a suitable algebraic setting to model discrete event systems involving synchronization and delay phenomena which are often found in transportation networks, communications systems, and manufacturing systems. One way of controlling this kind of systems consists in choosing the dates of input events in order to achieve the desired performances, e.g., to obtain output events in order to respect given dates. This kind of control is optimal, according to a just-in-time criterion, if the input-event dates are delayed as much as possible while ensuring the output events to occur before a desired reference date. This paper presents the observer-based controller for max-plus linear systems where only estimations of system states are available for the controller. As in the classical sense, this is a state-feedback control problem, which is solved in two steps: first, an observer computes an estimation of the state by using the input and the output measurements, then, this estimated state is used to compute the state-feedback control action. As a main result, it is shown that the optimal solution of this observer-based control problem leads to a greater control input than the one obtained with the output feedback strategy. A high throughput screening example in drug discovery illustrates this main result by showing that the scheduling obtained from the observer-based controller is better than the scheduling obtained from the output feedback controller.

## I. INTRODUCTION

Max-plus algebra is a suitable algebraic setting to model the temporal evolution of discrete event dynamic systems such as transportation networks ([22], [29]), communication networks [31], and manufacturing assembly lines [11]. Delay and synchronization are the phenomena mainly considered for such systems, but some more complex behaviors have been studied in this framework, such as conflict

L. Hardouin and B. Cottenceau are with the Laboratoire Angevin de Recherche en Ingénierie des Systèmes, Université d'Angers, France, e-mail : laurent.hardouin@univ-angers.fr,bertrand.cottenceau@univ-angers.fr.

Y. Shang is with the Department of Electrical and Computer Engineering, Southern Illinois University Edwardsville, Campus Box 1801, Edwardsville, IL 62026, USA, e-mail: yshang@siue.edu.

C.A. Maia is with the Departamento de Engenharia Elétrica, Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, Pampulha, 31270-010, Belo Horizonte, MG, Brazil, e-mail : maia@cpdee.ufmg.br.

phenomena [1], packing/unpacking phenomena [16], and partial synchronization [20]. This paper focuses on discrete event systems which can be represented by timed-event graphs (TEGs) which are timed Petri nets where each place has exactly one upstream transition and one downstream transition. Its description can be transformed into a max-plus or a min-plus linear model and *vice versa* ([3], [15]). This property has advantaged the emergence of a specific control theory for these systems and several control strategies have been proposed, for instance, optimal open loop control ([12], [34], [40]) and optimal state-feedback control in order to solve the model matching problem ([17], [33], [36], [37]), as well as control strategies allowing the state to stay in a specific subset are proposed in ([2], [30], [37], [41]).

In the past literature, state estimation has also been studied for max-plus linear systems ([21], [26], [27], [38]). The observer aims at estimating system states for a given plant by using input/output measurements. The state trajectories correspond to the transition firings of the corresponding TEGs. Their estimation is worthy of interest because it provides insight into internal properties of the system. For example, these state estimations are sufficient to reconstruct the markings of the graph, as it is done in [23] for Petri nets without temporization. Moreover, the state estimation has many potential applications, such as fault detection, diagnosis, and state-feedback control. In general, the complete measurement of all the state variables is not possible due to the lack of sensors. It is then classical to use the state estimation provided by the observer to feed the controller. The main contribution of this paper is the design of the observer-based controllers for max-plus linear systems, using the observer introduced in ([26], [27]). Such an observer is an analogy with the classical Luenberger observer [35] for linear systems. As in the classical sense, this is a state-feedback control problem, which is solved in two steps: first, an observer computes an estimation of the state by using the input and the output measurements, then, this estimated state is used to compute the state-feedback control action. As a main result, it is shown that the optimal solution of this observer-based control problem leads to a greater control input than the one obtained with the output feedback strategy. This means that, in spite of the lack of sensors, this control strategy leads to a better control than the output feedback control strategy according to the just-in-time criterion. For instance, in a manufacturing setting, the observer-based controller would provide a better scheduling by starting the process later than the output feedback control, while ensuring the same finishing time of the output parts. This scheduling would allow users to load the raw parts later rather than earlier to avoid unnecessary congestions in manufacturing lines.

This paper is organized as the following. Section II reviews some algebraic tools concerning max-plus algebraic structures. Some results about the residuation theory [3] and its applications over semirings are also given.

Section IV reviews the observer for max-plus linear systems as in [27] and develops new results on causal projections of the observer matrix achieving the equality between the observed outputs and the measured outputs. Section V presents further developments of the output feedback and the state-feedback controllers as introduced in ([36], [37]). Section VI presents the observer-based controller and compares its differences between the output feedback controller and the observer-based state-feedback controller. In particular, the observer-based control strategy leads to a better control than the output feedback control strategy according to the just-in-time criterion. Section VII illustrates the main results using a high throughput screening example in drug discovery. An observer-based controller is constructed and proved to have a better performance comparing with an output feedback controller.

## II. ALGEBRAIC SETTING

*Definition 1:* A *semiring* is a set  $\mathcal{S}$ , equipped with two operations, denoted as  $\oplus$  and  $\otimes$ , such that  $(\mathcal{S}, \oplus)$  is a commutative monoid (the zero element will be denoted  $\varepsilon$ ),  $(\mathcal{S}, \otimes)$  is a monoid (the unit element will be denoted  $e$ ), operation  $\otimes$  is right and left distributive over  $\oplus$ , and  $\varepsilon$  is absorbing for the product (i.e.  $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a$ ).

A semiring  $\mathcal{S}$  is *idempotent* if  $a \oplus a = a$  for all  $a \in \mathcal{S}$ . As in classical algebra, the operator  $\otimes$  will be often omitted in the equations, moreover,  $a^i = a \otimes a^{i-1}$  and  $a^0 = e$ . In this algebraic structure, a partial order relation is defined by  $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow b = a \wedge b$  (where  $a \wedge b$  is the greatest lower bound of  $a$  and  $b$ ), therefore, an idempotent semiring is a partially ordered set (see [3], [29] for a more detailed introduction). An idempotent semiring is said to be complete if it is closed for infinite  $\oplus$ -sums and if  $\otimes$  distributes over infinite  $\oplus$ -sums, it will be denoted  $\mathcal{S}$ . In particular,  $\top = \bigoplus_{x \in \mathcal{S}} x$  is the greatest element of  $\mathcal{S}$  ( $\top$  is called the top element of  $\mathcal{S}$ ).

*Example 1* ( $\overline{\mathbb{Z}}_{\max}$ ): Set  $\overline{\mathbb{Z}}_{\max} = \mathbb{Z} \cup \{-\infty, +\infty\}$  endowed with the max operator as sum and the classical sum  $+$  as product is a complete idempotent semiring, usually denoted  $\overline{\mathbb{Z}}_{\max}$ , of which  $\varepsilon = -\infty$  and  $e = 0$ .

*Definition 2 (Subsemiring):* Let  $\mathcal{S}$  be a semiring and  $\mathcal{S}_{sub} \subset \mathcal{S}$ . If  $\varepsilon, e \in \mathcal{S}_{sub}$  and if  $\mathcal{S}_{sub}$  is closed for laws  $\oplus$  and  $\otimes$ , then  $\mathcal{S}_{sub}$  is a subsemiring of  $\mathcal{S}$ . A subsemiring  $\mathcal{S}_{sub}$  is complete if it is closed for infinite sums.

*Theorem 1* (see [3], Th. 4.75): The implicit inequality  $x \succeq ax \oplus b$  as well as  $x = ax \oplus b$  defined over  $\mathcal{S}$ , admit  $x = a^*b$  as the least solution, where  $a^* = \bigoplus_{i \in \mathbb{N}} a^i$  (Kleene star operator).

*Definition 3 (Residual and residuated mapping):* Let  $\mathcal{D}, \mathcal{C}$  be two complete idempotent semirings and  $f : \mathcal{D} \rightarrow \mathcal{C}$  be an order preserving mapping,  $f$  is a *residuated mapping* if for all  $y \in \mathcal{C}$  there exists a

greatest solution to the inequality  $f(x) \preceq y$  (hereafter denoted  $f^\sharp(y)$ ). Obviously, if equality  $f(x) = y$  is solvable,  $f^\sharp(y)$  yields the greatest solution. The mapping  $f^\sharp$  is called the *residual* of  $f$  and  $f^\sharp(y)$  is the optimal solution of the inequality.

*Theorem 2 (see [3] Th. 4.50,[5]):* Let  $\mathcal{D}, \mathcal{C}$  be two complete idempotent semirings and  $f : \mathcal{D} \rightarrow \mathcal{C}$  be an order preserving mapping, the following statements are equivalent:

- (i)  $f$  is residuated.
- (ii) there exists an unique order preserving mapping  $f^\sharp : \mathcal{C} \rightarrow \mathcal{D}$  such that  $f \circ f^\sharp \preceq \text{Id}_{\mathcal{C}}$  and  $f^\sharp \circ f \succeq \text{Id}_{\mathcal{D}}$ .

*Theorem 3 ([3, Th. 4.56]):* Let  $\mathcal{D}, \mathcal{C}$  be two complete idempotent semirings and  $f : \mathcal{D} \rightarrow \mathcal{C}$  be an order preserving mapping, the following properties hold:

$$f \circ f^\sharp \circ f = f \text{ and } f^\sharp \circ f \circ f^\sharp = f^\sharp. \quad (1)$$

*Example 2:* Mappings  $\Lambda_a : x \mapsto a \otimes x$  and  $\Psi_a : x \mapsto x \otimes a$  defined over  $\mathcal{S}$  are both residuated (see [3], Section 4.4.4). Their residuals are order preserving mappings, denoted respectively by  $\Lambda_a^\sharp(x) = a \not\backslash x$  and  $\Psi_a^\sharp(x) = x \not\backslash a$ . This means that  $a \not\backslash b$  (resp.  $b \not\backslash a$ ) is the greatest solution of the inequality  $a \otimes x \preceq b$  (resp.  $x \otimes a \preceq b$ ).

*Definition 4 (Restricted mapping):* Let  $\mathcal{C}, \mathcal{B}, \mathcal{D}$ , and  $\mathcal{E}$  be complete idempotent semirings. Let  $f : \mathcal{D} \rightarrow \mathcal{C}$  be a mapping and  $\mathcal{B} \subset \mathcal{D}$ . We will denote by  $f|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{C}$  the mapping defined by  $f|_{\mathcal{B}} = f \circ \text{Id}_{\mathcal{B}}$  where  $\text{Id}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{D}, x \mapsto x$  is the canonical injection. Identically, let  $\mathcal{E} \subset \mathcal{C}$  be a set such that  $\text{Im} f \subset \mathcal{E}$ . Mapping  ${}_{\mathcal{E}}|f : \mathcal{D} \rightarrow \mathcal{E}$  is defined by  ${}_{\mathcal{E}}|f = \text{Id}_{\mathcal{E}} \circ {}_{\mathcal{E}}|f$ , where  $\text{Id}_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{C}, x \mapsto x$ .

*Definition 5 (Closure mapping):* A closure mapping is an order preserving mapping  $f : \mathcal{S} \rightarrow \mathcal{S}$  such that  $f \succeq \text{Id}_{\mathcal{S}}$  and  $f \circ f = f$ .

