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SYNTHESIS OF OPTIMAL CONTROLLED TRAJECTORIES WITH CHATTERING ARCS

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Abstract. A field theoretic approach that is based on the method of characteristics is outlined to prove the optimality of a synthesis of controlled trajectories that contains chattering arcs. Since the corresponding controls are not piecewise continuous - they switch infinitely many times on finite intervals - the classical argument by Boltyansky to prove differentiability of the associated value function is no longer applicable. This difficulty is overcome by matching the *two* parameterized families of extremal controlled trajectories that correspond to the constant controls $u = \pm 1$ at the switching surfaces in a continuously differentiable way.

Keywords. optimal control, Fuller problem, chattering controls, synthesis, tumor anti-angiogenesis.

AMS (MOS) subject classification: 49N35, 49L20, 93C10 .

1 Introduction

In the early sixties, in the design of an optimal filter for a communication problem in electronics, A.T. Fuller came across an innocently looking nonlinear optimal control problem with a puzzling solution [10]. Optimal controls alternate at increasing speeds between the extreme values $+1$ and -1 and the switchings converge in a geometric progression to the terminal time generating optimal controls that are no longer piecewise continuous, but only Lebesgue measurable. For a long time, this so-called Fuller phenomenon was considered an aberration and ignored although it was quite clear that chattering actually is a rather simple mechanism that can be found in many physical applications, the simplest one being a bouncing ball if friction is neglected in the mathematical model. It was only in the late eighties with the work of

I.A.K. Kupka [14] that it became understood that chattering is a generic and thus common place phenomenon in higher dimensional systems that simply has to be dealt with. Several examples from engineering and economics are given and analyzed in the research by M.I. Zelikin and his co-workers such as [26, 28] and especially in the monograph [27] about chattering controls jointly with V.F. Borisov. Some more recent examples include the control of an autonomous underwater vehicle in the work by M. Chyba et al. [7, 8] or moral hazard in a principal-agent problem in economics considered by J. Yang [25].

Mathematically, the analysis of chattering controls and their corresponding trajectories is quite challenging. For example, chattering invalidates the standard reasoning to prove differentiability of the associated value function as it is used in Boltyansky's theory of regular synthesis [3] or its more general extensions such as [22]. In the monograph [27], Zelikin and Borisov develop a geometric picture of chattering extremals in the cotangent bundle, the joint state-multiplier space, that is based on a careful analysis and generalizations of scaling properties that are present in the Fuller problem. However, it typically is quite difficult to analyze the projections of these extremal lifts into the state-space and this is needed in order to prove optimality. In this paper, we propose to use an alternative geometric field theoretic approach that is directly carried out in the state-space, not in the state-multiplier space, to prove the optimality of a synthesis of controlled trajectories that contain chattering arcs. The difficulty of dealing with controls that are merely Lebesgue measurable is overcome by matching the *two* parameterized families of extremal controlled trajectories that correspond to the two constant controls $u = \pm 1$ (the limits of the control set) in a continuously differentiable way at the switching surfaces.

Our interest in this topic was stirred up by the analysis of optimal control problems for tumor anti-angiogenesis, a novel cancer treatment approach, in which singular arcs determine the structure of optimal solutions. When a standard linear pharmacokinetic model is included to describe the concentrations of the anti-angiogenic agent in the plasma, the corresponding singular control is of intrinsic order 2 [27] and transitions to and from this singular arc are made through chattering controls. We briefly comment on the local synthesis around the singular arc in the last section of this paper and how the approach developed here can be used to establish the local optimality of these controls.

2 The Method of Characteristics in Optimal Control

We start with a definition of the optimal control problem that we consider in this paper. We think of a control system as a collection of time-dependent vector fields on a differentiable manifold parameterized by controls which, by means of the solutions of the corresponding ordinary differential equations, give rise to a family of controlled trajectories. An optimal control problem then is the task to minimize some functional over these controlled trajectories subject to additional constraints.

In order to keep the presentation simple, we take as the state space an open subset M of $\mathbb{R}^n$. Given an arbitrary subset $U \subset \mathbb{R}^m$, the control set, admissible controls are locally bounded, Lebesgue measurable functions $u : [0,T] \to U$, $t \mapsto u(t)$, that take values in this control set. The dynamics is described by an ordinary differential equation

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \tag{1}$$

and we assume that there exists some positive integer ℓ or $\ell = \infty$, so that f is ℓ-times continuously differentiable in x and u. Under these assumptions it follows from the standard Caratheodory conditions that for any admissible control the initial value problem has a unique solution which we call the corresponding trajectory. The pair (x,u) is a controlled trajectory. Here we only consider in addition a terminal constraint in the form

$$x(T) \in N = \{x \in \mathbb{R}^n : \Psi(x) = 0\}$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}^{n-k}$, $x \mapsto \Psi(t,x) = (\psi_1(x), \dots, \psi_{n-k}(x))^T$, and the functions also are ℓ-times continuously differentiable. Furthermore, we assume that the Jacobian matrix of the equations, $D\Psi$, is of full rank on N and thus N is a k-dimensional embedded C^ℓ-submanifold of $\mathbb{R}^n$. Finally, for an admissible controlled trajectory (x,u), we define the objective as

$$\mathcal{J}(u) = \int_0^T L(x(s), u(s))ds + \varphi(x(T))$$

making the same smoothness assumptions on the Lagrangian L as on f and the penalty function φ also is C^ℓ. We then consider the following optimal control problem:

[OC] for a given initial condition $x_0 \in \mathbb{R}^n$, minimize the functional $\mathcal{J}(u)$ over the class of controlled trajectories $(x.u)$ for which $x(T) \in N$. The terminal time T is free.

