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PERTURBATION FEEDBACK CONTROL: A GEOMETRIC INTERPRETATION

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Abstract. Perturbation feedback control is a classical procedure in control engineering that is based on linearizing a nonlinear system around some locally optimal nominal trajectory. In the presence of terminal constraints defined by a k-dimensional embedded submanifold, the corresponding flow of extremals for the underlying system gives rise to a canonical foliation in the (t, x)-space consisting of $(n - k + 1)$-dimensional leaves and k-dimensional cross sections. In this paper, the connections between the formal computations in the engineering literature and the geometric meaning underlying these constructions are described.

1. **Introduction.** *Perturbation feedback control* is probably the most important practical application of optimal control theory. It is the staple of any regulation mechanism and it is an essential feature in almost any engineering design. Practical examples are omnipresent and include, for example, auto pilots on commercial aircraft, chemical process control, or any other control or guidance scheme whose objective it is to regulate a system around some predetermined set point or nominal trajectory. Generally, these feedback laws are constructed as solutions to some linear-quadratic optimal control problem around some reference controlled trajectory. In the engineering literature (e.g., [5], [7, Chapter 6]), admittedly formal calculations abound for these *neighboring extremals* with little explanations and at times incomplete assumptions, especially if constraints and implicitly defined terminal times are involved. Mathematically, these ideas have their origin in the notion of a Jacobi-field and the Jacobi-equation in the calculus of variations. For optimal control problems, these results are often phrased in the context of sensitivity equations related to the coercivity of quadratic forms (e.g., [10]) with little geometric insight given. And if a geometric point of view is taken, the framework is one of Lagrangian manifolds in the cotangent bundle and projections onto the state-space [1, 2, 3] and the resulting geometric conditions are not related to the practically applicable criteria found in the engineering literature. Thus, even after 50 years

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of optimal control theory, there still exists a gap between the mathematical theory and engineering practice on this important topic.

In this paper, we bridge this gap by carefully establishing the connections between the formulas of the engineering literature and the geometric properties of the flow of extremals. In our point of view, which is the more classical one from a mathematical perspective, the *neighboring extremals* of the engineering literature are linearizations of the members in a parameterized family of extremals around some reference trajectory and the conditions that are imposed guarantee that there exists such a local embedding of the reference extremal into a *field of extremals*. The key property is the local injectivity of the flow of trajectories in the state-space and the formulas derived, for example, in Chapter 6 of the textbook by Bryson and Ho [7], indeed provide sufficient conditions for this to be the case around a nominal extremal. While this is rather obvious if there are no terminal constraints, it also holds in the presence of terminal constraints. In this case, it is related to a *foliation* structure that is obtained by desingularizing the flow of extremals (now in the cotangent bundle) near the terminal manifold. The procedure is canonical and it provides a geometric interpretation for some of the matrices arising in a perturbation feedback control scheme other than the solutions to the standard Riccati equations. We shall show how this implies to have one-to-one projections from the cotangent bundle into the state space which ensure the local optimality of trajectories. We also show that typically, i.e., near fold singular points, local optimality will be lost if these projections are no longer one-to-one. The results presented in this paper are classical, yet the connections between the formal calculations in the engineering literature and their fundamental geometric meaning are at best poorly explained in the literature.

2. **Parameterized Families of Extremals.** We consider a time-varying, fixed horizon, optimal control problem with a Bolza type objective and terminal constraints of the following form:

[OC]: For given initial conditions $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and a fixed terminal time T, minimize the functional

$$\mathcal{J}(u) = \int_{t_0}^{T} L(s, x(s), u(s))ds + \varphi(x(T)) \tag{1}$$

over all locally bounded, Lebesgue measurable functions $u : [t_0, T] \to U \subset \mathbb{R}^m$, $t \mapsto u(t)$, subject to the constraints that the solution x to the initial value problem

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0, \tag{2}$$

exists over the full interval $[t_0, T]$ and satisfies a terminal constraint of the form

$$\Psi(x(T)) = 0. \tag{3}$$

We assume that all the functions defining the data of the problem formulation are continuous and twice continuously differentiable in x and u. We call a locally bounded, Lebesgue measurable function u that takes values in the control set U a.e. an admissible control and the solution x of the differential equation (2) for u is the corresponding trajectory; the pair (x, u) is called a *controlled trajectory*. Local existence of solutions to the differential equation (2) is guaranteed by standard results on existence and uniqueness of solutions for ODEs, but existence over a prescribed interval cannot be guaranteed in general without strong extra assumptions on the dynamics. However, if a reference controlled trajectory $(\bar{x}, \bar{u})$ exists over an interval $[t_0, T]$, then it follows from the continuity of the general solution that this will also be true for close-by solutions and this is the framework of perturbation feedback control. Satisfying the terminal constraint then is a matter of controllability properties around this reference trajectory. We assume that the terminal set has a regular geometric structure, more specifically, that it is a k-dimensional embedded C^2-submanifold $N = \{x \in \mathbb{R}^n : \Psi(x) = 0\}$ defined by equations $\Psi : \mathbb{R}^n \to \mathbb{R}^{n-k}$, $x \mapsto \Psi(x) = (\psi_1(x), \dots, \psi_{n-k}(x))^T$, which have the property that the gradients of the functions $\psi_1(x), \dots, \psi_{n-k}(x)$ are linearly independent on N. Thus the Jacobian matrix $D\Psi$ is of full rank $n-k$ everywhere on N. Also, H denotes the control Hamiltonian defined by,

$$H = H(t, \lambda_0, \lambda, x, u) = \lambda_0 L(t, x, u) + \lambda f(t, x, u). \tag{4}$$

with $\lambda_0 \in \mathbb{R}$ and λ an n-dimensional row vector, $\lambda \in (\mathbb{R}^n)^*$. We denote the space of column vectors by $\mathbb{R}^n$ and write $(\mathbb{R}^n)^*$ for the corresponding space of row vectors.

Necessary conditions for optimality are given by the Pontryagin maximum principle [16]. (For some recent textbooks on the subject, see [4, 6, 18].) For problem $[OC]$, they state that, if (x_*, u_*) is an optimal controlled trajectory defined over the interval $[t_0, T]$, then there exist a constant $\lambda_0 \geq 0$ and a co-vector $\lambda : [t_0, T] \to (\mathbb{R}^n)^*$, the so-called *adjoint variable*, such that the following conditions are satisfied:

1. *Nontriviality* of the multipliers: $(\lambda_0, \lambda(t)) \neq 0$ for all $t \in [t_0, T]$;
2. *Adjoint equation:* the adjoint variable λ is a solution to the time-varying linear differential equation
$$\dot{\lambda}(t) = -\lambda_0 L_x(t, x_*(t), u_*(t)) - \lambda(t) f_x(t, x_*(t), u_*(t)); \tag{5}$$
3. *Minimum condition:* everywhere in $[t_0, T]$ we have that
$$H(t, \lambda_0, \lambda(t), x_*(t), u_*(t)) = \min_{v \in U} H(t, \lambda_0, \lambda(t), x_*(t), v). \tag{6}$$
4. *Transversality condition:* at the endpoint of the controlled trajectory, the covector $\lambda(T) - \lambda_0 \varphi_x(x_*(T))$ is orthogonal to the terminal manifold N.

By the rank condition on the Jacobian matrix $D\Psi$, the normal space to N at $x_*(T)$ is spanned by the gradients of the function ψ_i, $i = 1, \dots, n-k$, defining the constraint and thus the transversality condition can equivalently be expressed in the form that there exists a multiplier $\nu \in (\mathbb{R}^{n-k})^*$ such that

$$\lambda(T) = \lambda_0 \nabla \varphi(x_*(T)) + \nu D\Psi\left(x_*(T)\right). \tag{7}$$

A parameterized family of extremals is a collection of controlled trajectories and multipliers that satisfy these conditions. More specifically, we want that the parameterizations are smooth and we assume that with p a parameter, the controls $u = u(t, p)$ are continuous and for t fixed are r-times continuously differentiable in the parameter p with the partial derivatives continuous in (t, p). We write $u \in C^{0,r}$ for this class of functions and make the following formal definition:

Definition 2.1. **(C^r-parameterized family of extremals for $[OC]$)** Given an open subset P of $\mathbb{R}^n$ and an r-times continuously differentiable function $t_- : P \to (-\infty, T)$, $p \mapsto t_-(p)$, let $D = \{(t, p) : p \in P,\, t_-(p) \leq t \leq T\}$. A C^r-parameterized family $\mathcal{E}$ of extremals (or extremal lifts) with domain D consists of

1. a family of controlled trajectories (x, u), $u : D \to U$, $(t, p) \mapsto u(t, p)$, and $x : D \to \mathbb{R}^n$, $(t, p) \mapsto x(t, p)$, so that $u \in C^{0,r}(D)$ and
$$\dot{x}(t, p) = f(t, x(t, p), u(t, p)), \qquad x(T, p) \in N \tag{8}$$

2. a non-negative multiplier $\lambda_0 \in C^{r-1}(P)$, a function $\lambda : D \to (\mathbb{R}^n)^*$, $\lambda = \lambda(t,p)$, and a row vector $\nu \in (\mathbb{R}^{n-k})^*$, $\nu \in C^{r-1}(P)$, so that $(\lambda_0(p), \lambda(t,p)) \neq (0,0)$ for all $(t,p) \in D$ and the adjoint equation,

$$\dot{\lambda}(t,p) = -\lambda_0(p) L_x(t, x(t,p), u(t,p)) - \lambda(t,p) f_x(t, x(t,p), u(t,p)), \tag{9}$$

is satisfied on the interval $[t_-(p), T]$ with terminal conditions

$$\lambda(T,p) = \lambda_0(p) \nabla\varphi(x(T)) + \nu(p) D\Psi(x(T)), \tag{10}$$

3. and such that the controls $u = u(t,p)$ solve the minimization problem

$$H(t, \lambda_0(p), \lambda(t,p), x(t,p), u(t,p)) = \min_{v \in U} H(t, \lambda_0(p), \lambda(t,p), x(t,p), v). \tag{11}$$

This definition provides the framework for our constructions. It merely formalizes that all controlled trajectories in the family $\mathcal{E}$ satisfy the conditions of the maximum principle while the parameterization is sufficiently smooth. Analogous definitions can be given for more general formulations of the optimal control problem (see, for instance, [19]), but will not be needed for the purpose of this paper. Note that it has not been assumed that the parameterization $\mathcal{E}$ of extremals covers the state-space injectively and our objective here also is to use this framework to elucidate the connections between injectivity properties of the associated flow of extremals and local optimality. The degree r in the definition denotes the smoothness of the parameterization of the controls in the parameter p, $u \in C^{0,r}$. The control u extends as a $C^{0,r}$-function onto an open neighborhood of D and it thus follows that the trajectories $x(t,p)$ and their time-derivatives $\dot{x}(t,p)$ are r-times continuously differentiable in p and that these derivatives are continuous jointly in (t,p) in an open neighborhood of D, i.e., $x \in C^{1,r}(D)$. For the adjoint equation, there is a natural loss of differentiability since the Lagrangian L and the dynamics f are differentiated in x. The conditions $\lambda_0 \in C^{r-1}(P)$ and $\nu \in C^{r-1}(P)$ ensure that the boundary value $\lambda(T,p)$ is given by an $(r-1)$-times continuously differentiable function of p and thus $\lambda \in C^{1,r-1}$. In particular, for a C^1-parameterized family of extremals only continuity in p is required. If the data defining the problem $[OC]$ possess an additional degree of differentiability in x and if the the multipliers λ_0 and ν are r-times continuously differentiable with respect to p, then it follows that $\lambda \in C^{1,r}$ as well. In such a case, we call $\mathcal{E}$ a *nicely* C^r-parameterized family of extremals. Also, if $\lambda_0(p) > 0$ for all $p \in P$, then all extremals are normal and by diving by $\lambda_0(p)$ we may assume that $\lambda_0(p) \equiv 1$ and we call such a family *normal*.