*Theorem 4 (Projection in subsemiring [6] ):* Let  $\mathcal{S}$  be a complete semiring and  $\mathcal{S}_{sub}$  a complete subsemiring of  $\mathcal{S}$ . The canonical injection  $I_{\mathcal{S}_{sub}} : \mathcal{S}_{sub} \rightarrow \mathcal{S}, x \mapsto x$  is residuated. The residual is denoted as  $\text{Pr}_{\mathcal{S}_{sub}} = I_{\mathcal{S}_{sub}}^\sharp$  and is such that:

- (i)  $\text{Pr}_{\mathcal{S}_{sub}} \circ \text{Pr}_{\mathcal{S}_{sub}} = \text{Pr}_{\mathcal{S}_{sub}}$ ,
- (ii)  $\text{Pr}_{\mathcal{S}_{sub}} \preceq \text{Id}_{\mathcal{S}}$  where  $\text{Id}_{\mathcal{S}}$  is the identity mapping over  $\mathcal{S}$ ,
- (iii)  $x \in \mathcal{S}_{sub} \Leftrightarrow \text{Pr}_{\mathcal{S}_{sub}}(x) = x$ .

*Proposition 1 (see [17]):* Let  $f : \mathcal{S} \rightarrow \mathcal{S}$  be a closure mapping. Then,  ${}_{\text{Im} f}|f$  is a residuated mapping whose residual is the canonical injection  $\text{Id}_{|\text{Im} f}$ .

*Example 3:* Mapping  $K : \mathcal{S} \rightarrow \mathcal{S}, x \mapsto x^*$  is a closure mapping (indeed  $a \preceq a^*$  and  $a^* = (a^*)^*$  see (f.2) in Appendix). Then,  $(\text{lm}K|K)$  is residuated and its residual is  $(\text{lm}K|K)^\sharp = \text{Id}_{\text{lm}K}$ . In other words,  $x = a^*$  is the greatest solution of inequality  $x^* \preceq a$  if  $a \in \text{lm}K$ , that is  $x \preceq a^* \Leftrightarrow x^* \preceq a$ .

*Example 4:* Mapping  $P : \mathcal{S} \rightarrow \mathcal{S}, x \mapsto x^+ = \bigoplus_{i \in \mathbb{N}^+} x^i = xx^* = x^*x$  is a closure mapping (indeed  $a \preceq a^+$  and  $a^+ = (a^+)^+$  see (f.2) in Appendix). Then  $(\text{lm}P|P)$  is residuated and its residual is  $(\text{lm}P|P)^\sharp = \text{Id}_{\text{lm}P}$ . In other words,  $x = a^+$  is the greatest solution of inequality  $x^+ \preceq a$  if  $a \in \text{lm}P$ , that is  $x \preceq a^+ \Leftrightarrow x^+ \preceq a$ .

The set of  $n \times n$  matrices with entries in  $\mathcal{S}$  is an idempotent semiring. The sum, the product and the residuation of matrices are defined after the sum, the product and the residuation of scalars in  $\mathcal{S}$ , *i.e.*,

$$(A \otimes B)_{ik} = \bigoplus_{j=1}^n (a_{ij} \otimes b_{jk}) \quad (2)$$

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \quad (3)$$

$$(A \bowtie B)_{ij} = \bigwedge_{k=1}^n (a_{ki} \bowtie b_{kj}), \quad (B \phi A)_{ij} = \bigwedge_{k=1}^n (b_{ik} \phi a_{jk}). \quad (4)$$

The identity matrix of  $\mathcal{S}^{n \times n}$  is a matrix with entries equal to  $e$  on the diagonal and to  $\varepsilon$  elsewhere. This identity matrix will be denoted  $I_n$ , and the matrix with all its entries equal to  $\varepsilon$  will also be denoted  $\varepsilon$ .

*Properties 1:* ([24], [39]) Given a complete semiring  $\mathcal{S}$ , and two matrices  $A \in \mathcal{S}^{p \times n}$ ,  $B \in \mathcal{S}^{n \times p}$ , the following equations hold :

$$A \bowtie A = (A \bowtie A)^*, \quad B \phi B = (B \phi B)^*. \quad (5)$$

### III. THE TEG DESCRIPTION IN AN IDEMPOTENT SEMIRING

TEGs constitute a subclass of timed Petri nets in which each place has one upstream and one downstream transition. A TEG description can be transformed into a max-plus or a min-plus linear model and *vice versa*. To obtain an algebraic model in  $\overline{\mathbb{Z}}_{\max}$ , a “dater” function is associated to each transition. For transition labelled  $x_i$ ,  $x_i(k)$  represents the date of the  $k^{\text{th}}$  firing (see [3],[29]). In this paper, without loss of generality, TEGs are described by the following max-plus linear system:

$$\begin{aligned} x(k) &= \bigoplus_{j=0}^{N_a} A_j x(k-j) \oplus \bigoplus_{l=0}^{N_b} B_l u(k-l) \oplus R_0 w(k), \\ y(k) &= C_0 x(k), \end{aligned} \quad (6)$$

where  $u \in (\overline{\mathbb{Z}}_{\max})^p$ ,  $y \in (\overline{\mathbb{Z}}_{\max})^m$ , and  $x \in (\overline{\mathbb{Z}}_{\max})^n$  are the controllable inputs, outputs, and state vectors, respectively. Each of the entries in these vectors is a trajectory which represents the collection

of firing dates of the corresponding transition. The integer number of token  $N_a$  (resp.  $N_b$ ) is equal to a maximal number of tokens initially available in the internal places (resp. in the places between input and internal transitions). Matrices  $A_j \in (\overline{\mathbb{Z}}_{\max})^{n \times n}$ ,  $B_l \in (\overline{\mathbb{Z}}_{\max})^{n \times p}$ , and  $C_0 \in (\overline{\mathbb{Z}}_{\max})^{m \times n}$  represent the links between each transition and then describe the structure of the graph. Each output transition  $y_i$  is assumed to be linked to one and only one internal transition  $x_j$ , *i.e.* each row of matrix  $C_0$  has one entry equal to  $e$  and the others equal to  $\varepsilon$  and at most one entry equal to  $e$  on each column. All vectors  $w \in (\overline{\mathbb{Z}}_{\max})^l$  represents uncontrollable inputs (*i.e.* disturbances). Each entry of  $w$  corresponds to a transition which disables the firing of an internal transition of the graph, and consequently decreases the performance of the system. This vector is linked to the transitions through matrix  $R_0 \in (\overline{\mathbb{Z}}_{\max})^{n \times l}$ . Each uncontrollable input transition  $w_i$  is assumed to be connected to one and only one internal transition  $x_j$ , this means that each column of matrix  $R_0$  has one entry equal to  $e$  and the others equal to  $\varepsilon$  and at most one entry equal to  $e$  on each row. Requirements on  $C_0$  and  $R_0$  are satisfied without loss of generality, since it is sufficient to add extra input and output transitions. The following example illustrates how to model a TEG into a max-plus linear system.

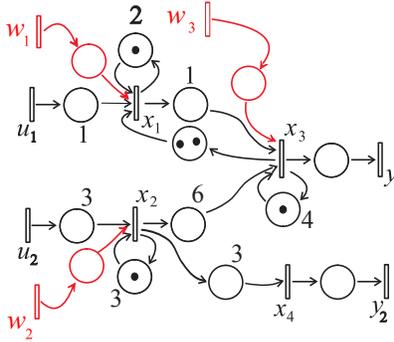


Fig. 1. TEG with 2 controllable transitions  $(u_1, u_2)$ , 3 uncontrollable transitions  $(w_1, w_2, w_3)$ , and 2 measurable transitions  $(y_1, y_2)$ .

*Example 5:*

In Fig. 1, a TEG with  $p = 2$  controllable inputs,  $l = 3$  uncontrollable inputs, and  $m = 2$  measurable outputs, is depicted. Clearly,  $N_a = 2$  and  $N_b = 0$ , hence, the TEG model can be represented as the

max-plus linear system in Eq. (6), where system matrices are

$$\begin{aligned}
 A_0 &= \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & 6 & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 3 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}, \\
 C_0 &= \begin{bmatrix} \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}, \quad R_0 = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix}.
 \end{aligned}$$

Notice that if matrix  $R_0 = I_n$ , where  $I_n \in \overline{\mathbb{Z}}_{\max}[\gamma]^{n \times n}$  is the identity matrix, then each transition is directly affected by independent disturbances. In this case, the disturbance  $w$  can represent the system initial state  $x(0)$  by considering  $w(0) = x(0)$  (see [3], p. 245, for a discussion about compatible initial conditions). Moreover, in ([21], [34]), model uncertainty is taken into account by assuming that matrix entries in Eq. (6) are uncertain and belong to interval with known bounds. For instance, in Fig. 1, let us assume that the minimal sojourn time between  $x_2$  and  $x_3$  is an uncertain value belonging to the interval  $[6, 8]$ , the uncertainty can be equivalently represented by considering a specific disturbance vector  $w$  in a subset of  $(\overline{\mathbb{Z}}_{\max})^l$ , e.g., disturbance  $w_3$  can be chosen as:

$$w_3(k) = w_3(k-1) \oplus x_2(k) \otimes \tilde{A}_{3,2}$$

where  $\tilde{A}_{3,2}$  is a random value in  $[6, 8]$  (see [26] for details).

A trajectory of a TEG transition is then a sequence of firing dates. This collection of dates can be represented by a formal series  $x_i(\gamma) = \bigoplus_{k \in \mathbb{Z}} (x_i(k) \otimes \gamma^k)$  where  $x_i(k) \in \overline{\mathbb{Z}}_{\max}$  and  $\gamma$  is a backward shift operator<sup>1</sup> in the event domain (formally  $\gamma x_i(k) = x_i(k-1)$ ). The set of formal series in  $\gamma$  is denoted by  $\overline{\mathbb{Z}}_{\max}[\gamma]$  and constitutes a complete idempotent semiring. The support of series  $x_i(\gamma)$  is defined by  $Supp(x_i) = \{k \in \mathbb{Z} | x_i(k) \neq \varepsilon\}$ . The valuation in  $\gamma$  of  $x_i(k)$  is defined as:  $val(x_i) = \min\{k | k \in Supp(x_i)\}$ . A series  $x_i \in \overline{\mathbb{Z}}_{\max}[\gamma]$  is said to be a polynomial if  $Supp(x_i)$  is finite. Furthermore, a polynomial is said to be a monomial if there is only one element.

<sup>1</sup>Operator  $\gamma$  plays a role similar to operator  $z^{-1}$  in the  $\mathcal{Z}$ -transform for the conventional linear systems theory.