The method of characteristics is a geometric approach to construct solutions to the Hamilton-Jacobi-Bellman (HJB) equation and thus to prove the optimality of a field of extremals that has been found through an analysis of necessary conditions for optimality The simple idea is to parameterize extremals by integrating the system and adjoint equation backward from the terminal manifold while maintaining the minimum condition of the maximum principle and then to investigate the mapping properties of the corresponding family of controlled trajectories. If this flow covers the state space injectively, then the objective evaluated along this family of trajectories, also called the *cost-to-go function*, will be the desired solution to the HJB equation. The construction itself clearly brings out the relationships between the necessary conditions of the Pontryagin maximum principle [23] (or, for a more modern treatment, see [5, 6]) and the dynamic programming principle [11].

Solutions to optimal control problems typically have a regular structure "piecewise" in the sense that the state-space can be decomposed into a finite or locally finite collection of embedded submanifolds with the property that the restrictions of the controls and trajectories to these domains are smooth and that there is sufficient regularity when the various "patches" are glued together to form the full solution. We only give the definitions that apply to *one* such patch and assume that the controls are continuous and for fixed time t depend r-times continuously differentiable on a parameter p with the partial derivatives $\frac{\partial u}{\partial p}(t,p)$ continuous in (t,p). It is not difficult to glue together various patches [21], but this will not be pursued here.

Definition 1 ***(parameterized family of controlled trajectories)*** *Given an open subset P of $\mathbb{R}^d$ with $0 \leq d \leq n-1$, let $t_- : P \to \mathbb{R}$, $p \mapsto t_-(p)$, and $t_+ : P \to \mathbb{R}$, $p \mapsto t_+(p)$, be two r-times continuously differentiable functions, $t_\pm \in C^r(P)$, that satisfy $t_-(p) < t_+(p)$ for all $p \in P$. We call t_- and t_+ the initial and terminal times of the parametrization and define its domain as*

$$D = \{(t,p) : p \in P,\, t_-(p) \leq t \leq t_+(p)\}.$$

Let $\xi_- : P \to \mathbb{R}^n$, $p \mapsto \xi_-(p)$, and $\xi_+ : P \to \mathbb{R}^n$, $p \mapsto \xi_+(p)$, be r-times continuously differentiable functions, $\xi_\pm \in C^r(P)$. A d-dimensional C^r-parameterized family $\mathcal{T}$ of controlled trajectories with domain D, initial conditions ξ_- and terminal conditions ξ_+ consists of:

1. *admissible controls, $u : D \to U$, $(t,p) \mapsto u(t,p)$, that are continuous on D, r-times continuously differentiable in p on the interior of D with these partial derivatives extending continuously onto D, ($u \in C^{0,r}(D)$),*
2. *and corresponding trajectories $x : D \to M$, $(t,p) \mapsto x(t,p)$, i.e., solutions of the dynamics*

$$\dot{x}(t,p) = f(x(t,p), u(t,p)), \tag{2}$$

that exist over the full interval $[t_-(p), t_+(p)]$ and satisfy the initial condition $x(t_-(p),p) = \xi_-(p)$ and terminal condition $x(t_+(p),p) = \xi_+(p)$.

Definition 2 ***(flow of controlled trajectories)*** *Given a C^r-parameterized family $\mathcal{T}$ of controlled trajectories, the associated flow is the mapping*

$$\digamma : D \to \mathbb{R}^n, \quad (t,p) \longmapsto \digamma(t,p) = x(t,p).$$

We say the flow $\digamma$ is a $C^{1,r}$-mapping on some open set $Q \subset D$ if the restriction of $\digamma$ to Q is continuously differentiable in (t,p) and r times differentiable in p with derivatives that are jointly continuous in (t,p). If $\digamma \in C^{1,r}(Q)$ is injective and the Jacobian matrix $D\digamma(t,p)$ is nonsingular everywhere on Q, then we say $\digamma$ is a $C^{1,r}$-diffeomorphism onto its image $\digamma(Q)$.