In principle, the parameter can be arbitrary, but for problem $[OC]$ there exists a canonical choice for the parameterization that consists in taking as parameter the combination of the terminal state $x(T) \in N$ and the multiplier $\nu \in (\mathbb{R}^{n-k})^*$. Since the terminal manifold is k-dimensional, together this provides an n-dimensional set. Then integrating the dynamics and adjoint equation backward from the terminal time while maintaining the minimum condition, defines a parameterized family of extremals. This procedure works for what are called nonsingular extremals in [7].

Definition 2.2. (nonsingular extremal) A normal extremal $\Lambda = ((x,u), \lambda)$ consisting of a controlled trajectory $\Gamma = (x,u)$ defined over an interval $[\tau, T]$ with corresponding multiplier λ is said to be nonsingular if for all $t \in [\tau, T]$

$$\frac{\partial H}{\partial u}(t, \lambda(t), x(t), u(t)) \equiv 0$$

and the matrix
$$\frac{\partial^2 H}{\partial u^2}(t, \lambda(t), x(t), u(t))$$
is positive definite. As in the calculus of variations, in this case we say the *strengthened Legendre condition* is satisfied along the extremal Λ.

Note that, if the control lies in the interior of the control set U, then it immediately follows from the minimization condition of the maximum principle that $\frac{\partial H}{\partial u}(t, \lambda(t), x(t), u(t)) \equiv 0$ and that the matrix $\frac{\partial^2 H}{\partial u^2}(t, \lambda(t), x(t), u(t))$ is positive semi-definite. We refer to this property also as the *Legendre condition*.

Theorem 2.3. *Let $\Lambda = ((\bar{x}, \bar{u}), \bar{\lambda})$ be a nonsingular extremal for problem $[OC]$ defined over an interval $[\tau, T]$ and suppose that for every $t \in [\tau, T]$* (i) *the control $\bar{u}(t)$ lies in the interior of the control set, $\bar{u}(t) \in \mathrm{int}(U)$, and* (ii) *$\bar{u}(t)$ is the unique minimizer of the function $v \mapsto H(t, \bar{\lambda}(t), \bar{x}(t), v)$ over the control set U. Then there exists a canonical, nicely C^1-parameterized family of nonsingular extremals $\mathcal{E}$ with domain $D = \{(t,p) : p \in P,\ t_-(p) \leq t \leq T\}$ and a parameter value $p_0 \in P$ such that for all $t \in [\tau, T]$ we have that*
$$x(t, p_0) \equiv \bar{x}(t), \quad u(t, p_0) \equiv \bar{u}(t), \quad \text{and} \quad \lambda(t, p_0) \equiv \bar{\lambda}(t).$$
We say the reference extremal Λ is embedded into the family $\mathcal{E}$.

Condition (ii) that $\bar{u}(t)$ is the unique minimizer over the control set is important in the construction. It holds, for instance, if the Hamiltonian H is strictly convex in u. If it is not satisfied, the construction below can still be carried out, but it does not guarantee that the resulting controlled trajectories are extremal since the minimizing controls might switch in any neighborhood of p_0. In such a case, the smoothness properties that we require will not be satisfied and separate parameterized families of extremals will need to be considered that then must be glued together. Generally, this is a much more involved procedure and the standard realizations of perturbation feedback control do not take this into account.

Proof. Let ρ denote the curve $\rho : [\tau, T] \to [\tau, T] \times \mathbb{R}^n \times (\mathbb{R}^n)^*$, $t \mapsto \rho(t) = (t, \bar{x}(t), \bar{\lambda}(t))$. Geometrically, this is the lift of the graph of the controlled trajectory $\bar{x}$ into the cotangent bundle defined by the extremal Λ. Since the strengthened Legendre condition is satisfied along ρ, it follows from the implicit function theorem that for every time $t \in [\tau, T]$ there exist neighborhoods V_t of $\rho(t)$ and W_t of $\bar{u}(t)$, which, without loss of generality, we take in the forms
$$V_t = (t - \varepsilon, t + \varepsilon) \times B_\varepsilon(\bar{x}(t)) \times B_\varepsilon(\bar{\lambda}(t)) \subset \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n)^*$$
and $W_t = B_\delta(\bar{u}(t)) \subset \mathrm{int}(U)$, so that for all $(s, x, \lambda) \in V_t$ the equation
$$H_u(s, \lambda, x, u) = 0$$
has a unique solution $u = u(s, x, \lambda) \in W_t$. This solution is continuous, continuously differentiable in x and λ, and satisfies
$$u(s, \bar{x}(s), \bar{\lambda}(s)) = \bar{u}(s) \quad \text{for all} \quad s \in (t - \varepsilon, t + \varepsilon).$$
The parameters ε and δ generally depend on the time t, $\varepsilon = \varepsilon(t)$ and $\delta = \delta(t)$, but it follows from a standard compactness argument that there exists a positive ϵ (independent of t) so that this function exists, is continuous and continuously differentiable in x and λ on a tubular neighborhood
$$V = \{(t, x, \lambda) :\ \tau \leq t \leq T,\ \|x - \bar{x}(t)\| < \epsilon,\ \left\|\lambda - \bar{\lambda}(t)\right\| < \epsilon\}$$

of the curve ρ. Furthermore, by choosing ϵ sufficiently small, we can also guarantee that for all $(t,x,\lambda) \in V$ the control $u(t,x,\lambda)$ is the unique minimizer of the function $v \mapsto H(t,\lambda,x,v)$ over the control set U. Note that this need not hold any longer if there exists a time $t \in [\tau, T]$ and a control value $v \in U$ so that $H(t,\bar{\lambda}(t),\bar{x}(t),\bar{u}(t)) = H(t,\bar{\lambda}(t),\bar{x}(t),v)$. In this case, it is possible that there exist control values near v that are better than the local solution $u(t,x,\lambda)$ near the point $(t,\bar{\lambda}(t),\bar{x}(t))$. For this reason assumption (ii) is needed.

We now define the flow of extremals by parameterizing the terminal points for the states and multipliers. Since N is an embedded submanifold, there exists a canonical parameterization of extremals in terms of the final points for the state x in the manifold N and the normal vectors to the manifold N at x, the so-called *normal bundle*. Let $\xi_0 = \bar{x}(T) \in N$ and let $\nu_0 \in (\mathbb{R}^{n-k})^*$ be the covector in the terminal condition for the multiplier $\bar{\lambda}(T)$, $\bar{\lambda}(T) = \nabla\varphi(\xi_0) + \nu_0 D\Psi(\xi_0)$. Let P_1 be an open neighborhood of the origin in $\mathbb{R}^k$ and let $\xi : P_1 \to N$, $y \mapsto \xi(y)$, be a C^2-coordinate chart for N centered at ξ_0, i.e., ξ is a C^2-diffeomorphisms so that $\xi(0) = \xi_0$. By our standing assumption, the rows of the matrix $D\Psi(\xi(y))$ are linearly independent and therefore form a basis for the normal space to N at $\xi(y)$. Hence the vector ν defines coordinates for this normal space. Thus, if P_2 is an open neighborhood of ν_0^T in $\mathbb{R}^{n-k}$ and $P = P_1 \times P_2$, then the mapping

$$\omega : P \to N \times \left(\mathbb{R}^{n-k}\right), \qquad p = (y, \nu^T) \mapsto \omega(p) = (\xi(y), \nu^T) \tag{12}$$

defines a C^2-coordinate chart for the normal bundle to N at the point (ξ_0, ν_0^T). Define the *end-point mapping for the states and costates* as $\Omega : P \to N \times (\mathbb{R}^n)^*$,

$$p = (y, \nu^T) \mapsto \Omega(p) = (\xi(y), \nabla\varphi(\xi(y)) + \nu D\Psi(\xi(y))) = (x(T,p), \lambda(T,p)). \tag{13}$$

Note that $\Omega(p_1) = \Omega(p_2)$ if and only if $\omega(p_1) = \omega(p_2)$ and thus Ω is a C^1-diffeomorphism onto its n-dimensional image. We then define the trajectories $x = x(t,p)$ and co-states $\lambda = \lambda(t,p)$ as the solutions to the combined dynamics

$$\dot{x} = f(t,x,u(t,x,\lambda)), \tag{14}$$

$$\dot{\lambda} = -L_x(t,x,u(t,x,\lambda)) - \lambda f_x(t,x,u(t,x,\lambda)), \tag{15}$$

with terminal values $(x(T,p), \lambda(T,p)) = \Omega(p)$. The control $u(t,p)$ is given by

$$u(t,p) = u(t,x(t,p),\lambda(t,p)). \tag{16}$$

By taking ϵ in the definition of the neighborhood V small enough, it follows from the continuous dependence of solutions to ordinary differential equations on initial conditions and parameters that $x(\cdot,p)$ and $\lambda(\cdot,p)$ exist and are continuously differentiable in the parameter p over some interval $[t_-(p), T]$ with $t_- : P \to (-\infty, T]$, $p \mapsto t_-(p)$, a smooth function that satisfies $t_-(p_0) = \tau$. This therefore defines a nicely C^1-parameterized family of nonsingular extremals $\mathcal{E}$ that reduces to the reference extremal Λ for $p_0 = \left(0, \nu_0^T\right)$. □

We henceforth consider this canonical, nicely C^1-parameterized family of nonsingular extremals $\mathcal{E}$ and denote its domain by $D = \{(t,p) : p \in P,\ t_-(p) \leq t \leq T\}$ with $t_-(p_0) = \tau$. For the parameterization of the terminal states and multipliers we write $\xi(p)$ and $\nu(p)$, but always assume that the mapping $\omega : P \to N \times \mathbb{R}^{n-k}$, $p \mapsto (\xi(p), \nu^T(p))$ is a diffeomorphism.