The TEGs can then be described by the following model:

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw, \\ y &= Cx, \end{aligned} \tag{7}$$

where  $u \in (\overline{\mathbb{Z}}_{\max}[\gamma])^p$ ,  $y \in (\overline{\mathbb{Z}}_{\max}[\gamma])^m$ ,  $x \in (\overline{\mathbb{Z}}_{\max}[\gamma])^n$ , and  $w \in (\overline{\mathbb{Z}}_{\max}[\gamma])^l$  are series of inputs, outputs, states, and disturbances, respectively. Matrices  $A \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{n \times n}$ ,  $B \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{n \times p}$ ,  $R \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{n \times l}$ , and  $C \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times n}$  represent the links between each transition, and are defined as follows:

$$A = \bigoplus_{j=0}^{N_a} \gamma^j A_j, \quad B = \bigoplus_{l=0}^{N_b} \gamma^l B_l, \quad R = \gamma^0 R_0 = R_0, \quad C = C_0.$$

Therefore, the  $\gamma$ -domain representation describes the same structure of the TEG model as the event domain equation in Eq (6). By considering Theorem 1, for this system, the state and the output trajectories can be rewritten as:

$$\begin{aligned} x &= A^*Bu \oplus A^*Rw \\ y &= CA^*Bu \oplus CA^*Rw, \end{aligned} \tag{8}$$

where  $CA^*B \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times p}$  (respectively,  $CA^*R \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{m \times l}$ ) is the input/output (respectively, the disturbance/output) transfer matrix. Matrix  $CA^*B$  represents the earliest behavior of the system, therefore, it must be underlined that the uncontrollable input vector  $w$  (initial conditions or disturbances) is only able to delay the transition firings, *i.e.* according to the order relation of the semiring, to increase the vectors  $x$  and  $y$ . Consequently, the model and the initial state correspond to the fastest behavior (e.g. ideal behavior of a manufacturing system without extra delays) and the disturbances can only delay the occurrence of events (e.g. disturbance due to machine breakdown or delay due to an unexpected failure in component supply).

*Example 6:* Model of the system depicted in Fig. 1 can be described concisely thanks to the represen-

tation in Eq. (7) where the system matrices are given as

$$A = \begin{bmatrix} 2\gamma & \varepsilon & \gamma^2 & \varepsilon \\ \varepsilon & 3\gamma & \varepsilon & \varepsilon \\ 1 & 6 & 4\gamma & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon \end{bmatrix}, B = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 3 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix},$$

$$C = \begin{bmatrix} \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}, R = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix}.$$

The entries  $A(3,3) = 4\gamma$  represents the place between transition  $x_3$  and itself, it means that there is one token in the place and a minimal sojourn time equal to 4 time units. The implicit model as given in Eq. (8) can be easily computed by using the toolbox `MinMaxGD`, a C++ library developed in order to handle periodic series (see [18]). It should be mentioned that this library has an interface with Scicoslab. The source code is available in [28] and yields:

$$CA^*B = \begin{bmatrix} 2(4\gamma)^* & 9(4\gamma)^* \\ \varepsilon & 6(3\gamma)^* \end{bmatrix},$$

$$CA^*R = \begin{bmatrix} 1(4\gamma)^* & 6(4\gamma)^* & (4\gamma)^* \\ \varepsilon & 3(3\gamma)^* & \varepsilon \end{bmatrix}.$$

In this paper, without loss of generality, we assume that the TEGs are structurally controllable and structurally observable. It is a classical assumption for max-plus linear systems which is of practical interests.

*Definition 6:* (Structural Controllability [3]) A TEG is said to be *structurally controllable* if every internal transition can be reached by a path from at least one input transition.

*Definition 7:* (Structural Observability [3]) A TEG is said to be *structurally observable* if, from every internal transition, there exists a path to at least one output transition.

Moreover, structural controllability (resp. Structural observability) of TEGs can be evaluated from these transfer relations thanks to the following Theorems :

*Theorem 5:* ([24], [43]) A TEG is structurally controllable if and only if the corresponding matrix  $A^*B$  is such that at least one entry on each row is different from  $\varepsilon$ .

*Theorem 6:* ([24], [43]) A TEG is structurally observable if and only if the corresponding matrix  $CA^*$  is such that at least one entry on each column is different from  $\varepsilon$ .

The entries of the transfer matrix representing a TEG have another important properties, they are periodic and causal series. Below definitions about periodic series are recalled.

*Definition 8 (Periodicity):* A series  $s \in \overline{\mathbb{Z}}_{\max}[[\gamma]]$  is said to be periodic if it can be written as  $s = p \oplus q r^*$ , with  $p = \bigoplus_{i=1}^m t_i \gamma^{n_i}$ ,  $q = \bigoplus_{j=1}^n T_j \gamma^{N_j}$  are polynomials and  $r = \tau \gamma^\nu$ , with  $\nu, \tau \in \mathbb{N}$ , is a monomial depicting the asymptotic slope of the series. Polynomial  $p$  depicts the transient behavior of the series, polynomial  $q$  represents a pattern which is repeated each  $\nu$  events and  $\tau$  time units. The ratio  $\sigma_\infty(s) = \nu/\tau$  is the production rate of the series. A matrix is said to be periodic if its entries are periodic.

*Definition 9 (Causality):* A series  $s \in \overline{\mathbb{Z}}_{\max}[[\gamma]]$  is causal if  $s = \varepsilon$  or if both  $\text{val}(s) \geq 0$  and  $s \succeq \gamma^{\text{val}(s)}$ . A matrix is causal if its entries are causal.

*Definition 10 (Realizability):* A series  $s \in \overline{\mathbb{Z}}_{\max}[[\gamma]]$  is said to be realizable if there exists three matrices  $A, B$  and  $C$  with entries in  $\mathbb{N} \cup \{-\infty, +\infty\}$  such that  $s = C A^* B$ . A matrix is said to be realizable if its entries are realizable.

In other words, a series  $s$  is realizable if it corresponds to the transfer relation of a timed event graph.

*Theorem 7 ([14], [4], Theorem 5.39):* Let  $H \in \overline{\mathbb{Z}}_{\max}[[\gamma]]^{q \times p}$  be a matrix with entries in  $\overline{\mathbb{Z}}_{\max}[[\gamma]]$ . The following statement holds :

$$H \text{ is realizable} \Leftrightarrow H \text{ is periodic and causal.}$$

*Definition 11 (Semiring  $\overline{\mathbb{Z}}_{\max}[[\gamma]]^+$ ):* The set of causal elements of  $\overline{\mathbb{Z}}_{\max}[[\gamma]]$  is a semiring denoted  $\overline{\mathbb{Z}}_{\max}[[\gamma]]^+$ .

It must be noted that  $\varepsilon, e \in \overline{\mathbb{Z}}_{\max}[[\gamma]]^+ \subset \overline{\mathbb{Z}}_{\max}[[\gamma]]$  and that  $\overline{\mathbb{Z}}_{\max}[[\gamma]]^+$  is closed for laws  $\oplus$  and  $\otimes$ , and also for infinite sums too. Hence, according to Definition 2,  $\overline{\mathbb{Z}}_{\max}[[\gamma]]^+$  is a complete subsemiring of  $\overline{\mathbb{Z}}_{\max}[[\gamma]]$ .

*Theorem 8:* The canonical injection  $\text{Id}_{\overline{\mathbb{Z}}_{\max}[[\gamma]]^+} : \overline{\mathbb{Z}}_{\max}[[\gamma]]^+ \rightarrow \overline{\mathbb{Z}}_{\max}[[\gamma]]$  is residuated and its residual is denoted  $\text{Pr}_+ : \overline{\mathbb{Z}}_{\max}[[\gamma]] \rightarrow \overline{\mathbb{Z}}_{\max}[[\gamma]]^+$ .

$\text{Pr}_+(s)$  is the greatest causal series less than or equal to series  $s \in \overline{\mathbb{Z}}_{\max}[[\gamma]]$ . From a practical point of view, for all series  $s \in \overline{\mathbb{Z}}_{\max}[[\gamma]]$ , the computation of  $\text{Pr}_+(s)$  is obtained by:

$$\text{Pr}_+(s) = \text{Pr}_+ \left( \bigoplus_{k \in \mathbb{Z}} s(k) \gamma^k \right) = \bigoplus_{k \in \mathbb{Z}} s_+(k) \gamma^k$$

where

$$s_+(k) = \begin{cases} s(k) & \text{if } (k, s(k)) \geq (0, 0), \\ \varepsilon & \text{otherwise.} \end{cases}$$

*Example 7:* Let  $s = -5\gamma^{-1}(3\gamma^2)^* \in \overline{\mathbb{Z}}_{\max}[[\gamma]]$  be a periodic series. It can be written  $s = -5\gamma^{-1} \oplus -2\gamma^1 \oplus 1\gamma^3 \oplus 4\gamma^5 \oplus \dots$ , hence the causal projection  $s_+ = \text{Pr}_+(s) = 1\gamma^3 \oplus 4\gamma^5 \oplus \dots = 1\gamma^3(3\gamma^2)^*$  is the greatest series in  $\overline{\mathbb{Z}}_{\max}[[\gamma]]^+$  such that  $s_+ \preceq s$  (see item (ii) in Theorem 4).

#### IV. MAX-PLUS OBSERVER

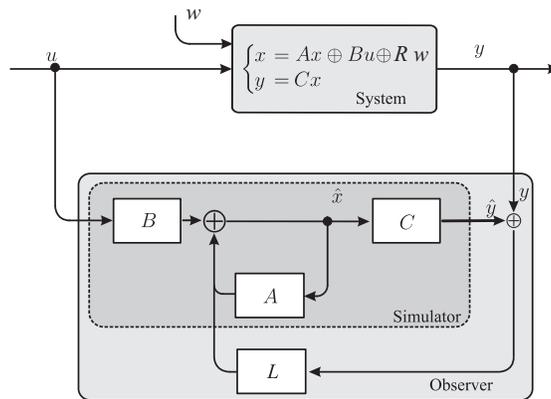


Fig. 2. The observer structure of max-plus linear systems.

Fig. 2 depicts the observer structure directly inspired from the Luenberger observer in classical linear system theory ([26],[27],[35]). The observer matrix  $L$  is used to provide information from the system output into the simulator in order to take the disturbances  $w$  acting on the system into account. The simulator is described by the model<sup>2</sup>(matrices  $A, B, C$ ) which is assumed to represent the fastest behavior of the real system in a guaranteed way<sup>3</sup>, furthermore, the simulator is initialized by the canonical initial conditions (*i.e.*,  $\hat{x}_i(k) = \varepsilon, \forall k \leq 0$ ).

By considering the configuration in Fig. 2 and these assumptions, the computation of the optimal observer matrix  $L$  will be proposed in order to achieve the constraint  $\hat{x} \preceq x$ . Optimality means that matrix  $L$  is the greatest one, according to the residuation theory (see Definition 3). Therefore, the estimated state  $\hat{x}$  is the greatest which achieves the objective, in other words, it is as close as possible to  $x$ . As in the development proposed in conventional linear systems theory, matrices  $A, B, C$  and  $R$  are assumed to be known, then the system trajectories are given by Eq. (8). According to Fig. 2, the observer equations, similarly as the Luenberger observer, are given by:

<sup>2</sup>Disturbances are uncontrollable and *a priori* unknown, then the simulator does not take them into account.

<sup>3</sup>Unlike in the conventional linear system theory, this assumption means that the fastest behavior of the system is assumed to be known and that the disturbances can only delay its behavior.

$$\begin{aligned}
\hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) = A\hat{x} \oplus Bu \oplus LC\hat{x} \oplus LCx \\
&= (A \oplus LC)\hat{x} \oplus Bu \oplus LCx, \\
&= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCx \text{ (see Theorem 1),} \\
&= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LC(A^*Bu \oplus A^*Rw) \\
&\quad \text{(according to Eq. (8)).}
\end{aligned} \tag{9}$$

By applying (f.1) the following equality is obtained:

$$(A \oplus LC)^* = A^*(LCA^*)^*, \tag{10}$$

by replacing in Eq. (9):

$$\begin{aligned}
\hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^*LCA^*Bu \\
&\quad \oplus A^*(LCA^*)^*LCA^*Rw,
\end{aligned} \tag{11}$$

and by recalling that  $(LCA^*)^*LCA^* = (LCA^*)^+$  (see Example 4), this equation may be written as follows:

$$\hat{x} = A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Bu \oplus A^*(LCA^*)^+Rw.$$

Since  $(LCA^*)^* \succeq (LCA^*)^+ = (LCA^*)^*LCA^*$ , the observer model may be written as follows:

$$\begin{aligned}
\hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Rw \\
&= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw,
\end{aligned} \tag{12}$$

due to Eq.(10).