It is easily seen that the control u extends as a $C^{0,r}$-function onto an open neighborhood of D and it thus follows that the trajectories $x(t,p)$ and their time-derivatives $\dot{x}(t,p)$ are r-times continuously differentiable in p and that these derivatives are continuous jointly in (t,p) in an open neighborhood of D, i.e., $x \in C^{1,r}(D)$. In particular, the flow $\mathcal{F}$ is a $C^{1,r}$-mapping, but not necessarily a $C^{1,r}$-diffeomorphism. The boundary sections

$$M_- = \{(t,p) : \ p \in P, \ t = t_-(p)\} \text{ and } M_+ = \{(t,p) : \ p \in P, \ t = t_+(p)\}$$

of a C^r-parameterized family of controlled trajectories are the graphs of the functions t_- and t_+, $M_- = \mathfrak{gr}\,(t_-)$ and $M_+ = \mathfrak{gr}\,(t_+)$. We call the images of these sections under the flow $\mathcal{F}$, $N_\pm = \mathcal{F}\,(M_\pm) = \{\xi_\pm(p) : \ p \in P\}$, the *source*, respectively the *target* of the parametrization. In the construction, generally one of these is specified and the trajectories are defined as the solutions of the associated initial (or terminal) value problem. The other then simply is defined by the flow of these solutions. If $\mathcal{T}$ is a C^r-parameterized family of controlled trajectories with source N_-, then (with the obvious modifications to the definition) we also allow that $t_+(p) \equiv +\infty$ and for a family with target N_+ we may have that $t_-(p) \equiv -\infty$. Depending on the particular situation, trajectories are either integrated forward in time (families of controlled trajectories with source N_-) or backward in time (families of controlled trajectories with target N_+).

In order to calculate the cost-to-go function along a parameterized family of controlled trajectories, we need to add a function γ that describes this cost at the source N_- or the target N_+.

Definition 3 ***(parameterized family of controlled trajectories with cost)*** *Suppose $\mathcal{T}$ is a d-dimensional C^r-parameterized family of controlled trajectories with domain D and initial and terminal values ξ_- and ξ_+. Given an r-times continuously differentiable function $\gamma_- : P \to \mathbb{R}$, $p \mapsto \gamma_-(p)$, (respectively, $\gamma_+ : P \to \mathbb{R}$, $p \mapsto \gamma_+(p)$,) we define the cost associated with $\mathcal{T}$ as*

$$C(t,p) = \gamma_-(p) - \int_{t_-(p)}^{t} L(x(s,p), u(s,p))\, ds,$$

(respectively, as

$$C(t,p) = \int_{t}^{t_+(p)} L(x(s,p), u(s,p))\, ds + \gamma_+(p),$$

when the terminal value is specified). We call $\mathcal{T}$ a C^r-parameterized family of controlled trajectories with cost γ.

The functions γ_+ and γ_- propagate the cost along trajectories from patch to patch and $C(t,p)$ represents the value of the objective for the control $u = u(\cdot,p)$ if the initial condition at time t is given by $x(t,p)$. Since the value of the optimal cost on the terminal manifold N is specified by the penalty term φ in the objective, integrating trajectories backward in time is the typical procedure. For syntheses where trajectories are successively integrated backward from the terminal manifold, these functions are easily computed like in the classical time-optimal control problem for the double integrator.

Example (double integrator). Consider the time-optimal control problem to steer a point $x = (x_1, x_2)^T$ into the origin in minimum time subject to the dynamics $\dot{x}_1 = x_2$ and $\dot{x}_2 = u$ and control constraint $|u| \leq 1$. The optimal solution to this standard textbook example is easily found from the conditions of the maximum principle (e.g., [11]) and we just use it as a simple illustration to show how parameterized families of various dimensions d easily allow to build up the optimal synthesis in an inductive way. For this problem optimal controls are bang-bang with at most one switching. The switching curves Γ_+ and Γ_- are 0-dimensional "parameterized families" which (integrating backward from the target $N_+ = \{0\}$ and dropping the 0-dimensional parameter $p = 0$) can be described as functions of t over the interval $(-\infty, 0]$ as the trajectories $x_\pm(t) = \pm(\frac{1}{2}t^2, t)^T$ that correspond to the constant controls $u(t) \equiv \pm 1$ while $\xi_+ = 0$. The full families of trajectories that cover the regions $R_+ = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 < -sgn(x_2)\frac{1}{2}x_2^2\}$ and $R_- = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > -sgn(x_2)\frac{1}{2}x_2^2\}$ then form 1-dimensional parameterized families $\mathcal{T}_+$ and $\mathcal{T}_-$ given by

$$\begin{aligned}\mathcal{T}_+ : &P = (0, \infty) \quad t_-(p) = -\infty, \ t_+(p) = -p, \quad D = \{(t, p) : \ t \leq -p < 0\}, \\ &\xi_+(p) = \begin{pmatrix} -\frac{1}{2}p^2 \\ p \end{pmatrix}, \quad u(t,p) \equiv +1, \quad x(t,p) = \begin{pmatrix} \frac{1}{2}t^2 + 2pt + p^2 \\ t + 2p \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_- : &P = (0, \infty) \quad t_-(p) = -\infty, \ t_+(p) = -p, \quad D = \{(t, p) : \ t \leq -p < 0\}, \\ &\xi_+(p) = \begin{pmatrix} \frac{1}{2}p^2 \\ -p \end{pmatrix}, \quad u(t,p) \equiv -1, \quad x(t,p) = \begin{pmatrix} -\frac{1}{2}t^2 - 2pt - p^2 \\ -t - 2p \end{pmatrix}.\end{aligned}$$

Since the problem is autonomous, there is flexibility in choosing one of the initial or terminal times t_+ or t_-. In the parametrizations above, we have chosen $t_+(p)$ so that the variable t in $x(t, p)$ is equal to the negative of the total time it takes to steer the initial condition $x(t, p)$ into the origin along the concatenated optimal trajectory. The cost functions on the targets of the parameterizations therefore are thus immediate and are given by $\gamma_+ = 0$ for the 0-dimensional strata $N = \{0\}$ and $\gamma_+(p) = p$ for the two 1-dimensional families $\mathcal{T}_\pm$. All these parameterizations are real analytic, C^ω.