3. **Sufficient Conditions for a Relative Minimum.** Given this embedding of a reference trajectory, the mapping properties of the flow determine whether or not the reference trajectory is optimal. In problem $[OC]$, the aim is to control the system over a prescribed interval. The terminal time restriction makes the overall problem time dependent (regardless of whether the original system is time-invariant or not) and we therefore need to consider the time-varying version for the Hamilton-Jacobi-Bellman equation of the optimal control problem. For this reason, the correct way to consider the flow is to look at the graphs of the trajectories.

Definition 3.1. (flow of controlled trajectories) Let $\mathcal{E}$ be a C^r-parameterized family of extremals. The flow associated with the controlled trajectories (x, u) is the mapping

$$\mathcal{F} : D \to \mathbb{R} \times \mathbb{R}^n, \quad (t, p) \longmapsto \mathcal{F}(t, p) = \begin{pmatrix} t \\ x(t, p) \end{pmatrix},$$

i.e., is defined in terms of the graphs of the corresponding trajectories. We say the flow $\mathcal{F}$ is a $C^{1,r}$-mapping on an open set $Q \subset D$ if the restriction of $\mathcal{F}$ to Q is continuously differentiable in (t, p) and r times differentiable in p with derivatives that are jointly continuous in (t, p). If $\mathcal{F} \in C^{1,r}(Q)$ is injective and the Jacobian matrix $D\mathcal{F}(t, p)$ is nonsingular everywhere on Q, then we say $\mathcal{F}$ is a $C^{1,r}$-diffeomorphism onto its image $\mathcal{F}(Q)$.

Definition 3.2. (parameterized cost or cost-to-go function) Given a C^r-parameterized family $\mathcal{E}$ of extremals with domain D, the corresponding parameterized cost or cost-to-go function is defined as

$$C(t, p) = \int_t^T L(s, x(s, p), u(s, p))\, ds + \varphi(x(T, p)).$$

It represents the value of the objective $\mathcal{J}(u)$ for the control $u = u(\cdot, p)$ if the initial condition at time t is given by $x(t, p)$.

We briefly recall some classical results that form the framework for a geometric approach to sufficient conditions for local optimality of a reference extremal. For proofs that utilize the framework presented here, we refer the interested reader to the paper [14] or our text [18]. Essential for the constructions is the following relation between the multiplier λ and the cost C that follows from the conditions of the maximum principle.

Lemma 3.3. ***(shadow-price lemma)*** [14] *Let $\mathcal{E}$ be a C^1-parameterized family of extremal lifts with domain D. Then for all $(t, p) \in D$*

$$\lambda_0(p) \frac{\partial C}{\partial p}(t, p) = \lambda(t, p) \frac{\partial x}{\partial p}(t, p) \tag{17}$$

For normal extremals, $\lambda_0(p) \equiv 1$, the shadow-price lemma implies that the cost-to-go function gives rise to a differentiable solution of the Hamilton-Jacobi-Bellman equation on regions G of the state-space that are covered injectively by the corresponding flow of trajectories.

Theorem 3.4. [14] *Let $\mathcal{E}$ be a C^r-parameterized family of normal extremals for problem $[OC]$ and suppose the restriction of the flow $\mathcal{F}$ to some open set $Q \subset D$ is a $C^{1,r}$-diffeomorphism onto an open subset $G \subset \mathbb{R} \times \mathbb{R}^n$ of the (t, x)-space. Then the function $V^{\mathcal{E}} : G \to \mathbb{R}$, $V^{\mathcal{E}} = C \circ \mathcal{F}^{-1}$, is continuously differentiable in (t, x) and r-times continuously differentiable in x for fixed t. The function $u_* : G \to \mathbb{R}$,*

$u_* = u \circ F^{-1}$, is an admissible feedback control that is continuous and r-times continuously differentiable in x for fixed t. Together, the pair $(V^{\mathcal{E}}, u_*)$ is a classical solution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t,x) + \min_{u\in U}\left\{\frac{\partial V}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\} \equiv 0$$

on G. Furthermore, the following identities hold in the parameter space on Q:

$$\frac{\partial V^{\mathcal{E}}}{\partial t}(t,x(t,p)) = -H(t,\lambda(t,p),x(t,p),u(t,p)) \tag{18}$$

$$\frac{\partial V^{\mathcal{E}}}{\partial x}(t,x(t,p)) = \lambda(t,p). \tag{19}$$

If $\mathcal{E}$ is nicely C^r-parameterized, then $V^{\mathcal{E}}$ is $(r+1)$-times continuously differentiable in x on G and we also have that

$$\frac{\partial^2 V^{\mathcal{E}}}{\partial x^2}(t,x(t,p)) = \frac{\partial \lambda^T}{\partial p}(t,p)\left(\frac{\partial x}{\partial p}(t,p)\right)^{-1}. \tag{20}$$

Definition 3.5. (local field of extremals) A C^r-parameterized local field of extremals, $\mathcal{F}$, is a C^r-parameterized family of normal extremals for which the associated flow $F : D \to \mathbb{R}\times\mathbb{R}^n$, $(t,p)\mapsto F(t,p)$, is a $C^{1,r}$-diffeomorphism on the interior of the domain D.

If there are no terminal constraints in the problem, then the flow F extends as a $C^{1,r}$-diffeomorphism onto a neighborhood of the full closed domain D. However, with terminal constraints present, the flow F collapses at the terminal time T and is not a diffeomorphism. But the value function $V^{\mathcal{E}}$ has a well-defined continuous extension to N given by $V^{\mathcal{E}}(x) = \varphi(x)$ since the terminal value of the cost only depends on the terminal point $x(T,p)$, but not on the parameter p itself.

Definition 3.6. (relative minimum) A controlled trajectory $\Gamma = (\bar{x},\bar{u})$ defined over a compact interval $[\tau,T]$ is a relative minimum over a domain G if the graph of the trajectory x over the right-open interval $[\tau,T)$ is contained in the interior of G and if for any other admissible controlled trajectory (x,u) with the same initial condition and graph in G the value for the cost is not better than for Γ, $\mathcal{J}(\bar{u}) \le \mathcal{J}(u)$.

Corollary 3.7. *Let $\mathcal{F}$ be a C^r-parameterized local field of extremals and assume its associated flow covers a domain G in the (t,x)-space, $G = F(D)$. Then, given any initial condition $(t_0,x_0)\in G$, $x_0 = x(t_0,p_0)$, the open-loop control $\bar{u}(t) = u(t,p_0)$, $t_0 \le t \le T$, with associated trajectory $\bar{x}(t) = x(t,p_0)$ is a relative minimum over G.*

Proposition 3.8. *If the matrix $\frac{\partial x}{\partial p}(t,p_0)$, is nonsingular on the interval $[\tau,T)$, then the canonical, nicely C^1-parameterized family of nonsingular extremals $\mathcal{E}$ constructed in Theorem 2.3 defines a* local field *around the reference trajectory.*

Proof. If the matrix $DF(t,p_0)$ is nonsingular on the interval $[\tau,T)$, then for every time t in this interval there exists a neighborhood O_t of (t,p_0), $O_t \subset D$, so that the restriction of the flow map F to O_t is a C^1-diffeomorphism. If we thus define $G = \bigcup_{\tau\le t<T} F(O_t)$, then G is an open set which, except for the terminal point $\bar{x}(T) = x(T,p_0)$, contains the controlled reference trajectory $\bar{\Gamma} = (\bar{x},\bar{u})$. By Corollary 3.7, $\bar{\Gamma}$ is a relative minimum over the domain G. □

4. **Perturbation Feedback Control and Regularity of the Flow $\mathcal{F}$.** Perturbation feedback control is based on formulas that establish the local injectivity of the flow $\mathcal{F}$. As we have seen, the boundary conditions at time T for $x(T,p)$ and $\lambda(T,p)$ define an n-dimensional manifold. The combined flow in the cotangent bundle, the combined state-multiplier space, is a C^1-diffeomorphism in (x,λ)-space since together (x,λ) are solutions to the differential equations (14) and (15) and thus the resulting flow map Φ_t, which maps the terminal conditions to the solutions at time t, is one-to-one. Hence the image of the flow for a given time t is always an n-dimensional manifold. *Local optimality* of the reference trajectory follows if one can ensure that the projection $\pi : (x,\lambda) \mapsto x$ is one-to-one away from the terminal time T. For in this case, for every time t the flow not only covers an n-dimensional set in the state-multiplier space, but an n-dimensional set in the state space as well. Hence the image covers a neighborhood of the reference trajectory and local optimality follows from Theorem 3.4 and Corollary 3.7.