As said previously, the objective considered is to compute the greatest observation matrix  $L$ , denoted as  $L_{opt}$ , such that the estimated state vector  $\hat{x}$  be as close as possible to state  $x$ , under the constraint  $\hat{x} \preceq x$ , formally it can be written as, finding the greatest  $L$  satisfying the following inequality,  $\forall u, w$ :

$$(A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw \preceq A^*Bu \oplus A^*Rw, \tag{13}$$

or equivalently :

$$(A \oplus LC)^*B \preceq A^*B \text{ and,}$$

$$(A \oplus LC)^*LCA^*R \preceq A^*R, \text{ respectively.}$$

*Lemma 1 ([27]):* The following equivalence holds:

$$(A \oplus LC)^*B = A^*B \Leftrightarrow L \preceq L_1 = (A^*B) \phi(CA^*B).$$

*Lemma 2 ([27]):* The following equivalence holds:

$$(A \oplus LC)^*LCA^*R \preceq A^*R \Leftrightarrow L \preceq L_2 = (A^*R) \phi(CA^*R).$$

*Proposition 2 ([27]):*  $L_{opt} = L_1 \wedge L_2$  is the greatest observer matrix  $L$  such that:

$$\hat{x} = A\hat{x} \oplus Bu \oplus Ly \preceq x = Ax \oplus Bu \oplus Rw, \quad \forall(u, w).$$

Notice that the preceding lemmas and proposition could be generalized to causal projections  $L_{opt_+} = \text{Pr}_+(L_{opt})$ , which is the the greatest causal solution for  $(A \oplus LC)^*B \preceq A^*B$  and  $(A \oplus LC)^*LCA^*R \preceq A^*R$ , with  $\hat{x} \preceq x$ .

*Proposition 3:* Matrix  $\tilde{L} = C^T$ , where  $(\cdot)^T$  denotes the matrix transpose, ensures the equality between estimated output  $\hat{y}$  and measured output  $y$ .

*Proof:* Based on Eq. (13),  $\hat{y} = y$  is equivalent to

$$C(A \oplus \tilde{L}C)^*B = CA^*B, \tag{14}$$

$$\text{and } C(A \oplus \tilde{L}C)^*\tilde{L}CA^*R = CA^*R. \tag{15}$$

By assumption, the entries of matrix  $C \in \overline{\mathbb{Z}}_{\max}[\gamma]^{m \times n}$  are in  $\{\varepsilon, e\}$  and at most one is equal to  $e$  on each row (see Section III). Hence, it must be noted that  $\tilde{L}C = C^T C \preceq I_n$ , where  $I_n \in \overline{\mathbb{Z}}_{\max}[\gamma]^{n \times n}$  is the identity matrix and that  $C\tilde{L} = CC^T = I_m$  where  $I_m \in \overline{\mathbb{Z}}_{\max}[\gamma]^{m \times m}$ . This implies that

$$\begin{aligned} (A \oplus \tilde{L}C)^* &\preceq (A \oplus I_n)^* = (A^*I_n)^*A^*, \text{ due to (f.1),} \\ &= A^*A^* = A^* \preceq (A \oplus \tilde{L}C)^*, \text{ due to (f.4).} \end{aligned}$$

Hence, the equality  $(A \oplus \tilde{L}C)^* = A^*$  holds. By replacing the equality  $(A \oplus \tilde{L}C)^* = A^*$  in Eq. (14), the first equality is proved. By replacing the equality  $(A \oplus \tilde{L}C)^* = A^*$  in Eq.(15), its left hand side term

can be written  $C(A \oplus \tilde{L}C)^* \tilde{L}CA^*R = CA^* \tilde{L}CA^*R$ . This leads to the two following inequalities :

$$CA^* \tilde{L}CA^*R \preceq CA^* I_n A^*R = CA^*R \text{ due to (f.4),}$$

$$CA^* \tilde{L}CA^*R \succeq C \tilde{L}CA^*R = I_m CA^*R = CA^*R$$

due to Kleene star definition,

hence, Eq. (15) holds. ■

*Corollary 1:* Matrix  $L_{opt_+} = \text{Pr}_+(L_{opt})$  is the greatest causal observer ensuring the equality between the estimated output  $\hat{y}$  and the measured output  $y$ .

*Proof:* In order to prove the equality between the estimated output  $\hat{y}$  and the measured output  $y$ , we recall that

$$y = CA^*Bu \oplus CA^*Rw,$$

$$\hat{y} = C(A \oplus L_{opt_+}C)^*Bu$$

$$\oplus C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*Rw,$$

so, we only need to show  $CA^*B = C(A \oplus L_{opt_+}C)^*B$  and  $CA^*R = C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R$ , respectively. First, it must be noted that  $\tilde{L} = C^T$  is causal, since  $C$  is causal, and recalled that  $(A \oplus \tilde{L}C)^* = A^*$ , hence  $(A \oplus \tilde{L}C)^*B = A^*B$  and  $C(A \oplus \tilde{L}C)^*B = CA^*B$ . Second,  $(A \oplus \tilde{L}C)^* \tilde{L}CA^*R = A^* \tilde{L}CA^*R \preceq A^*R$  due to  $\tilde{L}C \preceq I_n$ . According to Theorem 8  $L_{opt_+}$  is the greatest causal matrix such that  $L_{opt_+} \preceq L_{opt}$ , hence according to Lemma 1 and Lemma 2 matrix  $L_{opt_+}$  is the greatest causal matrix such that  $(A \oplus LC)^*B = A^*B$  and  $(A \oplus LC)^*LCA^*R \preceq A^*R$ , hence,  $\tilde{L} \preceq L_{opt_+}$ . Since matrix  $L_{opt_+}$  is such that  $C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R \preceq CA^*R$ , and since  $\tilde{L} \preceq L_{opt_+}$  the following inequality holds:

$$C(A \oplus \tilde{L}C)^* \tilde{L}CA^*R \preceq C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R \preceq CA^*R.$$

According to Proposition 3, the left hand side term is equal to  $C(A \oplus \tilde{L}C)^* \tilde{L}CA^*R = CA^*R$ . Therefore, we conclude  $C(A \oplus L_{opt_+}C)^*L_{opt_+}CA^*R = CA^*R$ , and  $y = \hat{y}$  when the observer matrix  $L_{opt_+}$  is considered. ■

## V. OUTPUT FEEDBACK AND STATE-FEEDBACK CONTROLLERS

This section presents how to synthesize the output feedback and state-feedback controllers as an optimization problem with constraints in order to solve the model matching problem, i.e. the model reference control problem ([25], [36], [37]) for the transfer function between the external input  $v$  and the output signal  $y$ . Let us mention that some other control strategies, e.g., the one taking the disturbances

into account in model matching ([25], [33], [32]) or the one ensuring the system stabilization [19], [10], [13] could also be considered.

### A. Output Feedback Controller

The output feedback control  $u_F = P(v \oplus Fy)$ , where  $P \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times p}$  and  $F \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times m}$ , is considered. According to Eq. (8), the output of the controlled system is

$$y = CA^*BPu \oplus CA^*Rw = CA^*BP(v \oplus Fy) \oplus CA^*Rw.$$

Hence, the input of the controlled system is

$$u_F = P(v \oplus Fy) = Pv \oplus PF(CA^*Bu_F \oplus CA^*Rw),$$

which yields, by applying Theorem 1, the following input and output:

$$\begin{aligned} u_F &= (PFCA^*B)^*Pv \oplus (PFCA^*B)^*PFCA^*Rw, \\ &\triangleq H_{uv}(P, F)v \oplus H_{uw}(P, F)w, \\ y &= (CA^*BPF)^*CA^*BPv \oplus (CA^*BPF)^+CA^*Rw, \\ &\triangleq H_{yv}(P, F)v \oplus H_{yw}(P, F)w. \end{aligned}$$

The aim of the model matching problem is to obtain the controllers  $P$  and  $F$  such that the transfer matrix  $H_{uv}(P, F)$  between  $v$  and  $u$  be the greatest, while the transfer matrix  $H_{yv}(P, F)$ , be smaller than, but as close as possible to, a reference transfer, denoted  $G_{ref}$ . This reference model describes the desired behaviour in the absence of disturbances. The term ‘‘smaller than’’ means that the controlled system must behave faster than the behaviour described by the specification.

Formally, the synthesis aims to find the greatest transfer matrix  $H_{uv}(P, F)$  by optimal choices of  $P$  and  $F$  such that the constraint  $H_{yv}(P, F) \preceq G_{ref}$  is satisfied according to the just-in-time criterion. Mathematically, this optimization problem is stated as follows:

$$\begin{aligned} \bigoplus_{P, F} \quad & H_{uv}(P, F) & (16) \\ \text{subject to} \quad & H_{yv}(P, F) \preceq G_{ref}. \end{aligned}$$

The maximization of transfer  $H_{uv}(P, F)$  corresponds to a just-in-time criterion which aims delaying as much as possible the inputs while respecting the constraint.

*Lemma 3:* The following equivalence holds :

$$H_{yv}(P, F) \preceq G_{ref} \Leftrightarrow H_{uv}(P, F) \preceq (CA^*B) \backslash G_{ref}. \quad (17)$$

Hence, the constraint of problem (16) can be replaced by the right hand side constraint in Eq. (17), and the optimal solution of the problem is  $H_{uv}(P, F) = (CA^*B) \backslash G_{ref}$ .

*Proof:* First, according to (f.3), the following equality holds

$$\begin{aligned} H_{yv}(P, F) &= (CA^*BPF)^*CA^*BP \\ &= CA^*B(PFCA^*B)^*P, \end{aligned}$$

then, the constraint  $(CA^*BPF)^*CA^*BP \preceq G_{ref}$  is equivalent to  $(PFCA^*B)^*P \preceq (CA^*B) \backslash G_{ref}$ , i.e.,  $H_{uv}(P, F) \preceq (CA^*B) \backslash G_{ref}$ . Hence, the optimality is achieved if equality holds. ■

*Lemma 4:* Controller  $P$  is bounded as follows:

$$P \preceq (CA^*B) \backslash G_{ref} = P_{opt}. \quad (18)$$

*Proof:* According to the Kleene star operator definition ( $e \preceq a^*$ ), hence the constraint  $(CA^*BPF)^*CA^*BP \preceq G_{ref}$  implies that  $CA^*BP \preceq G_{ref}$ . According to the residuation definition of the left product (see Example 2), this inequality is equivalent to  $P \preceq (CA^*B) \backslash G_{ref} = P_{opt}$ . ■

*Lemma 5:* The output feedback control  $u_F = P_{opt}(v \oplus Fy)$ , with  $P_{opt} = (CA^*B) \backslash G_{ref}$  and  $F = \varepsilon$ , solves the optimization problem (16) in an optimal way.