For a parameterized family like in this example, the cost functions $\gamma_\pm$ are easily obtained through backward integration from the targets. Naturally, this direct integration approach may not be convenient for problems when the synthesis becomes more complex or the switching curves Γ_+ and Γ_- no longer are trajectories of the system. But, as we shall see below, integrating

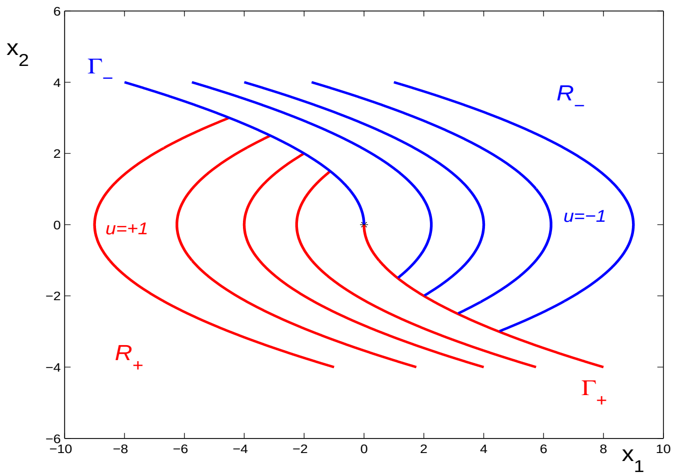

Figure 1: Synthesis of optimal controlled trajectories for the time-optimal control problem to the origin for the double integrator

the cost along the trajectories in the family is not the only method to find the correct functions γ that propagate the cost. This precisely is our main argument in this paper.

We are interested in optimal controlled trajectories and thus we now consider families of controlled trajectories that satisfy the conditions of the Pontryagin maximum principle [23], so-called extremals. Once more, we assume that $u \in C^{0,r}$ which naturally extends to the state as $x \in C^{1,r}$. However, since the adjoint is defined in terms of data that are differentiated in the state x, we generally only obtain one degree less of smoothness for the multipliers, $\lambda \in C^{1,r-1}$. The following definition is essential for the construction of solutions to the Hamilton-Jacobi-Bellman equation by means of the method of characteristics. As always,

$$H = H(\lambda_0, \lambda, x, u) = \lambda_0 L(x, u) + \lambda f(x, u).$$

Definition 4 ***(C^r-parameterized family of extremals)*** *As before, let P be an open subset of $\mathbb{R}^d$ with $0 \leq d \leq n-1$, let t_- and t_+, $t_\pm \in C^r(P)$, be the initial and terminal times for the parametrization and let $D = \{(t,p) : p \in P, t_-(p) \leq t \leq t_+(p)\}$. A d-dimensional C^r-parameterized family $\mathcal{E}$ of extremals (or extremal lifts) with domain D consists of*

1. *a C^r-parameterized family $\mathcal{T}$ of controlled trajectories (x, u) with domain D, initial and terminal conditions ξ_- and ξ_+, and cost γ_- (respectively, γ_+):*

$$\dot{x}(t,p) = f(x(t,p), u(t,p)), \qquad x(t_\pm(p), p) = \xi_\pm(p);$$

2. *a non-negative multiplier* $\lambda_0 \in C^{r-1}(P)$ *and co-state* $\lambda : D \to (\mathbb{R}^n)^*$, $\lambda = \lambda(t,p)$, *so that* $(\lambda_0(p), \lambda(t,p)) \neq (0,0)$ *for all* $(t,p) \in D$ *and the adjoint equation*

$$\dot{\lambda}(t,p) = -\lambda_0(p) L_x(x(t,p), u(t,p)) - \lambda(t,p) f_x(x(t,p), u(t,p)),$$

is satisfied on the interval $[\tau_-(p), \tau_+(p)]$ *with boundary condition* $\lambda_-(p) = \lambda(\tau_-(p), p)$ *(respectively,* $\lambda_+(p) = \lambda(\tau_+(p), p)$*) given by an* $(r-1)$*-times continuously differentiable function of* p*,*

such that the following conditions are satisfied:

3. *defining* $h(t,p) = H(\lambda_0(p), \lambda(t,p), x(t,p), u(t,p))$*, the controls* $u = u(t,p)$ *solve the minimization problem*

$$h(t,p) = \min_{v \in U} H(\lambda_0(p), \lambda(t,p), x(t,p), v);$$

4(a). *with* $h_\pm(p) = h(t_\pm(p), p))$*, the following transversality condition holds at the source (respectively, target)*

$$\lambda_\pm(p) \frac{\partial \xi_\pm}{\partial p}(p) = \lambda_0(p) \frac{\partial \gamma_\pm}{\partial p}(p) + h_\pm(p) \frac{\partial t_\pm}{\partial p}(p); \qquad (3)$$