For problem $[OC]$ (and even in the case when the terminal time becomes part of the constraints and is only defined implicitly) there exist classical results that characterize the local regularity of the flow along a reference extremal in terms of solutions to matrix Riccati and other linear differential equations [7, Chapter 6]. In the engineering literature, these calculations are usually treated formally without any relations to the underlying geometric properties they represent. These are the geometric facts that we want to elucidate. In the subsequent calculations, x, u and λ and their partial derivatives are evaluated at (t,p), partial derivatives of f are evaluated along the controlled trajectories of the family, $(t,x(t,p),u(t,p))$, and all partials of H are evaluated along the full extremals, $(t,\lambda(t,p),x(t,p),u(t,p))$. For notational clarity, however, we drop these arguments. The matrix $\frac{\partial x}{\partial p}(t,p)$ of the partial derivatives with respect to the parameter p is the solution of the variational equation of the dynamics, i.e.,

$$\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) = f_x\frac{\partial x}{\partial p} + f_u\frac{\partial u}{\partial p}.$$

In order to eliminate $\frac{\partial u}{\partial p}$, we differentiate the identity $H_u^T(t,\lambda(t,p),x(t,p),u(t,p)) = 0$ with respect to p. The notation is set up for column vectors and differentiating the m-dimensional column vector H_u^T with respect to x we get a $m \times n$ matrix whose row vectors are the x-gradients of the components of H_u^T We denote this matrix by H_{ux}. In particular, under our general differentiability assumptions the mixed partials are equal and we have that $H_{xu} = H_{ux}^T$. However, the multiplier λ is written as a row vector and, in order to be consistent in our notation, we need to differentiate H_u^T with respect to the column vector λ^T. For example, $H_{u\lambda^T}$ is the $m \times n$ matrix whose row vectors are the partial derivatives of the components of H_u^T with respect to λ^T. Since $H_u^T = f_u^T\lambda^T + L_u^T$, we simply have that $H_{u\lambda^T} = f_u^T$ etc. Differentiating $H_u^T = 0$ thus gives

$$0 = H_{u\lambda^T}\frac{\partial \lambda^T}{\partial p} + H_{ux}\frac{\partial x}{\partial p} + H_{uu}\frac{\partial u}{\partial p} = f_u^T\frac{\partial \lambda^T}{\partial p} + H_{ux}\frac{\partial x}{\partial p} + H_{uu}\frac{\partial u}{\partial p}.$$

Along a nonsingular extremal, it therefore follows that

$$\frac{\partial u}{\partial p} = -H_{uu}^{-1}\left(H_{ux}\frac{\partial x}{\partial p} + f_u^T\frac{\partial \lambda^T}{\partial p}\right). \tag{21}$$

Substituting into the variational equation for x, we obtain

$$\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) = \left(f_x - f_u H_{uu}^{-1} H_{ux}\right)\frac{\partial x}{\partial p} - f_u H_{uu}^{-1} f_u^T \frac{\partial \lambda^T}{\partial p}.$$

A differential equation for the partial derivative $\frac{\partial \lambda^T}{\partial p}$ follows by differentiating the adjoint equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \lambda^T}{\partial p}\right) &= \frac{\partial^2 \lambda^T}{\partial t \partial p} = \frac{\partial}{\partial p}\left(\dot{\lambda}^T\right) = \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial x}\right)^T \\ &= -H_{xx}\frac{\partial x}{\partial p} - H_{x\lambda^T}\frac{\partial \lambda^T}{\partial p} - H_{xu}\frac{\partial u}{\partial p} \\ &= -\left(H_{xx} - H_{xu}H_{uu}^{-1}H_{ux}\right)\frac{\partial x}{\partial p} - \left(f_x - f_u H_{uu}^{-1} H_{ux}\right)^T \frac{\partial \lambda^T}{\partial p}.\end{aligned} \tag{22}$$

Hence, overall we have the following version of the Jacobi-equation, i.e., the combined variational equation for (x, λ), for problem $[OC]$:

Proposition 4.1. *Given the canonical C^1-parameterized family of nonsingular extremals, the matrices $\frac{\partial x}{\partial p}(t,p)$ and $\frac{\partial \lambda^T}{\partial p}(t,p)$ are solutions to the following homogeneous linear matrix differential equation*

$$\begin{pmatrix} \frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) \\ \frac{d}{dt}\left(\frac{\partial \lambda^T}{\partial p}\right) \end{pmatrix} = \begin{pmatrix} f_x - f_u H_{uu}^{-1} H_{ux} & -f_u H_{uu}^{-1} f_u^T \\ -H_{xx} + H_{xu} H_{uu}^{-1} H_{ux} & -\left(f_x - f_u H_{uu}^{-1} H_{ux}\right)^T \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial p} \\ \frac{\partial \lambda^T}{\partial p} \end{pmatrix}. \tag{23}$$

We are interested in whether the first matrix of this combined equation is nonsingular. The following proposition gives the classical result that goes back to Legendre and Jacobi in the calculus of variations and characterizes this condition in terms of the existence of a solution to an associated Riccati differential equation. For completeness sake, we include the brief proof.

Proposition 4.2. *Suppose $A(\cdot), B(\cdot), M(\cdot)$ and $N(\cdot)$ are continuous $n \times n$ matrices defined on $[0,T]$ and let $(X,Y)^T$ be the solution to the initial value problem*

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} A & -M \\ -N & -B \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}. \tag{24}$$

Suppose X_0 is nonsingular. Then the solution $X(t)$ is nonsingular on the full interval $[0,T]$ if and only if the solution S to the Riccati equation

$$\dot{S} + SA(t) + B(t)S - SM(t)S + N(t) \equiv 0, \qquad S(0) = Y_0 X_0^{-1}, \tag{25}$$

exists on the full interval $[0,T]$ and in this case we have that

$$Y(t) = S(t)X(t). \tag{26}$$

The solution S to the Riccati equation (25) has a finite escape time at $t = \tau$ if and only if τ is the first time when the matrix $X(t)$ becomes singular.

Proof. If the matrix $X(t)$ is nonsingular for all $t \in [0,T]$, then the matrix $S(t) = Y(t)X(t)^{-1}$ is well-defined and it is a matter of direct verification to show that it satisfies the Riccati equation (25). Conversely, if a solution S to the Riccati equation exists on all of $[0,T]$, let $U = U(t)$ be the solution of the linear matrix initial value problem $\dot{U} = (A(t) - M(t)S(t))U$, $U(t_0) = X_0$, on the interval $[0,T]$. Defining $V(t) = S(t)U(t)$, it follows that $V(0) = S(0)X_0 = Y_0$ and $\dot{V} = -NU - BV$. Thus the pair $(U,V)^T$ is a solution to the initial value problem (24). But so is $(X,Y)^T$ and by the uniqueness of solutions we have that $(X,Y) = (U,V)$, i.e., $Y(t) = S(t)X(t)$.

Suppose now there exists a time τ so that $X(\tau)$ is singular. Pick $x_0 \neq 0$ so that $X(\tau)x_0 = 0$ and let $x(t) = X(t)x_0$ and $y(t) = Y(t)x_0$. Then $x(\tau) = 0$ and $y(\tau) = Y(\tau)x_0 = S(\tau)x(\tau) = 0$ and thus, since $(x,y)^T$ satisfies a homogeneous linear differential equation, both x and y vanish identically. But $x(0) = X(0)x_0 \neq 0$ since $X(0)$ is nonsingular. Contradiction. Thus $X(t)$ is nonsingular over all of $[0,T]$. □

In the presence of terminal constraints, the flow map $\mathcal{F}$ is singular at the terminal time T and it becomes necessary to desingularize it. This is the geometric meaning of the transversality condition at the end point. Since $\frac{\partial x}{\partial p}(T,p) = \frac{\partial \xi}{\partial p}(p)$, the image of $\frac{\partial x}{\partial p}(T,p)$ is given by the tangent space of N at $\xi(p)$. It is the second part of the parameter, the multiplier ν, that desingularizes the flow $\mathcal{F}$ around the terminal manifold. Differentiating the transversality condition (10) for $\lambda(T,p)$ with respect to p, and taking the transpose, we have that

$$\frac{\partial \lambda^T}{\partial p}(T,p) = \left[\varphi_{xx}(\xi(p)) + \nu(p)\Psi_{xx}(\xi(p))\right]\frac{\partial \xi}{\partial p}(p) + D\Psi(\xi(p))^T\frac{\partial \nu^T}{\partial p}(p). \tag{27}$$

In this equation, $\nu\Psi_{xx}$ is a convenient short-cut notation for $\nu\Psi_{xx} = \sum_{i=1}^{n-k}\nu_i\frac{\partial^2\psi_i}{\partial x^2}$ with $\frac{\partial^2\psi_i}{\partial x^2}$ denoting the Hessian matrices of the functions ψ_i that define the terminal constraint, $\Psi = (\psi_1,\ldots,\psi_{n-k})^T$. Thus the first term in (27) is the sum of $1+n-k$ vectors which each are obtained by an $n\times n$ matrix acting on the tangent vector $\frac{\partial \xi}{\partial p}(p)$ to N at $\xi(p)$. Equation (27) therefore points to the following ansatz for the solutions to the variational equations,

$$\frac{\partial \lambda^T}{\partial p}(t,p) = S(t,p)\frac{\partial x}{\partial p}(t,p) + R^T(t,p)\frac{\partial \nu^T}{\partial p}(p).$$

Substituting this relation into the differential equations (23) for $\frac{\partial x}{\partial p}$ and $\frac{\partial \lambda^T}{\partial p}$, differential equations for S and R can be derived that indeed generate this relation.

Lemma 4.3. *If the solution $S = S(t,p) \in \mathbb{R}^{n\times n}$ to the Riccati equation*

$$\dot{S} + Sf_x + f_x^T S + H_{xx} - (Sf_u + H_{xu})H_{uu}^{-1}(H_{ux} + f_u^T S) \equiv 0, \tag{28}$$

with terminal condition

$$S(T,p) = \varphi_{xx}(\xi(p)) + \nu(p)\Psi_{xx}(\xi(p)). \tag{29}$$

exists over the full interval $[\tau,T]$ and $R \in \mathbb{R}^{(n-k)\times n}$ denotes the solution to the linear ODE

$$\dot{R} = R\left(-f_x + f_u H_{uu}^{-1}H_{ux} + f_u H_{uu}^{-1}f_u^T S\right) \tag{30}$$

with terminal condition

$$R(T,p) = D\Psi(\xi(p)), \tag{31}$$

then for all $t \in [\tau,T]$ we have that

$$\frac{\partial \lambda^T}{\partial p}(t,p) = S(t,p)\frac{\partial x}{\partial p}(t,p) + R^T(t,p)\frac{\partial \nu^T}{\partial p}(p). \tag{32}$$