*Proof:* It is obvious that

$$H_{uv}(P_{opt}, \varepsilon) = (P_{opt}\varepsilon CA^*B)^*P_{opt} = P_{opt} = (CA^*B) \backslash G_{ref},$$

which is the optimal solution to the optimization problem (16), according to Lemma 3. ■

*Proposition 4:* The output feedback control  $u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y)$ , with  $P_{opt} = (CA^*B) \backslash G_{ref}$  and

$$F_{opt} = P_{opt} \backslash P_{opt} \not\backslash (CA^*BP_{opt}), \quad (19)$$

solves the optimization problem (16) in an optimal way.

*Proof:* By considering  $P_{opt}$  and  $H_{uv}(P_{opt}, F) = (P_{opt}FCA^*B)^*P_{opt}$ , the following equivalences

hold:

$$\begin{aligned}
& (P_{opt}FCA^*B)^*P_{opt} \preceq (CA^*B)\setminus G_{ref} = P_{opt} \\
& \Leftrightarrow P_{opt}(FCA^*BP_{opt})^* \preceq (CA^*B)\setminus G_{ref} = P_{opt} \\
& \Leftrightarrow (FCA^*BP_{opt})^* \preceq P_{opt}\setminus P_{opt}, \text{ (see Example 2),} \\
& \Leftrightarrow (FCA^*BP_{opt})^* \preceq P_{opt}\setminus P_{opt} = (P_{opt}\setminus P_{opt})^*, \text{ (see Eq.(5)),} \\
& \Leftrightarrow FCA^*BP_{opt} \preceq (P_{opt}\setminus P_{opt})^*, \text{ (see Example 3),} \\
& \Leftrightarrow F \preceq P_{opt}\setminus P_{opt}\setminus (CA^*BP_{opt}), \text{ (see Example 2).}
\end{aligned}$$

Then, the control  $u_{F_{opt}}$  leads to  $H_{uv}(P_{opt}, F_{opt}) = (CA^*B)\setminus G_{ref}$  that is, according to Lemma 3, it solves the optimization problem (16) in an optimal way. ■

*Remark 1:* According to Proposition 4 and Lemma 5, the output feedback control  $u_F = P_{opt}(v \oplus F_{opt}y)$  solves the optimization problem in Eq.(16) in an optimal way for any  $F \in [\varepsilon, F_{opt}]$ . The control  $u_{F_{opt}}$  is the one which maximizes the information coming from the output.

### B. State Feedback Controller

It can be noticed that a state-feedback control can be considered, *i.e.*  $u_K = P(v \oplus Kx)$ , with  $K \in (\overline{\mathbb{Z}}_{\max}[\gamma])^{p \times n}$ . According to Eq. (8), the output of the controlled system is

$$x = A^*BPu \oplus A^*Rw = A^*BP(v \oplus Fx) \oplus A^*Rw.$$

This control applied to the system can be expressed as

$$u_K = P(v \oplus Kx) = P(v \oplus K(A^*Bu_K \oplus A^*Rw)),$$

which yields, by applying Theorem 1, the following transfer relations:

$$\begin{aligned}
u_K &= (PKA^*B)^*Pv \oplus (PKA^*B)^*PKA^*Rw, \\
&\triangleq N_{uv}(P, K)v \oplus N_{uw}(P, K)w. \\
x &= A^*B(PKA^*B)^*Pv \oplus A^*B(PKA^*B)^*PKA^*Rw, \\
&= A^*B(PKA^*B)^*Pv \oplus (A^*BPK)^+A^*Rw, \\
y &= CA^*BP(KA^*BP)^*v \oplus C(A^*BPK)^+A^*Rw, \\
&\triangleq N_{yv}(P, K)v \oplus N_{yw}(P, K)w.
\end{aligned}$$

The model matching problem can then be translated as the following optimization problem :

$$\begin{aligned} & \bigoplus_{P,K} N_{uv}(P, K) \\ & \text{subject to } N_{yv}(P, K) \preceq G_{ref}. \end{aligned} \quad (20)$$

*Proposition 5:* The state feedback control  $u_{K_{opt}} = P_{opt}(v \oplus K_{opt}x)$ , with  $P_{opt} = (CA^*B) \setminus G_{ref}$  and

$$K_{opt} = P_{opt} \setminus P_{opt} \setminus (A^*BP_{opt}), \quad (21)$$

solves the optimization problem in Eq. (20) in an optimal way.

*Proof:* The proof takes the same steps as the ones in Proposition 4 by replacing  $CA^*BP_{opt}$  by  $A^*BP_{opt}$ . Hence, the control  $u_{K_{opt}}$  leads to  $N_{uv}(P_{opt}, K_{opt}) = (CA^*B) \setminus G_{ref}$ , i.e., the control  $u_{K_{opt}}$  solves the optimization problem (20) in an optimal way. ■

## VI. OBSERVER-BASED CONTROLLERS

As in the classical theory, the state often is not measurable or it is too expensive to measure all the states. Hence, in this section, we propose to use the estimated state  $\hat{x}$ , obtained thanks to the observer proposed in section IV, to compute the observer-based state-feedback control law. Then this control strategy is compared with the output feedback control as given in Proposition 4. Formally, the observer-based control  $u_M = P(v \oplus M\hat{x})$  is considered, where

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus L_{opt}(Cx \oplus C\hat{x}) \\ &= (A \oplus L_{opt}C)^*Bu \oplus (A \oplus L_{opt}C)^*L_{opt}CA^*Rw \end{aligned}$$

(See Eq.(12)).

The optimal observer matrix  $L_{opt}$  as given in Proposition 2 is clearly independent of the control law  $u_M$ .



The system state can be written, as

$$\begin{aligned}
x &= A^*Bu_M \oplus A^*Rw \\
&= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \\
&\oplus A^*BPM((A \oplus L_{opt}C)^*BPM)^*(A \oplus L_{opt}C)^*L_{opt}CA^*Rw \\
&\oplus A^*Rw, \\
&= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \\
&\oplus A^*BPM((A^*L_{opt}C)^*A^*BPM)^*(A^*L_{opt}C)^*A^*L_{opt}CA^*Rw \\
&\oplus A^*Rw \\
&= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \\
&\oplus (A^*BPM(A^*L_{opt}C)^*)^*A^*BPM(A^*L_{opt}C)^+A^*Rw \\
&\oplus A^*Rw, \\
&= A^*BP(M(A \oplus L_{opt}C)^*BP)^*v \\
&\oplus ((A^*BPM(A^*L_{opt}C)^*)^+A^*L_{opt}CA^*R \oplus A^*R)w \\
&\triangleq T_{xv}(P, M)v \oplus T_{xw}(P, M)w,
\end{aligned}$$

and the system output is then

$$\begin{aligned}
y &= CA^*Bu_M \oplus CA^*Rw \\
&= CA^*BP(M(A \oplus L_{opt}C)^*BP)^*v \\
&\quad \oplus (C(A^*BPM(A^*L_{opt}C)^*)^+A^*L_{opt}CA^*R \oplus CA^*R)w \\
&\triangleq T_{yv}(P, M)v \oplus T_{yw}(P, M)w.
\end{aligned} \tag{23}$$

As in Section V, the aim of the model matching problem, using the observer-based control  $u_M = P(v \oplus M\hat{x})$ , is to obtain the controllers  $P$  and  $M$  such that the transfer relation  $T_{uv}(P, M)$  be the greatest while respecting the constraint  $T_{yv}(P, M) \preceq G_{ref}$ . Formally, this model matching problem is stated as the following optimization problem:

$$\begin{aligned}
&\bigoplus_{P, M} T_{uv}(P, M) \\
&\text{subject to } T_{yv}(P, M) \preceq G_{ref}.
\end{aligned} \tag{24}$$

Lemmas 3 and 4 can be transposed in a straightforward way.

*Lemma 6:* The following equivalence holds:

$$T_{yw}(P, M) \preceq G_{ref} \Leftrightarrow T_{uw}(P, M) \preceq (CA^*B) \setminus G_{ref}. \quad (25)$$

*Lemma 7:* Controller  $P$  is bounded as follows:

$$P \preceq (CA^*B) \setminus G_{ref} = P_{opt}. \quad (26)$$

*Proposition 6:* The control  $u_M = P_{opt}(v \oplus M_{opt}\hat{x})$ , with  $P_{opt} = (CA^*B) \setminus G_{ref}$  and

$$M_{opt} = P_{opt} \setminus P_{opt} \setminus (A^*BP_{opt}) = K_{opt}, \quad (27)$$

solves the optimization problem in Eq. (24) in an optimal way.

*Proof:* By considering Lemma 1 and Proposition 2, the equality  $(A \oplus L_{opt}C)^*B = A^*B$  holds. Then, the following equivalence holds as well:

$$\begin{aligned} CA^*BP_{opt}(M(A \oplus L_{opt}C)^*BP_{opt})^* &\preceq G_{ref} \\ \Leftrightarrow CA^*BP_{opt}(MA^*BP_{opt})^* &\preceq G_{ref}. \end{aligned}$$

Hence, according to Proposition 5,  $M_{opt} = K_{opt}$ . ■

Thanks to the Separation Principle, Proposition 6 shows that the controller synthesis and the observer synthesis can be obtained independently. In another words, first, we can find the greatest observer matrix  $L_{opt}$  to ensure that the estimated output is the same as the original output. Second, we can find the greatest state feedback matrix  $M_{opt}$  to ensure that the greatest closed-loop transfer relation is smaller than the desired transfer matrix  $G_{ref}$ . After combining the greatest observer matrix  $L_{opt}$  and the state feedback matrix  $M_{opt}$ , the observer-based controller is constructed and denoted as  $u_{M_{opt}} = P_{opt}(v \oplus M_{opt}\hat{x})$ , where  $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt}y$ . Both of the observer-based control,  $u_{M_{opt}} = P_{opt}(v \oplus M_{opt}\hat{x})$  and the output feedback control,  $u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y)$  optimize the transfer function matrix between  $v$  and  $u$  by obtaining  $T_{uw}(P_{opt}, M_{opt}) = H_{uw}(P_{opt}, F_{opt}) = (CA^*B) \setminus G_{ref}$ .

*Proposition 7:* The output feedback control law  $u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y)$ , the observer-based control law  $u_{M_{opt}} = P_{opt}(v \oplus M_{opt}\hat{x})$ , and the state feedback control law  $u_{K_{opt}} = P_{opt}(v \oplus K_{opt}x)$  are ordered as follows:

$$u_{F_{opt}} \preceq u_{M_{opt}} \preceq u_{K_{opt}}.$$

*Proof:* According to Eq. (19) and Eq. (27), the following equality holds:

$$F_{opt} = M_{opt} \not\prec C.$$

Hence,  $F_{opt}C \preceq M_{opt} \Rightarrow F_{opt}C\hat{x} \preceq M_{opt}\hat{x}$ . According to Corollary 1,  $C\hat{x} = \hat{y} = y$ , hence  $F_{opt}y \preceq M_{opt}\hat{x}$ . This means that the feedback control taking the output into account is smaller than or equal to the observer-based control using the estimated state. By recalling that the addition and product laws are order preserving, it appears that:

$$u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y) \preceq u_{M_{opt}} = P_{opt}(v \oplus M_{opt}\hat{x}).$$

According to Proposition 2,  $\hat{x} \preceq x$ , and, according to Proposition 6,  $M_{opt} = K_{opt}$ . Hence,

$$u_{M_{opt}} = P_{opt}(v \oplus M_{opt}\hat{x}) \preceq u_{K_{opt}} = P_{opt}(v \oplus K_{opt}x).$$

■

According to the just-in-time criterion, Proposition 7 shows that the observer-based control strategy is better than the output feedback control strategy. For instance, in a manufacturing setting, the observer-based control would provide a better scheduling by starting the process later than the output feedback control, while ensuring the same output parts finishing time. This scheduling would allow users to load the raw parts later rather than earlier to avoid unnecessary congestions in the manufacturing lines.