4(b). *if the target* N_+ *is a part of the terminal manifold* N*,* $N_+ \subset N$*, then setting* $T(p) = t_+(p)$*, with* $\xi_+(p) = x(T(p), p)$ *we have that* $\gamma_+(p) = \varphi(\xi_+(p))$ *and* $h(T(p), p) = 0$*; furthermore, there exists an* $(r-1)$*-times continuously differentiable multiplier* $\nu : P \to \left(\mathbb{R}^{n+1-k}\right)^*$ *so that*

$$\lambda(T(p), p) = \lambda_0(p) \varphi_x(\xi_+(p)) + \nu(p) \Psi_x(\xi_+(p)), \qquad (4)$$

This definition merely formalizes that all controlled trajectories in the family $\mathcal{E}$ satisfy the conditions of the Pontryagin maximum principle while some smoothness properties are satisfied by the parametrization and natural geometric regularity assumptions are met at the terminal manifold N. Note that it has not been assumed that the parametrization $\mathcal{E}$ of extremals covers the state-space injectively and one of the advantages of this framework is that it allows to analyze the geometry of the flow of the associated controlled trajectories as injectivity becomes lost (e.g., see [9, 15]).

The degree r in the definition denotes the smoothness of the parametrization of the controls in the parameter p, $u \in C^{0,r}$, and this together with the smoothness assumptions on the initial, respectively, terminal data guarantees that $x \in C^{1,r}$ and $\lambda \in C^{1,r-1}$. Note that if $\lambda_0(p) > 0$ for all $p \in P$, then by diving by $\lambda_0(p)$ we may assume that $\lambda_0(p) \equiv 1$ and we call such a family *normal.*

Example (double integrator, ctd.) By construction, the families $\mathcal{T}_\pm$ of controlled trajectories defined earlier are real analytic families of extremals:

All extremals are normal, $\lambda_0(p) \equiv 1$, and the adjoint variables $\lambda(t,p)$ are the solutions to the equations $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = -\lambda_1$ with terminal conditions $\lambda_\pm(p) = \lambda(t_\pm(p),p) = \left(-\frac{\varepsilon}{p}, 0\right)$ where $\varepsilon = +1$ for $\mathcal{T}_+$ and $\varepsilon = -1$ for $\mathcal{T}_-$. [Reason: For this problem the multiplier λ_2 determines the switchings and vanishes on the switching curves $\Gamma_\pm$, $\lambda_2(t_\pm(p),p) = 0$. The value of $\lambda_1(t_\pm(p),p)$ is determined by the fact that the Hamiltonian H vanishes identically, i.e.,

$$0 \equiv 1 + \lambda_\pm(p) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi_\pm(p) = 1 + \lambda_1(t_\pm(p),p)p\varepsilon,$$

so that $\lambda_1(t_\pm(p),p) = -\frac{\varepsilon}{p}$.] With this specification, the controls satisfy the minimum condition by construction. Furthermore, $t_+(p) = -p$, $\gamma_+(p) = p$, and $\xi_+(p) = \varepsilon\left(-\frac{1}{2}p^2, p\right)^T$ which gives

$$\lambda_\pm(p)\frac{\partial \xi_\pm}{\partial p}(p) = \left(-\frac{\varepsilon}{p}, 0\right)\begin{pmatrix} -p \\ 1 \end{pmatrix}\varepsilon = \varepsilon^2 = 1 = \frac{\partial \gamma_\pm}{\partial p}(p),$$

i.e., the transversality condition (3) is satisfied (recall that $h \equiv 0$ and $\lambda_0(p) \equiv 1$). Since $p \neq 0$, both parameterizations are real-analytic.

The transversality condition (3) is the key to identifying the multiplier λ with the gradient of the value function, the key relation in making the transition from necessary to sufficient conditions for optimality. This is summarized in the following fundamental relation below that is proved in [21] and is known as the shadow price lemma in economics.

Lemma 1 ***(Shadow-Price Lemma)*** [21] *Let $\mathcal{E}$ be a C^1-parameterized family of extremals with domain D. Then for all $(t,p) \in D$*

$$\lambda_0(p)\frac{\partial C}{\partial p}(t,p) = \lambda(t,p)\frac{\partial x}{\partial p}(t,p) \tag{5}$$

For normal extremals, if the corresponding family of trajectories covers a region G injectively, then the Shadow-Price lemma implies that the associated cost is a classical solution to the Hamilton-Jacobi-Bellman equation on G.

Theorem 1 *Let $\mathcal{E}$ be a C^r-parameterized family of normal extremals and suppose the restriction of its flow $\digamma$ to some open set $Q \subset D$ is a $C^{1,r}$-diffeomorphism onto an open subset $G \subset \mathbb{R}^n$ of the state-space. Then*

$$V : G \to \mathbb{R}, \quad V = C \circ \digamma^{-1},$$

is r-times continuously differentiable in x and

$$u_* : G \to \mathbb{R}, \quad u_* = u \circ \digamma^{-1},$$

is an r-times continuously differentiable admissible feedback control. Together, the pair (V, u_*) is a classical solution of the Hamilton-Jacobi-Bellman equation