Proof. For simplicity of notation, we use the abbreviations

$$\tilde{A} = f_x - f_u H_{uu}^{-1}H_{ux}, \quad \tilde{B} = f_u H_{uu}^{-1} f_u^T, \quad \text{and} \quad \tilde{C} = H_{xx} - H_{xu}H_{uu}^{-1}H_{ux} \tag{33}$$

where, as always, the partial derivatives are evaluated along the extremals $t \mapsto (t, \lambda(t,p), x(t,p), u(t,p))$. Note that S, $\tilde{B}$, and $\tilde{C}$ are symmetric matrices. Defining

$$\Delta(t,p) = \frac{\partial \lambda^T}{\partial p}(t,p) - S(t,p)\frac{\partial x}{\partial p}(t,p) - R^T(t,p)\frac{\partial \nu^T}{\partial p}(p),$$

it follows from the specifications of the terminal values that $\Delta(T,p) = 0$. Using (23), we have that

$$\begin{aligned}\dot{\Delta}(t,p) &= \frac{d}{dt}\left(\frac{\partial \lambda^T}{\partial p}\right) - \dot{S}\frac{\partial x}{\partial p} - S\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) - \dot{R}^T\frac{\partial \nu^T}{\partial p} \\
&= -\tilde{A}^T\frac{\partial \lambda^T}{\partial p} - \tilde{C}\frac{\partial x}{\partial p} - \left(-S\tilde{A} - \tilde{A}^T S + S\tilde{B}S - \tilde{C}\right)\frac{\partial x}{\partial p} \\
&\qquad - S\left(f_x\frac{\partial x}{\partial p} + f_u\frac{\partial u}{\partial p}\right) - \left(-\tilde{A} + \tilde{B}S\right)^T R^T\frac{\partial \nu^T}{\partial p} \\
&= -\tilde{A}^T\frac{\partial \lambda^T}{\partial p} + S\tilde{A}\frac{\partial x}{\partial p} - \left(-\tilde{A} + \tilde{B}S\right)^T\left(S\frac{\partial x}{\partial p} + R^T\frac{\partial \nu^T}{\partial p}\right) \\
&\qquad - S\left(f_x\frac{\partial x}{\partial p} + f_u\frac{\partial u}{\partial p}\right).\end{aligned}$$

Since all extremals are nonsingular, (21) implies that

$$\begin{aligned}\dot{\Delta}(t,p) &= -\tilde{A}^T\frac{\partial \lambda^T}{\partial p} + S\tilde{A}\frac{\partial x}{\partial p} - \left(-\tilde{A} + \tilde{B}S\right)^T\left(S\frac{\partial x}{\partial p} + R^T\frac{\partial \nu^T}{\partial p}\right) \\
&\qquad - S\left(\tilde{A}\frac{\partial x}{\partial p} - \tilde{B}\frac{\partial \lambda^T}{\partial p}\right) \\
&= \left(-\tilde{A} + \tilde{B}S\right)^T\left(\frac{\partial \lambda^T}{\partial p} - S\frac{\partial x}{\partial p} - R^T\frac{\partial \nu^T}{\partial p}\right) \\
&= \left(-\tilde{A} + \tilde{B}S\right)^T \Delta\end{aligned}$$

and thus $\Delta(t,p) \equiv 0$. □

Lemma 4.4. *With $Q = Q(t,p) \in \mathbb{R}^{(n-k)\times(n-k)}$ the integral of*

$$\dot{Q} = Rf_u H_{uu}^{-1} f_u^T R^T, \qquad Q(T,p) = 0, \tag{34}$$

we also have that

$$R(t,p)\frac{\partial x}{\partial p}(t,p) + Q(t,p)\frac{\partial \nu^T}{\partial p}(p) \equiv 0. \tag{35}$$

Proof. Define

$$\Theta(t,p) = R(t,p)\frac{\partial x}{\partial p}(t,p) + Q(t,p)\frac{\partial \nu^T}{\partial p}(p).$$

It follows from $Q(T,p) = 0$ that

$$\Theta(T,p) = D\Psi(\xi(p))\frac{\partial \xi}{\partial p}(p) + 0 = 0$$

where, in the last step, we use the fact that the columns of $\frac{\partial \xi}{\partial p}(p)$ are tangent to N at $\xi(p)$ while the rows of $D\Psi(\xi(p))$ are a basis for the normal space to N at $\xi(p)$.

Furthermore,

$$\begin{aligned}\dot{\Theta}(t,p) &= \dot{R}\frac{\partial x}{\partial p} + R\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) + \dot{Q}\frac{\partial \nu^T}{\partial p} \\ &= R\left(-\tilde{A} + \tilde{B}S\right)\frac{\partial x}{\partial p} + R\left(\tilde{A}\frac{\partial x}{\partial p} - \tilde{B}\frac{\partial \lambda^T}{\partial p}\right) + R\tilde{B}R^T\frac{\partial \nu^T}{\partial p} \\ &= R\tilde{B}\left(S\frac{\partial x}{\partial p} + R^T\frac{\partial \nu^T}{\partial p} - \frac{\partial \lambda^T}{\partial p}\right) \equiv 0\end{aligned}$$

and thus also $\Theta(t,p) \equiv 0$. □

Corollary 4.5. *The matrix $\frac{\partial x}{\partial p}(t,p)$ is nonsingular if and only if $Q(t,p)$ is nonsingular.*

Proof. The matrix $R(t,p) \in \mathbb{R}^{(n-k)\times n}$ is the solution of the homogeneous linear matrix differential equation (30). Since the rows of the matrix $R(T,p) = D\Psi(\xi(p))$ are linearly independent, it follows that the rows of $R(t,p)$ are linearly independent for all times t and thus $R(t,p)$ is of full rank $n-k$ everywhere. Thus, if $\frac{\partial x}{\partial p}(t,p)$ is nonsingular, then the rank of the product $R(t,p)\frac{\partial x}{\partial p}(t,p)$ is $n-k$. By (35), this then is also the rank of the product $Q(t,p)\frac{\partial \nu^T}{\partial p}(p)$. Hence both $Q(t,p)$ and $\frac{\partial \nu^T}{\partial p}(p)$ must be of full rank $n-k$.

Conversely, suppose $Q(t,p)$ is nonsingular. If $z \in \ker\left(\frac{\partial x}{\partial p}(t,p)\right)$, then by (35) we have that $\frac{\partial \nu^T}{\partial p}(p)z = 0$ and by (32) also $\frac{\partial \lambda^T}{\partial p}(t,p)z = 0$. The vector functions $s \to \frac{\partial x}{\partial p}(s,p)z$ and $s \to \frac{\partial \lambda^T}{\partial p}(s,p)z$ are thus solutions to the homogeneous linear differential equation (23) which vanish for $s = t$. Hence these functions vanish identically and at the terminal time T we have that $\frac{\partial \xi}{\partial p}(p)z = \frac{\partial x}{\partial p}(T,p)z = 0$. But the mapping $\omega : p \mapsto \omega(p) = (\xi(p), \nu(p)^T)$ is a diffeomorphism and therefore

$$\left(\frac{\partial \xi}{\partial p}(p), \frac{\partial \nu^T}{\partial p}(p)\right) z = 0$$

implies that $z = 0$. Hence $\frac{\partial x}{\partial p}(t,p)$ is nonsingular. □

Summarizing the construction, we have the following result, stated in a form typical in the engineering literature:

Theorem 4.6. *Let $\mathcal{E}$ be the canonical, nicely C^1-parameterized family of nonsingular extremals constructed in Theorem 2.3 with domain $D = \{(t,p) : p \in P,\, t_-(p) \leq t \leq T\}$ for the optimal control problem [OC] with terminal constraints given by $N = \{x \in \mathbb{R}^n : \Psi(x) = 0\}$. Suppose the solution $S = S(t,p)$ to the matrix Riccati differential equation*

$$\dot{S} + Sf_x + f_x^T S + H_{xx} - (Sf_u + H_{xu})H_{uu}^{-1}(H_{ux} + f_u^T S) \equiv 0,$$

with terminal condition

$$S(T,p) = \varphi_{xx}(\xi(p)) + \nu(p)\Psi_{xx}(\xi(p)),$$

exists over the full interval $[\tau, T]$ and let $R = R(t,p)$ and $Q = Q(t,p)$ be the solutions to the terminal value problems

$$\dot{R} = R\left(-f_x + f_u H_{uu}^{-1} H_{ux} + f_u H_{uu}^{-1} f_u^T S\right), \qquad R(T,p) = D\Psi(\xi(p))$$

and

$$\dot{Q} = Rf_u H_{uu}^{-1} f_u^T R^T, \qquad Q(T,p) = 0,$$

over the interval $[\tau, T]$. *If* $Q(t,p)$ *is nonsingular on the interval* $[\tau, T)$, *then the matrix* $\frac{\partial x}{\partial p}(t,p)$ *is nonsingular over the interval* $[\tau, T)$ *as well. In this case, there exists a domain* G *in* (t,x)*-space which, with the exception of the terminal point* $\bar{x}(T)$, *contains the graph of the controlled reference trajectory* $\bar{\Gamma} = (\bar{x}, \bar{u})$ *and* $\bar{\Gamma}$ *is a relative minimum over the set* G. □

The last statement of the theorem follows from Corollary 3.7. Recall (see Theorem 3.4) that the Hessian matrix of the associated value $V^{\mathcal{E}}$ is given by

$$\frac{\partial^2 V^{\mathcal{E}}}{\partial x^2}(t, x(t,p)) = \frac{\partial \lambda^T}{\partial p}(t,p)\left(\frac{\partial x}{\partial p}(t,p)\right)^{-1}.$$

For $t < T$ we thus have that

$$\begin{aligned}\frac{\partial^2 V^{\mathcal{E}}}{\partial x^2}(t, x(t,p)) &= S(t,p) + R^T(t,p)\frac{\partial \nu^T}{\partial p}(p)\left(\frac{\partial x}{\partial p}(t,p)\right)^{-1} \\ &= S(t,p) - R^T(t,p)Q(t,p)^{-1}R(t,p) \qquad (36)\end{aligned}$$

which is the Schur complement of Q for the block matrix $\begin{pmatrix} S & R^T \\ R & Q \end{pmatrix}$. Note that the function

$$Z(t,p) = \frac{\partial \lambda^T}{\partial p}(t,p)\left(\frac{\partial x}{\partial p}(t,p)\right)^{-1}$$

satisfies the same Riccati differential equation as S does. For, if $\frac{\partial x}{\partial p}(t,p)$ is nonsingular, then it always holds that (see (22))

$$\begin{aligned}\dot{Z} &= \left[\frac{d}{dt}\left(\frac{\partial \lambda^T}{\partial p}\right)\right]\left(\frac{\partial x}{\partial p}\right)^{-1} + \left(\frac{\partial \lambda^T}{\partial p}\right)\left[\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right)^{-1}\right] \\ &= \left[\frac{d}{dt}\left(\frac{\partial \lambda^T}{\partial p}\right)\right]\left(\frac{\partial x}{\partial p}\right)^{-1} - \left(\frac{\partial \lambda^T}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1}\left[\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right)\right]\left(\frac{\partial x}{\partial p}\right)^{-1} \\ &= \left(-H_{xx}\frac{\partial x}{\partial p} - f_x^T\frac{\partial \lambda^T}{\partial p} - H_{xu}\frac{\partial u}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1} - Z\left(f_x\frac{\partial x}{\partial p} + f_u\frac{\partial u}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1} \\ &= -Zf_x - f_x^T Z - H_{xx} - (Zf_u + H_{xu})\left(\frac{\partial u}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1} \qquad (37)\end{aligned}$$

and along a nonsingular extremal we have that

$$\left(\frac{\partial u}{\partial p}\right)\left(\frac{\partial x}{\partial p}\right)^{-1} = -H_{uu}^{-1}\left(H_{ux} + f_u^T Z\right).$$

Naturally, in this case $Z(t,p)$ has a singularity as $t \to T$ and (36) resolves the behavior near the terminal time.