## VII. APPLICATIONS TO A HIGH THROUGHPUT SCREENING SYSTEM

High throughput screening (HTS) is a standard technology in drug discovery. In HTS systems, the optimal scheduling is required to finish the drug screening in the shortest time, as well as to preserve the consistent time spending on each activity in the screening. If we are interested in the release event time of each activity, then we can model the HTS system as a TEG model (see [8], [9]). In [7] a real industrial HTS system modelled by TEGs with more than one hundred transitions was considered. A smaller system is presented in this section to illustrate the main results of this paper. The HTS consists of four activities: activity 1, executed on the resource *Pipettor*, is filling the chemical compound A into the wells of a microplate, which lasts for 2 time units. Activity 2, executed on the resource *Pipettor* as well, is filling the chemical compound B into the wells of another microplate, which lasts for 3 time units. After 1 unit waiting time for the compound A and 6 units waiting time for the compound B, activity 3 is mixing the compound B into the microplate containing the compound A for 4 time units. The mixed compound AB will be released after activity 3 right away. In activity 4, the remaining compound B will be released after 3 time units.

This system can be represented by the TEG model given in Fig. 1, in which  $x_1$  denotes the release time of activity 1 on the *Pipettor* for the compound A,  $x_2$  denotes the release time of activity 2 on the *Pipettor* for the compound B,  $x_3$  denotes the release time of activity 3 after mixing the compounds A and B, and  $x_4$  denotes the release time of activity 4 for the remaining compound B. The inputs  $u_1$  and  $u_2$  are the controls for the loading times of activity 1 and 2, respectively, so that the users can decide when to load the chemical compounds A and B. The compound A is loaded after 1 time unit when it is ready. The compound B is loaded after 3 time units when the compound B is ready. The disturbance  $w_1$  delays the release time of the compound A after activity 1, the disturbance  $w_2$  delays the release time of the compound B after activity 2, the disturbance  $w_3$  delays the release time of the mixed compound AB after activity 3, and the disturbance  $w_4$  delays the release time of the remaining compound B after activity 4. The output  $y_1$  is the release time of the mixed compound AB. The output  $y_2$  is the release time of the unused compound B. In Fig. 1, each black token in the places represents that the corresponding resource is available, *i.e.* the activity is ready to start. The model of system shown in Fig. 1 is given in Example 6. In this paper, the control objective is to maintain the system performance, *i.e.* to obtain a just-in time control while preserving the system's full speed. Hence, the reference model transfer function series is chosen such that  $G_{ref} = CA^*B$ .

By Proposition 4 and Proposition 4, we can obtain the optimal output feedback controller as  $u_{F_{opt}} = P_{opt}(v \oplus F_{opt}y)$ , where

$$P_{opt} = (CA^*B) \setminus G_{ref} = \begin{bmatrix} (4\gamma)^* & 7(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{bmatrix},$$

$$F_{opt} = P_{opt} \setminus P_{opt} \setminus (CA^*BP_{opt}) = \begin{bmatrix} -2(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & -6(3\gamma)^* \end{bmatrix},$$

which solves the model matching problem. The pre-filter  $P_{opt}$  is causal, *i.e.*  $P_{opt+} = P_{opt}$ , but the feedback mapping  $F_{opt}$  is not causal (see Definition 9). According to Theorem 7 the realization of this control law needs to be causal, hence the causal projection must to be considered (see Theorem 8) : The greatest causal feedback matrix less than or equal to  $F_{opt}$  is

$$F_{opt+} = \text{Pr}_+(F_{opt}) = \begin{bmatrix} 2\gamma(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & \gamma^2(3\gamma)^* \end{bmatrix}.$$

The output feedback control  $u_{F_{opt+}} = P_{opt}(v \oplus F_{opt+}y)$  can be realized using a TEG model shown in Fig. 4. The pre-filter  $P_{opt}$  and the output-feedback control  $F_{opt+}$  are marked in gray areas. For instance,  $F_{opt+}(1, 1) = 2\gamma(4\gamma)^*$  implies that, in the TEG shown in Fig. 4, there is a cyclic component with one

token and 4 time delays for a new transition  $\xi_3$  and the output  $y_1$  is delayed for 2 time units and one token before going through the transition  $\xi_3$ . The other entries of  $F_{opt+y}$  and the prefilter  $P_{opt}$  can be constructed similarly in the TEG model. Practically, the output feedback control law  $u_{F_{opt+y}} = P_{opt}(v \oplus F_{opt+y})$  can be given in the event domain by considering the (max-plus)-algebra as follows:

$$F_{opt+y} : \begin{cases} \xi_3(k) = 4\xi_3(k-1) \oplus 2y_1(k-1) \oplus 1y_2(k), \\ \xi_4(k) = 3\xi_4(k-1) \oplus y_2(k-2), \end{cases}$$

$$P_{opt}(v \oplus F_{opt+y}) :$$

$$\begin{cases} \xi_1(k) = 4\xi_1(k-1) \oplus \xi_3(k) \oplus 7\xi_4(k) \oplus v_1(k) \oplus 7v_2(k), \\ \xi_2(k) = 3\xi_2(k-1) \oplus \xi_4(k) \oplus v_2(k), \end{cases}$$

and the controls  $u_1(k) = \xi_1(k)$  and  $u_2(k) = \xi_2(k)$ .

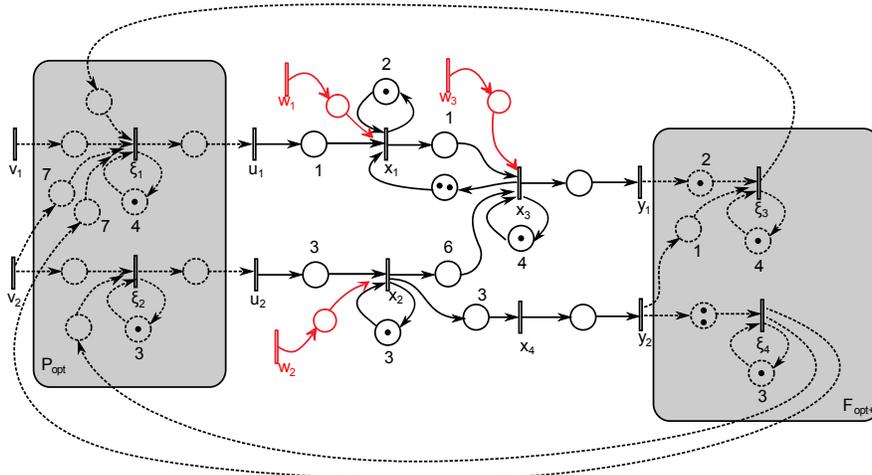


Fig. 4. The TEG realization of the causal output feedback controller  $u_{F_{opt+y}} = P_{opt}(v \oplus F_{opt+y})$  for the HTS system.

The output feedback control law  $u_{F_{opt+y}} = P_{opt}(v \oplus F_{opt+y})$  can alternatively be given in the time domain by considering the min-plus algebra as follows:

$$F_{opt+y} : \begin{cases} \xi_3(t) = 1\xi_3(t-4) \oplus 1y_1(t-2) \oplus y_2(t-1), \\ \xi_4(t) = 1\xi_4(t-3) \oplus 2y_2(t), \end{cases}$$

$$P_{opt}(v \oplus F_{opt+y}) :$$

$$\begin{cases} \xi_1(t) = 1\xi_1(t-4) \oplus \xi_3(t) \oplus \xi_4(t-7) \oplus v_1(t) \oplus v_2(t-7), \\ \xi_2(t) = 1\xi_2(t-3) \oplus \xi_4(t) \oplus v_2(t), \end{cases}$$

and the controls  $u_1(t) = \xi_1(t)$  and  $u_2(t) = \xi_2(t)$ , where  $\oplus$  means the min operation.

Next, we construct the observer-based control  $u_{M_{opt}} = P_{opt}(v \oplus M_{opt}\hat{x})$  with  $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt}y$ , where  $P_{opt} = (CA^*B) \setminus G_{ref}$  is the same as above. According to Lemma 1, Lemma 2, and Eq. (21), the observer matrix  $L_{opt}$  and the state-feedback matrix  $M_{opt}$  are computed, respectively, as follows:

$$\begin{aligned} L_{opt} &= L_1 \wedge L_2 = (A^*B) \setminus (CA^*B) \wedge (A^*R) \setminus (CA^*R) \\ &= \begin{bmatrix} \gamma^2(4\gamma)^* & 3\gamma^2(4\gamma)^* \\ \varepsilon & -3(3\gamma)^* \\ (4\gamma)^* & 3(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{bmatrix}, \\ M_{opt} &= P_{opt} \setminus P_{opt} \setminus (A^*BP_{opt}) \\ &= \begin{bmatrix} -1(4\gamma)^* & 4(4\gamma)^* & -2(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & -3(3\gamma)^* & \varepsilon & -6(3\gamma)^* \end{bmatrix}. \end{aligned}$$

Then, the causal observer matrix  $L_{opt+}$  is

$$L_{opt+} = \text{Pr}(L_{opt+}) = \begin{bmatrix} \gamma^2(4\gamma)^* & 3\gamma^2(4\gamma)^* \\ \varepsilon & \gamma(3\gamma)^* \\ (4\gamma)^* & 3(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{bmatrix},$$

and the causal state-feedback matrix  $M_{opt+} = \text{Pr}_+(M_{opt})$  is

$$M_{opt+} = \begin{bmatrix} 3\gamma(4\gamma)^* & 4(4\gamma)^* & 2\gamma(4\gamma)^* & 1(4\gamma)^* \\ \varepsilon & \gamma(3\gamma)^* & \varepsilon & \gamma^2(3\gamma)^* \end{bmatrix}.$$

The observer-based control  $u_{M_{opt+}} = P_{opt}(v \oplus M_{opt+}\hat{x})$  with  $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt+}y$  can be realized using a TEG model shown in Fig. 5. The pre-filter  $P_{opt}$ , the observer mapping  $L_{opt+}$ , and the state-feedback control  $M_{opt+}$  are marked in gray areas. For instance,  $L_{opt+}(1,1) = \gamma^2(4\gamma)^*$  implies that, in the TEG model shown in Fig. 5, there is a cyclic component with one token and 4 time delays for a new transition  $\xi_3$  and the output  $y_1$  has two tokens before going through the transition  $\xi_3$ . The other entries of matrix  $L_{opt+}$ , the observer-based state feedback matrix  $M_{opt+}$  and the prefilter  $P_{opt}$  can be constructed in a similar manner.