$$\min_{u \in U} \{\nabla V(x) f(x,u) + L(x,u)\} \equiv 0$$

on G. Furthermore, in the parameter space we have on Q that

$$\lambda(t,p) = \nabla V(x(t,p)) \tag{6}$$

and

$$H(\lambda(t,p), x(t,p), u(t,p)) \equiv 0 \tag{7}$$

Proof. [21] Since $\digamma \restriction Q : Q \to G$ is a $C^{1,r}$-diffeomorphism, the function V and the control u_* are well-defined and the stated smoothness properties carry over from the parametrization. Since $C = V \circ \digamma$, we have that $\nabla C(t,p) = \nabla V(x(t,p)) \cdot D\digamma(t,p)$. Furthermore, by the Shadow-Price lemma and since $H \equiv 0$, we also have that

$$\begin{aligned} \nabla C(t,p) &= \left(\frac{\partial C}{\partial t}(t,p), \frac{\partial C}{\partial p}(t,p) \right) = \left(-L(x(t,p), u(t,p)), \lambda(t,p) \frac{\partial x}{\partial p}(t,p) \right) \\ &= \lambda(t,p) \left(f(x(t,p), u(t,p)), \frac{\partial x}{\partial p}(t,p) \right) \\ &= \lambda(t,p) \left(\frac{\partial x}{\partial t}(t,p), \frac{\partial x}{\partial p}(t,p) \right) = \lambda(x(t,p)) D\digamma(t,p). \end{aligned}$$

Since $D\digamma(t,p)$ is nonsingular, equation (6) follows.

But then the minimum condition in the definition of extremals implies that the pair (V, u_*) solves the Hamilton-Jacobi-Bellman equation: indeed, for $x(t,p) \in G$ and an arbitrary control value $v \in U$ we have that

$$\begin{aligned} & V_x(x) f(x,v) + L(x,v) = V_x(x(t,p)) f(x(t,p), v) + L(x(t,p), v) \\ & = \lambda(t,p) f(x(t,p), v) + L(x(t,p), v) \\ & = H(\lambda(t,p), x(t,p), v) \geq H(\lambda(t,p), x(t,p), u(t,p)) = 0 \end{aligned}$$

with equality for $v = u(t,p)$. □

Definition 5 ***(local field of extremals)*** *A C^r-parameterized local field of extremals, $\mathcal{F}$, is a C^r-parameterized family of normal extremals for which the associated flow $\digamma : D \to \mathbb{R}^n$, $(t,p) \mapsto \digamma(t,p)$, is a $C^{1,r}$-diffeomorphism from the interior of D, int$(D) = \{(t,p) : p \in P,\ t_-(p) < t < t_+(p)\}$, onto a region $G = \digamma(\mathrm{int}(D))$.*

We do not require that the flow $\digamma$ is a diffeomorphism on the source or target of the parametrization. If these are codimension 1 embedded submanifolds, and if the flow $\digamma$ is transversal to them, then the flow extends as a $C^{1,r}$-diffeomorphism onto a neighborhood of the full closed domain D.

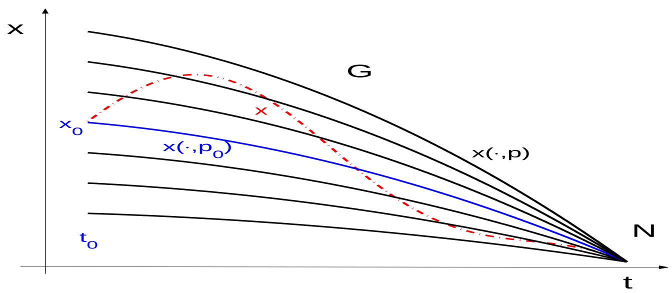

Figure 2: A relative minimum - $\mathcal{J}(u_{(t_0,x_0)}) \leq \mathcal{J}(v)$

However, if the target parameterizes a section N_T of the terminal manifold N, $N_T \subset N$, and N is of codimension greater than 1, then the flow F is not a diffeomorphism for the terminal time $t = t_+(p) = T(p)$. In this case, regardless of the dimension of N, the value function V constructed in Theorem 1 has a well-defined continuous extension to N_T since the terminal value of the cost,

$$C(T(p), p) = \varphi(T(p), \xi_+(p)) = \varphi(F(T(p), p)),$$

only depends on the terminal point $F(T(p), p)$, but not on the parameter p itself. Thus, if parameterizations for $p_1 \neq p_2$ have the same terminal point, $(T, x) = F(T(p_1), p_1) = F(T(p_2), p_2)$, this still gives rise to a unique specification of the value as

$$V(T, x) = \varphi(F(T(p), p)).$$

Hence we can extend the definition of $V = C \circ F^{-1}$ to the target $N_T \subset N$ of the parameterized family by taking any of the pre-images of F in the parametrization of the target through $D_T = \{(T, p) : T = T(p), p \in P\}$.