The condition that $Q(t,p)$ be nonsingular for all $t < T$, called a normality condition in [7], actually is a controllability assumption on the linearized, time-varying system

$$\dot{y} = \frac{\partial f}{\partial x}(t, x(t,p), u(t,p))y + \frac{\partial f}{\partial u}(t, x(t,p), u(t,p))v.$$

This is best seen through the connection with solutions to the adjoint equation for this system.

Lemma 4.7. *The matrix $Q(\tau,p)$ is singular if and only if there exists a nontrivial solution $\mu=\mu(t,p)$ of the linear adjoint equation*

$$\dot{\mu} = -\mu \frac{\partial f}{\partial x}(t, x(t,p), u(t,p))$$

with terminal condition $\mu(T,p)$ that is perpendicular to N at $\xi(p) = x(T,p)$ so that

$$\mu(t,p)\frac{\partial f}{\partial u}(t, x(t,p), u(t,p)) \equiv 0$$

on the interval $[\tau, T]$.

Proof. The matrix $Q = Q(\tau,p)$ is negative semi-definite: for any vector $z \in \mathbb{R}^{n-k}$ we have that

$$z^T Q(t,p) z = -\int_\tau^T z^T \dot{Q} z ds = -\int_\tau^T z^T R f_u H_{uu}^{-1} f_u^T R^T z ds \leq 0.$$

Since the matrix H_{uu} is positive definite, $Q(\tau,p)$ is singular if and only if there exists a non-zero vector $z \in \mathbb{R}^{n-k}$ such that

$$z^T R(t,p) f_u(t, x(t,p), u(t,p)) \equiv 0 \qquad \text{on} \quad [\tau, T].$$

In this case, $\mu(t,p) = z^T R(t,p)$ is a nontrivial solution of the linear adjoint equation,

$$\dot{\mu} = z^T \dot{R} = z^T R \left(-f_x + f_u H_{uu}^{-1} H_{ux} + f_u H_{uu}^{-1} f_u^T S\right) = -z^T R f_x = -\mu f_x$$

that satisfies $\mu(T,p) \perp N$ and $\mu f_u = z^T R f_u \equiv 0$. Conversely, if such a solution exists, then μ is also a solution to the ODE

$$\dot{\mu} = \mu \left(-f_x + f_u H_{uu}^{-1} H_{ux} + f_u H_{uu}^{-1} f_u^T S\right).$$

Since $\mu(T,p)$ is orthogonal to N, there exists a vector $z \in \mathbb{R}^{n-k}$ so that $\mu(T) = z^T D\Psi(\xi(p))$ and thus μ is given by $\mu(t) = z^T R(t,p)$. Thus $z^T Q(\tau,p) z = 0$ and $Q(\tau,p)$ is singular. □

It is a well-known result from the theory of time-varying linear systems that the equation $\dot{y} = A(t)y + B(t)v$ is controllable over an interval $[\tau, T]$ if and only if for every nontrivial solution μ of the adjoint equation $\dot{\mu} = -\mu A(t)$, the function $\mu(t)B(t)$ does not vanish identically on the interval $[\tau, T]$ (for example, see [11, 12]). A system is completely controllable if this holds for any subinterval $[\tau, T]$. Our situation is different in the sense that the problem only requires to steer the system into the terminal manifold N and for this reason it is not required that the linearization be completely controllable, but we only need that the system is fully controllable with respect to the normal directions at the terminal manifold. This is the significance of the terminal constraint $\mu(T) \perp N$ and controllability of the linearization along tangent directions to N is not required. In fact, depending on the values of the parameter, two different solutions $x(\cdot, p_1)$ and $x(\cdot, p_2)$ may or may not end at the same terminal point.

The assumption that $Q(t,p)$ is nonsingular enforces a regular geometric structure in the form of a foliation that clarifies these relations. Going back to the canonical, nicely C^1-parameterized family of nonsingular extremals $\mathcal{E}$ constructed in Theorem 2.3, the parameter set P, $P = P_1 \times P_2$, is the direct product of an open neighborhood P_1 of the origin in $\mathbb{R}^k$ which is the domain of a coordinate chart for the terminal manifold N, and an open neighborhood P_2 of ν_0^T in $\mathbb{R}^{n-k}$ that defines coordinates for the normal space and is used to parameterize the terminal conditions on the multiplier λ. For $p_1 \in P_1$ fixed, the controlled trajectories

$$(t, p_2) \mapsto (t, x(t, p_1, p_2)), \qquad p_2 \in P_2,\ t < T,$$

form $(n-k+1)$-dimensional integral submanifolds $M(p_1)$ of graphs of controlled trajectories which all steer the system into the point $(T, \xi(p_1))$ at the terminal time T. On the other hand, if we freeze a parameter $p_2 \in P_2$, then the image of the terminal manifold N under the flow $\mathcal{F}$ for a fixed time t,

$$(t, p_1) \mapsto (t, x(t, p_1, p_2)), \qquad p_1 \in P_1,\ t < T,$$

defines a k-dimensional submanifold $N(t, p_2)$ that is transversal to all the manifolds $M(p_1)$ at the point $(t, x(t, p_1, p_2))$. Mathematically, such a decomposition is called a *foliation* with the surfaces $M(p_1)$ the *leaves of the foliation* and the manifolds $N(t, p_2)$ the *transversal sections*.

Thus there exist both an intuitive control engineering interpretation and an elegant geometric picture underlying the formal computations of neighboring extremals done in [7, Chapter 6]. These interpretations can be extended to the case when the terminal time is only defined implicitly through one of the constraints in Ψ, but a third equation that models the time evolution is required and the notation becomes more cumbersome. Also, if one wants to make the constructions mathematically rigorous, as it was the case for the model considered here with condition (ii) in Theorem 2.3, some minor extra assumptions need to be made so that the formal computations in [7] are valid. For an alternative, mathematically rigorous formulation for optimal control problems with free terminal time, we refer the reader to the paper by Maurer and Oberle [13].

The equations derived above form the basis for the linearization of a nonlinear control system around a locally optimal controlled reference trajectory, the *perturbation feedback control* of the engineering literature. The nominal path is given by the controlled reference trajectory $(\bar{x}, \bar{u}) = (x(\cdot, p_0), u(\cdot, p_0))$ corresponding to the parameter p_0. For a real system, because of disturbances and high-order aspects in the true dynamics that are not modelled, generally at time t the system will not be in its specified location $\bar{x}(t) = x(t, p_0)$, but is expected to have some small deviation with the actual position x. If x lies in the region covered by the flow $\mathcal{F}$ of the parameterized family of extremals, then there exists a parameter $p \in P$ so that $x = x(t, p)$. The corresponding optimal solution therefore is given by $(x(\cdot, p), u(\cdot, p))$. Since the family is C^1-parameterized, a first order Taylor expansion for this control around the reference value is given by

$$u(t, p) = u(t, p_0) + \frac{\partial u}{\partial p}(t, p_0)(p - p_0) + o(\|p - p_0\|)$$

and by (21) we have that

$$\frac{\partial u}{\partial p}(t, p_0) = -H_{uu}^{-1}\left(H_{ux}\frac{\partial x}{\partial p}(t, p_0) + f_u^T \frac{\partial \lambda^T}{\partial p}(t, p_0)\right)$$

with the partial derivatives of H all evaluated along the reference extremal for parameter p_0. It follows from equations (32) and (35) that

$$\frac{\partial \lambda^T}{\partial p}(t, p_0) = \left(S(t, p_0) - R^T(t, p_0)Q(t, p_0)^{-1}R(t, p_0)\right)\frac{\partial x}{\partial p}(t, p_0)$$

and thus we have that

$$\frac{\partial u}{\partial p}(t,p_0) = -H_{uu}^{-1}\left(H_{ux} + f_u^T\left(S - R^TQ^{-1}R\right)\right)\frac{\partial x}{\partial p}(t,p_0).$$

If we denote the deviation of the state from the reference trajectory by $\Delta x(t) = x(t,p) - x(t,p_0)$ and the deviation of the reference control by $\Delta u(t) = u(t,p) - u(t,p_0)$, then overall this gives

$$\Delta u(t) = -H_{uu}^{-1}\left(H_{ux} + f_u^T\left(S - R^TQ^{-1}R\right)\right)\Delta x(t) + o(\|\Delta x(t)\|) \tag{38}$$

This equation provides the linearization of the optimal control $u(t,p)$ around the nominal control $u(t,p_0)$ as a time-varying linear feedback control of the deviation from the nominal trajectory, information that typically is readily available with today's sensor technology. For a nonlinear control system with reasonable local stability properties around the reference such a control scheme generally is highly effective and it has the advantage that it only requires computations along one trajectory, the reference controlled trajectory, since all the matrices S, R, Q and all the partial derivatives of H are evaluated along the controlled reference trajectory for parameter p_0. Naturally, if the deviation becomes too large, this control law no longer is applicable. Even with the best technology, if turbulence occurs, the pilot needs to turn off the auto-pilot and return to manual control since a linearization based control scheme no longer is able to handle deviations of such a magnitude. Summarizing the construction, we have the following result about local optimality:

Corollary 4.8. *Let $\Lambda = ((\bar{x},\bar{u}),\lambda)$ be a nonsingular extremal defined over $[\tau,T]$ and suppose that* (i) *for every $t \in [\tau,T]$ the control $\bar{u}(t)$ lies in the interior of the control set, $\bar{u}(t) \in \text{int}(U)$,* (ii) *$\bar{u}(t)$ is the unique minimizer of the function $v \mapsto H(t,\bar{\lambda}(t),\bar{x}(t),v)$ over the control set U, and,* (iii) *along Λ the Riccati equation (28) with terminal condition (29) has a solution $\bar{S}$ over the full interval $[\tau,T]$ and* (iv) *the matrix $\bar{Q}$ is nonsingular on $[\tau,T)$. Then $(\bar{x},\bar{u})$ is a relative minimum for the optimal control problem $[OC]$.* ■

While it is assumed in the classical formulation that the control lies in the interior of the control set, Fisher, Grantham and Teo give a formulation of neighboring extremals for optimal control problems in the presence of control constraints [8].