The estimated states  $\hat{x} = A\hat{x} \oplus Bu \oplus L_{opt+}y$  can be written in the event domain by considering the

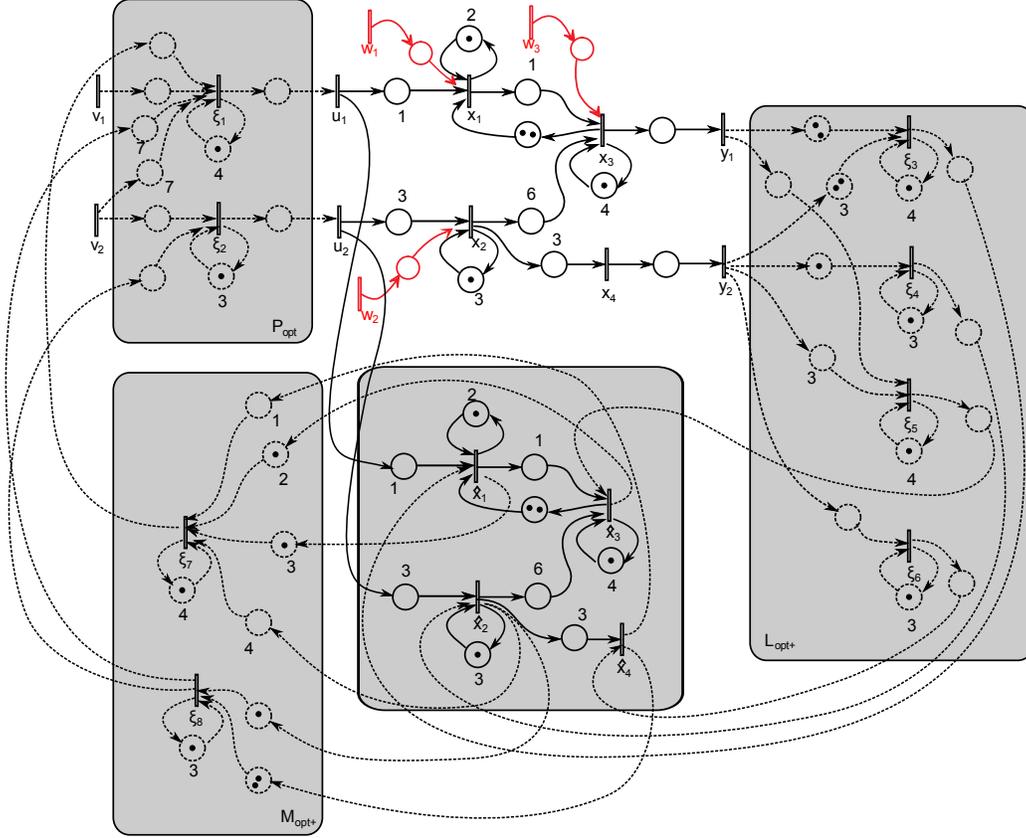


Fig. 5. The TEG realization of the causal observer-based controller  $u_{M_{opt+}} = P_{opt}(v \oplus M_{opt+}\hat{x})$  for the HTS system.

(max-plus)-algebra as follows:

$$L_{opt+y} : \begin{cases} \xi_3(k) = 4\xi_3(k-1) \oplus y_1(k-2) \oplus 3y_2(k-2), \\ \xi_4(k) = 3\xi_4(k-1) \oplus y_2(k-1), \\ \xi_5(k) = 4\xi_5(k-1) \oplus y_1(k) \oplus 3y_2(k), \\ \xi_6(k) = 3\xi_6(k-1) \oplus y_2(k), \end{cases}$$

$$\hat{x} : \begin{cases} \hat{x}_1(k) = 2\hat{x}_1(k-1) \oplus \hat{x}_3(k-2) \oplus 1u_1(k) \oplus \xi_3(k), \\ \hat{x}_2(k) = 3\hat{x}_2(k-1) \oplus 3u_2(k) \oplus \xi_4(k), \\ \hat{x}_3(k) = 1\hat{x}_1(k) \oplus 6\hat{x}_2(k) \oplus 4\hat{x}_3(k-1) \oplus \xi_5(k), \\ \hat{x}_4(k) = 3\hat{x}_2(k) \oplus \xi_6(k). \end{cases}$$

where  $\xi_i$ ,  $i = 3, \dots, 6$ , are the intermediate transitions in the TEG shown in Fig. 5. Then the event domain representation for the observer-based control law  $u_{M_{opt+}} = P_{opt}(v \oplus M_{opt+}\hat{x})$  is obtained as

follows:

$$\begin{aligned}
& M_{opt+\hat{x}} : \\
& \left\{ \begin{aligned} \xi_7(k) &= 4\xi_7(k-1) \oplus 3\hat{x}_1(k-1) \\ &\oplus 4\hat{x}_2(k) \oplus 2\hat{x}_3(k-1) \oplus 1\hat{x}_4(k), \xi_8(k) \\ &= 3\xi_8(k-1) \oplus \hat{x}_2(k-1) \oplus \hat{x}_4(k-2), \end{aligned} \right. \\
& P_{opt}(v \oplus M_{opt+\hat{x}}) : \\
& \left\{ \begin{aligned} \xi_1(k) &= 4\xi_1(k-1) \oplus \xi_7(k) \oplus 7\xi_8(k) \oplus v_1(k) \oplus 7v_2(k), \\ \xi_2(k) &= 3\xi_2(k-1) \oplus \xi_8(k) \oplus v_2(k), \end{aligned} \right.
\end{aligned}$$

and  $u_1(k) = \xi_1(k)$  and  $u_2(k) = \xi_2(k)$ , where  $\xi_i$ ,  $i = 1, 2, 7, 8$ , are the intermediate transitions in the TEG shown in Fig. 5. Similarly, the estimated

state  $\hat{x}$  and the observer-based control law  $u_{M_{opt+\hat{x}}} = P_{opt}(v \oplus M_{opt+\hat{x}})$  could be written in time-domain equations by considering the min-plus algebra. The extended developments and the source code are available in [28].

By Proposition 7, the observer-based control law  $u_{M_{opt+\hat{x}}} = P_{opt}(v \oplus M_{opt+\hat{x}})$  is greater than the output feedback control law  $u_{F_{opt+y}} = P_{opt}(v \oplus F_{opt+y})$ , for any external input  $v$ . This result can be verified in this example, because

$$F_{opt+}CA^*B \preceq M_{opt+}(A \oplus L_{opt+}C)^*B$$

$$\text{and } F_{opt+}CA^*R = M_{opt+}(A \oplus L_{opt+}C)^*L_{opt+}CA^*R$$

hold, then  $F_{opt+y} \preceq M_{opt+\hat{x}}$  (see the proofs in Proposition 7), where

$$\begin{aligned}
F_{opt+}CA^*B &= \begin{bmatrix} 4\gamma(4\gamma)^* & 7(4\gamma)^* \\ \varepsilon & 6\gamma^2(3\gamma)^* \end{bmatrix}, \\
M_{opt+}(A \oplus L_{opt+}C)^*B &= \begin{bmatrix} (4\gamma)^* & 7(4\gamma)^* \\ \varepsilon & 3\gamma(3\gamma)^* \end{bmatrix}, \\
F_{opt+}CA^*R &= M_{opt+}(A \oplus L_{opt+}C)^*L_{opt+}CA^*R \\
&= \begin{bmatrix} 3\gamma(4\gamma)^* & 4(4\gamma)^* & 2\gamma(4\gamma)^* & \varepsilon \\ \varepsilon & 3\gamma^2(3\gamma)^* & \varepsilon & \varepsilon \end{bmatrix}.
\end{aligned}$$

## VIII. CONCLUSION

The main contribution of this paper is the design of an observer-based controller for max-plus linear systems, where only a subset of the states obtained from measurement is available for the controller.

This paper first constructs the observer structure for max-plus linear systems, and then finds the greatest observer matrix such that the estimated output preserves the original output behaviors. Second, this paper calculates the greatest output feedback and state-feedback control laws such that the closed-loop transfer relation is smaller than the reference transfer relation in a model matching problem. Then, an observer-based controller is constructed using the estimated state in the TEG model of max-plus linear systems. Moreover, it is proved that the observer-based controller provides a greater control than the output feedback control, i.e. a better performance in terms of the just-in-time control criterion. At last, this paper applies the observer-based controller and the output feedback synthesis to a practical application of a HTS system in drug discovery. Both of the observer-based controller and the output feedback controller are constructed in the TEG models. The scheduling obtained from the observer-based controller yields better performance because it allows users to load the chemical compounds as late as possible to avoid unnecessary congestions according to the just-in-time criterion. In conclusion, the estimated state could also be useful in fault detection procedure [42] and then could be used to elaborate diagnosis for max-plus linear systems. Indeed differences between expected state computed thanks to the simulator and the one estimated thanks to the observer could be used to generate residue to detect abnormal behaviors, this would be the purpose of next works.

## REFERENCES

- [1] B. Addad, S. Amari, and J-J Lesage. Networked conflicting timed event graphs representation in  $(\max,+)$  algebra. *Discrete Event Dynamic Systems*, 22(4):429–449, December 2012.
- [2] S. Amari, I. Demongodin, J.J. Loiseau, and C. Martinez. Max-plus control design for temporal constraints meeting in timed event graphs. *IEEE Transactions on Automatic Control*, 57(2):462–467, 2012.
- [3] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronisation and Linearity: An Algebra for Discrete Event Systems*. John Wiley and Sons, New York, 1992.
- [4] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity : An Algebra for Discrete Event Systems*. Wiley and Sons, 1992.
- [5] T.S. Blyth. *Lattices and Ordered Algebraic Structures*. Springer, 2005.
- [6] T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Pergamon press, 1972.
- [7] T. Brunch, L. Hardouin, and J. Raisch. *Modeling and Control of Complex Systems in a Dioid Framework*. Thesis, LARIS - Université d’Angers, TU Berlin, 2014.
- [8] T. Brunsch, L. Hardouin, and J. Raisch. Control of Cyclically Operated High-Throughput Screening Systems. In *10th International Workshop on Discrete Event Systems, 2010, Berlin, WODES’10.*, pages 177–183, 2010.
- [9] T. Brunsch, J. Raisch, and L. Hardouin. Modeling and control of high-throughput screening systems. *Control Engineering Practice*, 20:1:14–23, January 2012. doi:10.1016/j.conengprac.2010.12.006.
- [10] G. Cohen. Dioids and Discrete Event Systems. In *Proc. of the 11th Workshop On Discrete Event Systems, WODES*, number 199 in Lect. Notes. in Control and Inf. Sci, Sophia Antipolis, June 1994. Springer.