Combining Theorem 1 with classical theorems that a differentiable solution for the Hamilton-Jacobi-Bellman equation implies optimality of the corresponding control [11], we therefore have the following result (see Fig. 2):

Corollary 1 *Let $\mathcal{F}$ be a C^r-parameterized local field of extremals with target N_T in the terminal manifold N, $N_T \subset N$, and assume its associated flow covers a domain G. Then, given any initial condition $x_0 \in G$, $x_0 = F(t_0, p_0)$, the open-loop control $u_{(t_0,x_0)}(t) = u(t, p_0)$, $t_0 \leq t \leq T(p_0)$, is optimal when compared with any other admissible control v for which the corresponding trajectory x lies in G.* ■

Corollary 1 is tailored to the formulation of sufficient conditions for local minima. For questions about global optimality, however, it is generally necessary to consider various local fields of extremals and then glue them together. For a specific problem, this often is easily accomplished once the extremal trajectories have been constructed. We illustrate this for the Fuller problem, a classical and rather innocently looking problem whose solutions are chattering controls.

3 The Fuller Problem

We use the constructions above to give a direct proof of the optimality of the chattering controlled trajectories for the Fuller problem, a classical optimal control problem [10, 1, 2].

(Fuller) Given a point $p \in \mathbb{R}^2$, find a Lebesgue measurable function with values in the interval $[-1, 1]$ that steers p into the origin under the dynamics $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ and minimizes the objective

$$J(u) = \frac{1}{2}\int_0^T x_1^2(t)dt.$$

The time T of transfer is free. Since the problem is time-invariant, we can arbitrarily shift the interval of definition for the control and for this problem it is more convenient to normalize the terminal time to be 0. We thus consider the controls and trajectories to be defined over intervals $[-T, 0] \subset (-\infty, 0]$. The following theorem gives the optimal solution to this classical problem.

Theorem 2 [10, 24] *Let* $\zeta = \sqrt{\frac{\sqrt{33}-1}{24}} = 0.4446236\ldots$, *the unique positive root of the equation* $z^4 + \frac{1}{12}z^2 - \frac{1}{18} = 0$, *and define*

$$\begin{aligned}
\Gamma_+ &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 = \zeta x_2^2, \ x_2 < 0\},\\
\Gamma_- &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 = -\zeta x_2^2, \ x_2 > 0\},\\
R_+ &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 < -sgn(x_2)\zeta x_2^2\},\\
R_- &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > -sgn(x_2)\zeta x_2^2\}.
\end{aligned}$$

Then the optimal control for the Fuller problem is given in feedback form as

$$u_*(x) = \begin{cases} +1 & \text{for } x \in R_+ \cup \Gamma_+ \\ -1 & \text{for } x \in R_- \cup \Gamma_- \end{cases}. \tag{8}$$

Corresponding trajectories cross the switching curves Γ_+ *and* Γ_- *transversally changing from* $u = -1$ *to* $u = +1$ *at points on* Γ_+ *and from* $u = +1$ *to* $u = -1$ *at points on* Γ_-. *These trajectories are chattering arcs with an infinite number of switchings that accumulate with a geometric progression at the final time* $T = 0$.

Fig. 3 depicts the optimal synthesis for the Fuller problem. It very much looks like the synthesis for the double integrator, but with the significant difference that the switching curve $\Gamma = \Gamma_+ \cup \{(0,0)\} \cup \Gamma_-$ now is NOT a trajectory of the system. Trajectories in the field always cross Γ transversally and cannot enter the origin along Γ. This leads to optimal controls that switch infinitely many times as they accumulate at the final time $T = 0$. While this problem with its solution given by chattering arcs was considered an aberration for a long time, Kupka has shown that this is far from the truth and that chattering extremals indeed are a generic phenomenon, i.e., are in some very precise mathematical sense "typical" in higher state-space dimensions [14, Thm 0., pg. 219].

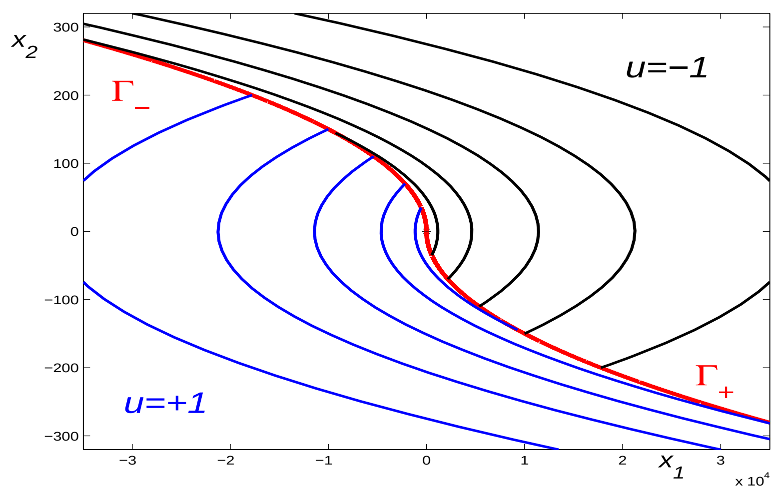

Figure 3: Optimal synthesis for the Fuller problem

The reason for the optimal chattering controls can be understood if one embeds the Fuller problem into a time-optimal control problem in $\mathbb{R}^3$ by adding the objective as a third variable, $\dot{x}_3 = \frac{1}{2}x_1^2$, i.e., the drift vector field f and control vector field g are then given by

$$f(x) = \begin{pmatrix} x_2 \\ 0 \\ \frac{1}{2}x_1^2 \end{pmatrix} \qquad \text{and} \qquad g(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For this 3-dimensional time-optimal control problem, there exists a locally optimal singular arc Γ_F given by $x_1 \equiv x_2 \equiv 0$ with singular control $u_{\sin} \equiv 0$. But this arc is of intrinsic order 2 [5] and thus it follows from classical junction conditions that it cannot be concatenated with the constant bang controls $u = 1$ or $u = -1$ [19, 20]. These transitions can only be accomplished by means of a chattering control. Indeed, the solutions to the Fuller problem are optimal abnormal extremals for this 3-dimensional time optimal control problem.