5. **Envelopes and Conjugate Points.** Thus, for problem $[OC]$, the existence of a solution to the terminal value problem (28) and (29) on the compact interval $[\tau,T]$ combined with the regularity of the matrix Q is the generalization of the *strengthened Jacobi-condition* from the calculus of variations. As in the calculus of variations, the existence of a solution on the semi-open interval $(\tau,T]$ is a necessary condition for a local minimum and for the least degenerate, i.e., typical case, we now show that local optimality indeed ceases at $t=\tau$ when the matrix $\frac{\partial x}{\partial p}(\tau,p)$ becomes singular. We make the following assumption:

(A): The singular set $S = \left\{(t,p) \in D : \det\left(\frac{\partial x}{\partial p}(t,p)\right) = 0\right\}$ is a codimension 1 embedded submanifold of D that entirely consists of corank 1 singular points and can be described as the graph of a continuously differentiable function $\sigma: P \to D$, $p \mapsto \sigma(p)$,

$$S = \{(t,p) \in \text{int}(D) :\ t = \sigma(p)\} = \mathfrak{gr}(\sigma).$$

Lemma 5.1. *There exists a C^1 vector field $v: P \to \mathbb{S}^{k-1} = \{z \in \mathbb{R}^k :\ z^Tz = 1\}$, $p \mapsto v(p)$, so that $\left(\frac{\partial x}{\partial p}(\sigma(p),p)\right)v(p) = 0$.* ■

The corresponding eigenvector field for the full flow $\mathcal{F}$ is obtained by adding zero as first coordinate,

$$\hat{v}: \; P \to \mathbb{S}^k, \; p \mapsto \hat{v}(p) = \begin{pmatrix} 0 \\ v(p) \end{pmatrix}.$$

Corank 1 singularities are broadly classified as fold or cusp points depending on whether the vector field $\hat{v}$ is transversal to the tangent space of the singular set S at (t_0, p_0), $T_{(t_0,p_0)}S$, or not [9]. A corank 1 singular point is a *fold* point if

$$T_{(t_0,p_0)}S \;\oplus\; lin\ span\ \{\hat{v}(p_0)\} = \mathbb{R}^{n+1};$$

and it is a *cusp* point if

$$\hat{v}(p_0) \in T_{(t_0,p_0)}S.$$

For our set-up, we have the following simple criterion:

Lemma 5.2. *The point (t_0, p_0), $t_0 = \sigma(p_0)$, is a fold point if and only if the Lie-derivative of the function σ along the vector field v does not vanish at p_0,*

$$\left(L_v\sigma\right)(p_0) = \nabla\sigma(p_0)v(p_0) \neq 0.$$

Proof. Since S is the graph of the function σ, $S = \{(t,p) : t - \sigma(p) = 0\}$, the tangent space $T_{(t_0,p_0)}S$ consists of all vectors that are orthogonal to $(1, -\nabla\sigma(p_0))$. □

By reversing the orientation of the vector field v, without loss of generality, we may always *assume* that $\nabla\sigma(p_0)v(p_0)$ *is positive* for a fold point. The local geometry of a flow of extremals near a fold singularity is identical with the one for the flow of catenaries near the envelope in the problem of minimum surfaces of revolution in the classical calculus of variations (e.g., [18]). In fact, *if we define the manifold M_f of fold points as the image of the singular set S under the flow $\mathcal{F}$, $M_f = \mathcal{F}(S)$, then trajectories in the the parameterized family $\mathcal{E}$ touch M_f in exactly one point.*

Definition 5.3. (classical envelope) Let $\mathcal{E}$ be a C^1-parameterized family of normal extremals with domain D. A classical envelope for the parameterized family $\mathcal{E}$ is a (possibly small) portion of an admissible controlled trajectory (ξ, η) of the control system defined over some interval $[a, b]$ with the property that there exists a differentiable curve $p : [a, b] \to P$, $t \mapsto p(t)$, so that $\xi(t) = x(t, p(t))$ and for $p = p(t)$ we have that

$$H(t, \lambda(t,p), \xi(t), \eta(t)) = H(t, \lambda(t,p), x(t,p), u(t,p)). \tag{39}$$

The main property of envelopes is the agreement of the cost along concatenations of portions of the envelope with trajectories of the parameterized family and the trajectories in the field itself.

Theorem 5.4. ***(envelope theorem)*** [21, 17] *Let $(\xi, \eta) : [a, b] \to M \times U$ be a classical envelope for a C^1-parameterized family $\mathcal{E}$ of normal extremals with domain $D = \{(t,p) : p \in P,\, t_-(p) \leq t \leq t_+(p)\}$. For any interval $[t_1, t_2] \subset (a, b)$, setting $p_1 = p(t_1)$ and $p_2 = p(t_2)$, we then have that*

$$C(t_1, p_1) = \int_{t_1}^{t_2} L(t, \xi(t), \eta(t))dt + C(t_2, p_2). \tag{40}$$

Proof. For $s \in [t_1, t_2]$, define a 1-parameter family of admissible controlled trajectories (ξ_s, η_s) as

$$\eta_s(t) = \begin{cases} \eta(t) & \text{for } t_1 \le t \le s, \\ \\ u(t, p(s)) & \text{for } s < t \le t_+(s), \end{cases}$$

and

$$\xi_s(t) = \begin{cases} \xi(t) & \text{for } t_1 \le t \le s, \\ \\ x(t, p(s)) & \text{for } s < t \le t_+(s). \end{cases}$$

Thus the controlled trajectories (ξ_s, η_s) follow (ξ, η) over the interval $[t_1, s]$ until the point $\xi(s)$ is reached and then switch to the controlled trajectory of the parameterized family $\mathcal{E}$ determined by the parameter $p(s)$ that passes through $\xi(s)$ at time s, $\xi(s) = x(s, p(s))$. In particular, these concatenations satisfy the terminal constraints since the controlled trajectories in the family $\mathcal{E}$ do so. The corresponding cost $\Gamma(s)$ is given by

$$\Gamma(s) = \int_{t_1}^{s} L(t, \xi(t), \eta(t))dt + C(s, p(s)).$$

and we have that $\Gamma(t_1) = C(t_1, p_1)$ and $\Gamma(t_2) = \int_{t_1}^{t_2} L(t, \xi(t), \eta(t))dt + C(t_2, p_2)$. It follows from our general regularity assumptions that the integrand is bounded over a compact interval and thus Γ is an absolutely continuous function. Hence Γ is differentiable almost everywhere on $[t_1, t_2]$ and it suffices to show that $\Gamma'(s) = \frac{d\Gamma}{ds}(s) \equiv 0$, so that Γ is constant on $[t_1, t_2]$. Differentiating $\Gamma(\cdot)$ gives

$$\begin{aligned} \Gamma'(s) &= L(s, \xi(s), \eta(s)) + \frac{\partial C}{\partial t}(s, p(s)) + \frac{\partial C}{\partial p}(s, p(s))\frac{dp}{ds}(s) \\ &= L(s, \xi(s), \eta(s)) - L(s, x(s, p(s)), u(s, p(s))) + \lambda(s, p(s))\frac{\partial x}{\partial p}(s, p(s))\frac{dp}{ds}(s) \end{aligned}$$

where, in the last equation, we use the formulas for the derivatives of C and the shadow-price lemma, lemma 3.3. Since $x(s, p(s)) \equiv \xi(s)$ and ξ is a controlled trajectory, it follows that

$$\frac{\partial x}{\partial t}(s, p(s)) + \frac{\partial x}{\partial p}(s, p(s))\frac{dp}{ds}(s) = \frac{d\xi}{ds}(s) = f(s, \xi(s), \eta(s)),$$

so that

$$\frac{\partial x}{\partial p}(s, p(s))\frac{dp}{ds}(s) = f(s, \xi(s), \eta(s)) - f(s, x(s, p(s)), u(s, p(s))).$$

Hence

$$\Gamma'(s) = H(s, \lambda(s, p(s)), \xi(s), \eta(s)) - H(s, \lambda(s, p(s)), x(s, p(s)), u(s, p(s))) \equiv 0$$

and the result follows. □

Recall that $C(t, p)$ denotes the cost of the trajectories in the parameterized family $\mathcal{E}$, i.e., $C(t, p)$ is the cost for the control $u(\cdot, p)$ with trajectory $x(\cdot, p)$ and initial condition $x(t, p)$ at initial time t. In terms of the boundary points $(t_1, x_1) = (t_1, x(t_1, p_1))$ and $(t_2, x_2) = (t_2, x(t_2, p_2))$, the envelope condition (40) can thus equivalently be expressed as

$$V^{\mathcal{E}}(t_1, x_1) = \int_{t_1}^{t_2} L(t, \xi(t), \eta(t))dt + V^{\mathcal{E}}(t_2, x_2).$$

As in the calculus of variations, (40) relates the cost of the controlled trajectories in the family $\mathcal{E}$ with concatenations along the envelope.