- [11] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. A linear system theoretic view of discrete event processes and its use for performance evaluation in manufacturing. *IEEE Trans. on Automatic Control*, AC-30:210–220, 1985.
- [12] G. Cohen, S. Gaubert, and J.-P. Quadrat. Max-plus algebra and system theory : Where we are and where to go now. *Annual Reviews in Control*, 23:207–219, 1999.
- [13] G. Cohen, S. Gaubert, and J.P. Quadrat. From first to second-order theory of linear discrete event systems. In *12th IFAC*, Sydney, Jul. 1993.
- [14] G. Cohen, P. Moller, J.-P. Quadrat, and M. Viot. Algebraic Tools for the Performance Evaluation of Discrete Event Systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1):39–58, January 1989.
- [15] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Linear system theory for discrete-event systems. In *23rd IEEE Conf. on Decision and Control*, Las Vegas, Nevada, 1984.
- [16] B. Cottenceau, L. Hardouin, and J.-L. Boimond. Modeling and control of weight-balanced timed event graphs in dioids. *IEEE Transactions on Automatic Control*, 59(5):1219 – 1231, 2014.
- [17] B. Cottenceau, L. Hardouin, J.-L. Boimond, and J.-L. Ferrier. Model Reference Control for Timed Event Graphs in Dioids. *Automatica*, 37(9):1451–1458, September 2001. Available at doi:10.1016/S0005-1098(01)00073-5.
- [18] B. Cottenceau, L. Hardouin, M. Lhommeau, and J.-L. Boimond. Data processing tool for calculation in dioid. In *Proceedings of the 5th International Workshop on Discrete Event Systems, WODES 2000*, Ghent, Belgium, 2000. cf. [www.istia.univ-angers.fr/~hardouin/outils.html](http://www.istia.univ-angers.fr/~hardouin/outils.html).
- [19] B. Cottenceau, M. Lhommeau, L. Hardouin, and J.-L. Boimond. On Timed Event Graph Stabilization by Output Feedback in Dioid. *Kybernetika*, 39(2):165–176, 2003.
- [20] X. David-Henriet, L. Hardouin, J. Raisch, and B. Cottenceau. Optimal control for timed event graphs under partial synchronization. In *IEEE Conference on Decision and Control, 52th CDC*, Firenze, December 2013.
- [21] M. DiLoreto, S. Gaubert, R. Katz, and J-J Loiseau. Duality between invariant spaces for max-plus linear discrete event systems. *SIAM Journal on Control and Optimization*, 48(8):5606–5628, December 2010.
- [22] N. Farhi, M. Goursat, and J.-P. Quadrat. Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and petri net modeling. In *CDC'05*, Sevilla, 2005.
- [23] A. Giua and C. Seatzu. Observability of place/transition nets. *IEEE Trans. on Automatic Control*, 47(9):1424 –1437, Sep. 2002.
- [24] L. Hardouin. Sur la commande linéaire de systèmes à événements discrets dans l’algèbre (max,+). *Habilitation à Diriger des Recherches*, 2004. [http://www.istia.univ-angers.fr/~hardouin/hdr\\_hardouin.pdf](http://www.istia.univ-angers.fr/~hardouin/hdr_hardouin.pdf).
- [25] L. Hardouin, M. Lhommeau, and Y. Shang. Towards Geometric Control of Max-Plus Linear Systems with Applications to Manufacturing Systems. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 171–176, Orlando, Florida, USA, December 2011. Available at [istia.univ-angers.fr/~hardouin/YShangDDPCDC2011.pdf](http://istia.univ-angers.fr/~hardouin/YShangDDPCDC2011.pdf).
- [26] L. Hardouin, C. A. Maia, B. Cottenceau, and M. Lhommeau. Max-plus Linear Observer: Application to manufacturing Systems. In *Proceedings of the 10th International Workshop on Discrete Event Systems, WODES'10*, pages 171–176, Berlin, September 2010. Available at [istia.univ-angers.fr/~hardouin/Observer.html](http://istia.univ-angers.fr/~hardouin/Observer.html).
- [27] L. Hardouin, C. A. Maia, B. Cottenceau, and M. Lhommeau. Observer Design for (max,plus) Linear Systems. *IEEE Transactions on Automatic Control*, 55(2), February 2010. doi:10.1109/TAC.2009.2037477.
- [28] Laurent Hardouin, Ying Shang, Carlos Andrey Maia, and Bertrand Cottenceau. Extended version with source code of the

- paper observer-based controllers for max-plus linear systems. In *Laris, Internal Report*, 2016. <http://perso-laris.univ-angers.fr/~hardouin/ObservBasedController.html>.
- [29] B. Heidergott, G.-J. Olsder, and J. van der Woude. *Max Plus at Work : Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and Its Applications*. Princeton University Press, 2006.
- [30] R.D. Katz. Max-plus (a,b)-invariant spaces and control of timed discrete-event systems. *IEEE Transactin on Automatic Control*, 52(2):229–241, 2007.
- [31] J.-Y. LeBoudec and P. Thiran. *Network Calculus*. Springer Verlag, 2002.
- [32] M. Lhommeau, L. Hardouin, and B. Cottenceau. Disturbance Decoupling of Timed Event Graphs by Output Feedback Controller. In *Proceedings of the 6th International Workshop on Discrete Event Systems, WODES'02*, Zaragoza, Spain, October 2002. Available at [istia.univ-angers.fr/~hardouin/Wodes02.pdf](http://istia.univ-angers.fr/~hardouin/Wodes02.pdf).
- [33] M. Lhommeau, L. Hardouin, and B. Cottenceau. Optimal control for (max,+)-linear systems in the presence of disturbances. *Positive Systems: Theory and Applications, POSTA, Springer LNCIS 294*, pages 47–54, 2003.
- [34] M. Lhommeau, L. Hardouin, J.-L. Ferrier, and I. Ouerghi. Interval Analysis in Dioid : Application to Robust Open Loop Control for Timed Event Graphs. In *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference*, pages 7744–7749, Seville, Spain, December 2005. Available at doi:10.1109/CDC.2005.1583413.
- [35] D. Luenberger. An introduction to observers. *IEEE Trans. on Automatic Control*, 16(6):596–602, Dec. 1971.
- [36] C. A. Maia, L. Hardouin, R. Santos Mendes, and B. Cottenceau. Optimal Closed-Loop Control for Timed Event Graphs in Dioid. *IEEE Transactions on Automatic Control*, 48(12):2284–2287, 2003. Available at doi:10.1109/TAC.2003.820666.
- [37] C.A. Maia, L. Hardouin, R. Santos Mendes, and B. Cottenceau. On the Model Reference Control for Max-Plus Linear Systems. In *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference*, pages 7799–7803, Seville, Spain, December 2005. Available at doi:10.1109/CDC.2005.1583422.
- [38] Vinicius Goncalves Mariano, Carlos Andrey Maia, Laurent Hardouin, and Ying Shang. An observer for tropical linear event-invariant dynamical systems. In *53rd IEEE Conference on Decision and Control, CDC 2014*, Los Angeles, 2014.
- [39] MaxPlus. Second Order Theory of Min-linear Systems and its Application to Discrete Event Systems. In *Proceedings of the 30th CDC*, Brighton, England, December 1991.
- [40] E. Menguy, J.-L. Boimond, L. Hardouin, and J.-L. Ferrier. Just in time control of timed event graphs: update of reference input, presence of uncontrollable input. *IEEE Transactions on Automatic Control*, 45(11):2155–2159, 2000. Available at doi:10.1109/9.887652.
- [41] I. Necoara, B. DeSchutter, T.J.J. van den Boom, and H. Hellendoorn. On the Schein rank of matrices over linear lattices. *Int. J. Robust Nonlinear Control*, 19:218242, 2009.
- [42] S. Franck and X. Ding. Survey of robust residual generation and evaluation methods in observer-based fault detection system. *J. Process Control*, 7(6):403–424, 1990.
- [43] Pavel Spacek, A. El Moudni, S. Zerhouni, and M. Ferney. Max-plus algebra for discrete event systems-some links to structural controllability and structural observability. In *IEEE International Symposium on Intelligent Control*, pages 579 – 584, 1995.

## IX. APPENDIX

A. *Formulas of Star Operations ([3], Chapter 4)*

B. *Formulas of Residuations([3], Chapter 4)*

$$a^*(ba^*)^* = (a \oplus b)^* = (a^*b)^*a^* \quad (\text{f.1})$$

$$(a^*)^* = a^* \quad (a^+)^+ = a^+ \quad (\text{f.2})$$

$$(ab)^*a = a(ba)^* \quad (\text{f.3})$$

$$a^*a^* = a^* \quad a^+a^* = a^+ \quad (\text{f.4})$$

$$(a^*)^+ = (a^+)^* = a^* \quad a^+ \preceq a^* \quad (\text{f.5})$$

$$a(a \dot{\setminus} x) \preceq x \quad (\text{f.6}) \quad (x \not\phi a)a \preceq x \quad (\text{f.14})$$

$$a \dot{\setminus} (ax) \succeq x \quad (\text{f.7}) \quad (xa) \not\phi a \succeq x \quad (\text{f.15})$$

$$a(a \dot{\setminus} (ax)) = ax \quad (\text{f.8}) \quad ((xa) \not\phi a)a = xa \quad (\text{f.16})$$

$$a \dot{\setminus} (x \wedge y) = a \dot{\setminus} x \wedge a \dot{\setminus} y \quad (\text{f.9}) \quad (x \wedge y) \not\phi a = x \not\phi a \wedge y \not\phi a \quad (\text{f.17})$$

$$(a \oplus b) \dot{\setminus} x = a \dot{\setminus} x \wedge b \dot{\setminus} x \quad (\text{f.10}) \quad x \not\phi (a \oplus b) = x \not\phi a \wedge x \not\phi b \quad (\text{f.18})$$

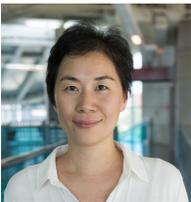
$$(ab) \dot{\setminus} x = b \dot{\setminus} (a \dot{\setminus} x) \quad (\text{f.11}) \quad x \not\phi (ba) = (x \not\phi a) \not\phi b \quad (\text{f.19})$$

$$b(a \dot{\setminus} x) \preceq (a \not\phi b) \dot{\setminus} x \quad (\text{f.12}) \quad (x \not\phi a)b \preceq x \dot{\setminus} (b \not\phi a) \quad (\text{f.20})$$

$$(a \dot{\setminus} x)b \preceq a \dot{\setminus} (xb) \quad (\text{f.13}) \quad b(x \not\phi a) \preceq (bx) \not\phi a \quad (\text{f.21})$$



**Laurent Hardouin** received the degrees of Master and Ph.D. in Acoustic and Control from the University of Poitiers, France, in 1990 and 1993, respectively. He received the Habilitation Diriger des Recherches from the University of Angers, France, in 2004. He is currently a Full Professor of ISTIA Engineering School at the University of Angers, France. He specializes in discrete-event systems, max-plus algebra, and interval analysis with applications to computer networks, manufacturing systems, transportation systems, and robotics.



**Ying Shang** received the B.S. degree in Control Engineering from Shandong University, China, in 1998, and M.S. and Ph.D. degrees in Electrical Engineering from the University of Notre Dame, USA, in 2003 and 2006, respectively. She is currently an Associate Professor in the Department of Electrical and Computer Engineering at Southern Illinois University Edwardsville, USA. Her research areas include discrete-event systems, max-plus linear systems, and hybrid systems with applications in queueing networks, manufacturing systems, and transportation networks.



**Bertrand Cottenceau** was born in 1973. He received the Ph.D. degree from the University of Angers, France, in 1999. He is currently an Associate Professor at the University of Angers. His research interests include modeling, simulation and control of timed discrete event systems with applications in manufacturing systems and computer networks.



**Carlos Andrey Maia** received the B.S. and the M.S. degrees in electrical engineering from the Universidade Federal de Minas Gerais (UFMG), Belo Horizonte, Brazil, respectively in 1994 and 1996, and the Ph.D. degree from the University of Angers, France, and from the Universidade Estadual de Campinas, Campinas, Brazil, in 2003. Since 1997, he is with the Escola de Engenharia, UFMG, where he is a Professor of electrical engineering. His current research interests include modeling, simulation, analysis, and control of time and event-driven dynamic systems.