It is shown in [17] that *integral curves of the vector field* v defined in Lemma 5.1 *in a canonical way generate classical envelopes at fold singularities.* Let $\pi : (-\varepsilon, \varepsilon) \to P$, $s \mapsto \pi(s)$, be the integral curve of the vector field v that passes through the point p_0 at time $t = 0$, i.e.,

$$\frac{d\pi}{ds}(s) = v(\pi(s)), \qquad \pi(0) = p_0, \tag{41}$$

and let

$$\chi : P \to S, \quad p \mapsto (\sigma(p), p),$$

be the C^1-diffeomorphism that injects the parameter space into the singular set. With $x = x(t,p)$ denoting the parameterized trajectories in the family $\mathcal{E}$, let Γ denote the curve

$$\Gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, \qquad s \mapsto \Gamma(s) = (x \circ \chi \circ \pi)(s) = x(\sigma(\pi(s)), \pi(s)). \tag{42}$$

We claim that a reparameterization of this curve is a controlled trajectory of the system. Let $\theta = \sigma \circ \pi$, so that $\frac{d\theta}{ds}(0) = \nabla\sigma(p_0) \cdot v(p_0) > 0$. For ε small enough, $t = \theta(s)$ is a strictly increasing function that maps $(-\varepsilon, \varepsilon)$ onto some interval (a, b) with inverse $s = \theta^{-1}(t)$. Let $p = p(t)$ be the curve

$$p : (a,b) \to P, \qquad t \mapsto p(t) = \left(\pi \circ \theta^{-1}\right)(t)$$

and define

$$\xi : (a,b) \to \mathbb{R}^n, \qquad t \mapsto \xi(t) = x(t, p(t))$$

and

$$\eta : (a,b) \to \mathbb{R}^n, \qquad t \mapsto \eta(t) = u(t, p(t)).$$

Proposition 5.5. [17] *The pair* (ξ, η) *is a portion of an admissible controlled trajectory.*

Proof. By construction, η is a continuous function that takes values in the control set U. (The latter is automatic since $u = u(t,p)$ is the control of the parameterized families of extremals.) Hence η is an admissible control. Differentiating ξ, it follows that

$$\dot{\xi}(t) = \dot{x}(t, p(t)) + \frac{\partial x}{\partial p}(t, p(t))\dot{p}(t) = f(t, \xi(t), \eta(t)) + \frac{\partial x}{\partial p}(t, p(t))\dot{p}(t).$$

But the second term vanishes by construction since $\dot{p}(t)$ is a multiple of the vector $v(p(t))$ in the nullspace of $\frac{\partial x}{\partial p}(t, p(t))$: writing $t = \theta(s)$, we have that

$$\dot{p}(t) = \pi'(s)\ \frac{ds}{dt}(t) = v(\pi(s))\ \frac{ds}{dt}(t) = v(p(t))\ \frac{ds}{dt}(t)$$

and thus

$$\frac{\partial x}{\partial p}(t, p(t))\dot{p}(t) = \frac{\partial x}{\partial p}(t, p(t)) v(p(t))\ \frac{ds}{dt}(t) = 0. \tag{43}$$

This segment can then be concatenated at the point $\xi(t) = x(t, p(t))$ with the controlled trajectory in the parameterized family $\mathcal{E}$ for the parameter $p(t)$ to give an admissible controlled trajectory. □

Thus the controlled trajectory (ξ, η) is a classical envelope. Note that it makes no difference for this argument whether the curve Γ can be reparameterized as an increasing or a decreasing function since we can simply reverse the orientation of the vector field v. But the result is no longer valid if a reversal of orientation occurs at p_0. This happens at a simple cusp singularity and thus this result does not extend to cusp points. Indeed, it is this property that is responsible for the fact that controlled trajectories can still be optimal at a cusp point [15, 20] while this is never true for fold points.

Theorem 5.6. *Let $\mathcal{E}$ be the n-dimensional, nicely C^1-parameterized, canonical family of nonsingular extremals with domain $D = \{(t,p) : p \in P,\, t_-(p) \le t \le T\}$ for the optimal control problem $[OC]$ constructed in Theorem 2.3. (Recall that we assume that the controls are the unique minimizers of the Hamiltonian and take values in the interior of the control set.) Suppose there exists a function $\sigma : P \to (t_-(p), T)$, $p \mapsto \sigma(p)$, so that the associated flow*

$$\digamma : D \to \mathbb{R}^{n+1}, \qquad (t,p) \mapsto \digamma(t,p) = (t, x(t,p)),$$

is a diffeomorphism when restricted to $D_{opt} = \{(t,p) : p \in P,\, \sigma(p) < t < T\}$, and that it has fold singularities at the points in $S = \{(t,p) : p \in P,\, t = \sigma(p)\}$. Then every controlled trajectory $(x(\cdot,p), u(\cdot,p))$ defined over an interval $[\tau, T]$ with $\tau > \sigma(p)$ provides a relative minimum over the domain $G = \digamma(D_{opt})$, but is no longer optimal over the interval $[\sigma(p), T]$.

Proof. The statements about local optimality of the controlled trajectories on D_{opt} follow from Corollary 3.7. It remains to show that the controlled trajectory $(x(\cdot,p), u(\cdot,p))$ is no longer optimal over the full interval $[\sigma(p), T]$.

Let $p_1 = p$, $t_1 = \sigma(p)$ and $x_1 = x(\sigma(p), p)$. Since (t_1, p_1) is a fold point, on some small interval $[t_1, t_2]$ there exists a classical envelope (ξ, η) through the point $(t_1, x_1) \in S$. Let $x_2 = \xi(t_2)$; since $(t_2, x_2) \in S$, there exists a parameter p_2 so that $t_2 = \sigma(p_2)$ and $x_2 = x(t_2, p_2)$. Define a second controlled trajectory $(\hat{\xi}, \hat{\eta})$ by

$$\hat{\xi}(t) = \begin{cases} \xi(t) & \text{if} \quad t_1 \le t < t_2, \\ x(t,p_2) & \text{if} \quad t_2 \le t \le T, \end{cases} \quad \text{and} \quad \hat{\eta}(t) = \begin{cases} \eta(t) & \text{if} \quad t_1 \le t < t_2, \\ u(t,p_2) & \text{if} \quad t_2 \le t \le T. \end{cases}$$

By Theorem 5.4, we have that

$$C(t_1, p_1) = \int_{t_1}^{t_2} L(t, \xi(t), \eta(t))dt + C(t_2, p_2)$$

and thus the controlled trajectories $(x(\cdot,p_1), u(\cdot,p_1))$ and $(\hat{\xi}, \hat{\eta})$ have the same cost. Hence, if $(x(\cdot,p_1), u(\cdot,p_1))$ is optimal over the interval $[t_1, T]$, then so is $(\hat{\xi}, \hat{\eta})$.

In order to show that this is not the case, we construct a curve of extremals in $\mathcal{E}$ that all project onto the *one* controlled trajectory $(x(\cdot,p_2), u(\cdot,p_2))$ on the interval $[t_2, T]$. For t fixed, the combined (x, λ) flow in the cotangent bundle always has an n-dimensional image since this is the dimension of the manifold describing the terminal conditions at time T. By our earlier results, for times t, $t_2 < t < T$, we already know that this flow has an n-dimensional projection into the state space. Contradiction.

If $(\hat{\xi}, \hat{\eta})$ is optimal, then on the interval $[t_2, T]$ there exists a nontrivial solution $\hat{\lambda}$ to the homogeneous linear equation

$$\frac{d\hat{\lambda}}{dt}(t) = -\hat{\lambda}(t) f_x(t, x(t,p_2), u(t,p_2))$$

with terminal condition $\hat{\lambda}(T) = \hat{\nu}\Psi_x(\xi(p_2))$ for some $\hat{\nu} \in \left(\mathbb{R}^{n-k}\right)^*$ that satisfies

$$0 = \hat{\lambda}(t) f_u(t, x(t,p_2), u(t,p_2)).$$

This directly follows from the maximum principle if $(\hat{\gamma}, \hat{\eta})$ is an abnormal extremal. If $(\hat{\gamma}, \hat{\eta})$ is a normal extremal, then these relations are satisfied by the difference between the adjoint vector for $(\hat{\gamma}, \hat{\eta})$ and the multiplier $\lambda(\cdot, p_2)$ from the parameterized family $\mathcal{E}$. Furthermore, because of the different structures of the controls over the interval $[t_1, t_2)$, the two multipliers cannot have the same values for $t = t_2$. In either case, there exists a nontrivial abnormal extremal lift of $(\hat{\xi}, \hat{\eta})$ to the cotangent bundle. For $|\varepsilon|$ sufficiently small, the co-vector

$$\lambda(t; \varepsilon) = \lambda(t, p_2) + \varepsilon \hat{\lambda}(t)$$

is thus a solution of the adjoint equation

$$\dot{\lambda}(t; \varepsilon) = -\lambda(t; \varepsilon) f_x(t, x(t, p_2), u(t, p_2)) - L_x(t, x(t, p_2), u(t, p_2))$$

on the interval $[t_2, T]$ that satisfies the terminal condition

$$\lambda(T; \varepsilon) = \varphi_x(\xi(p_2)) + (\nu(p_2) + \varepsilon \hat{\nu})\, \Psi_x(\xi(p_2)).$$

Recall from the construction of the parameterized family $\mathcal{E}$ in Theorem 2.3 that the function $u(t, x, \lambda)$ defining the parameterized controls is the unique local solution of the equation $H_u(t, \lambda, x, u) = 0$ for u in a neighborhood of the reference extremal. For t_2 and ε small enough, the pair $(x(t, p_2), \lambda(t; \varepsilon))$ will lie in this neighborhood. Since $u(\cdot, p_2)$ is a minimizing control that lies in the interior of the control set, we have that

$$0 = L_u(t, x(t, p_2), u(t, p_2)) + \lambda(t; p_2) f_u(t, x(t, p_2), u(t, p_2))$$

and since $0 = \hat{\lambda}(t) f_u(t, x(t, p_2), u(t, p_2))$ this gives that

$$0 = L_u(t, x(t, p_2), u(t, p_2)) + \lambda(t; \varepsilon) f_u(t, x(t, p_2), u(t, p_2)).$$

Thus $u(t, p_2)$ is a local solution of the equation

$$0 = H_u(t, \lambda(t; \varepsilon), x(t, p_2), u(t, p_2))$$

as well and so it follows that

$$u(t, p_2) = u(t, x(t, p_2), \lambda(t; \varepsilon)).$$

By our assumption that controls lie in the interior of the control set and are the unique minimizers of the Hamiltonian around the reference extremal, it furthermore follows from the proof of Theorem 2.3 that for given terminal values

$$(x(T, p), \lambda(T, p)) = (\xi(p_2), \varphi(\xi(p_2)) + (\nu(p_2) + \varepsilon \hat{\nu})^T\, \Psi(\xi(p_2))),$$

extremals (λ, x, u) are the unique solutions to the following combined system of differential equations and minimality condition:

$$\dot{x} = f(t, x, u), \qquad \dot{\lambda} = -\lambda f_x(t, x, u) - L_x(t, x, u), \qquad 0 = H_u(t, \lambda, x, u).$$

The triple consisting of the multiplier $\lambda(\cdot; \varepsilon)$, state $x(\cdot, p_2)$ and control $u(\cdot, p_2)$ satisfies these equations and thus for ε near 0, the curves $t \mapsto (\lambda(t; \varepsilon), x(t, p_2), u(t, p_2))$ are extremals for the optimal control problem $[OC]$, which, for $\varepsilon = 0$, reduce to the reference extremal. But extremals around the reference are unique and hence these extremals are members of the parameterized family $\mathcal{E}$ that all project onto the controlled trajectory $(x(\cdot, p_2), u(\cdot, p_2))$ on the interval $[t_2, T]$. This proves the theorem. $\square$

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