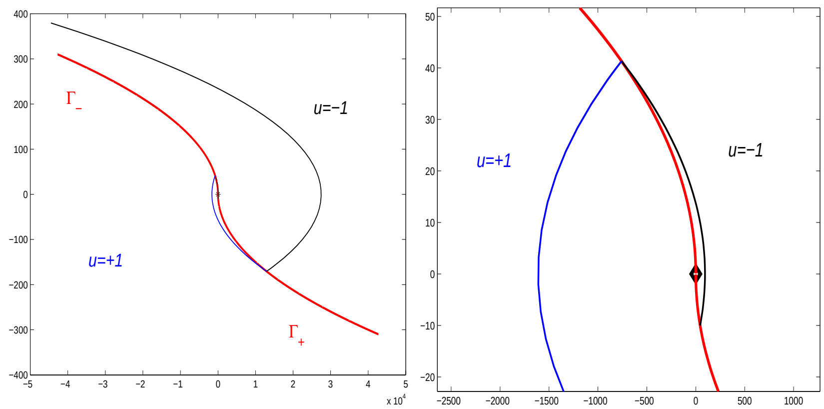

Figure 4: An example of an optimal controlled trajectory (a, left) and a blow-up near the final time (b, right)

The optimal solution is well-known [10] and a first proof of optimality was given by Wonham through an explicit construction of a solution to the Hamilton-Jacobi-Bellman equation taking advantage of symmetries the problem has [24]. We shall give a totally different proof of the optimality of this synthesis using the *method of characteristics*. Rather than pursuing an explicit formula for the solution as it is given in [24], we follow a geometric construction that defines this solution implicitly, but is generally applicable to chattering arcs. However, it is based on an analysis of the extremals for the problem and we briefly recall the construction of this synthesis below (but without proofs). The presentation follows ideas of Kupka who constructs this field using symmetries in the family of extremals.

We first recall more elementary properties of extremals. Let (x, u) be an optimal controlled trajectory that transfers p into the origin and minimizes the integral $\frac{1}{2}\int_0^T x_1^2(t)dt$. By the Pontryagin maximum principle there exist a constant $\lambda_0 \geq 0$ and an adjoint vector $\lambda = (\lambda_1, \lambda_2)$ such that (i) $(\lambda_0, \lambda_1, \lambda_2)$ do not vanish simultaneously, (ii) $\dot{\lambda}_1 = -\lambda_0 x_1$, $\dot{\lambda}_2 = -\lambda_1$, and (iii) the control minimizes the Hamiltonian $H = \frac{1}{2}\lambda_0 x_1^2 + \lambda_1 x_2 + \lambda_2 u$ over the interval $[-1, 1]$ with the minimum value identically zero. It is not difficult to see that extremals are normal and hence we set $\lambda_0 = 1$. Optimal controls are given by $u(t) = 1$ if $\lambda_2(t) < 0$ and by $u(t) = -1$ if $\lambda_2(t) > 0$. Defining the switching function Φ as $\Phi = \lambda_2$, we have $u(t) = -sgn\ \Phi(t)$ and the derivatives of Φ are given by

$$\dot{\Phi}(t) = -\lambda_1(t), \quad \ddot{\Phi}(t) = x_1(t), \quad \Phi^{(3)}(t) = x_2(t), \quad \Phi^{(4)}(t) = u(t)$$

Thus the switching function is a solution to the non-smooth differential equation $\Phi^{(4)}(t) = -sgn\ \Phi(t)$. Essentially, it is a consequence of this fact that switchings, the zeros of Φ, are isolated and must accumulate at the final time.

The family of all extremals possesses two groups of symmetries, one continuous, the other discrete, which very much can be used to advantage in calculating the synthesis. Without loss of generality, we define all extremals over the full interval $(-\infty, 0]$ with the terminal time T normalized to be $T = 0$. The next two propositions are verified by direct explicit computations. Let $\mathcal{G}_\alpha$ denote the multiplicative group of positive reals and define a 1-parameter group of scaling symmetries on $(-\infty, 0] \times [-1, 1] \times \mathbb{R}^2 \times \left(\mathbb{R}^2\right)^*$ by

$$\mathcal{G}_\alpha : \ (t, u, x_1, x_2, \lambda_1, \lambda_2) \mapsto (\frac{t}{\alpha}, \alpha^0 u, \alpha^2 x_1, \alpha x_2, \alpha^3 \lambda_1, \alpha^4 \lambda_2).$$

Proposition 1 *Given an extremal lift $\Gamma = ((x, u), \lambda)$ for the Fuller problem and $\alpha > 0$, define $\Gamma^\alpha = ((x^\alpha, u^\alpha), \lambda^\alpha)$ as the controlled trajectory (x^α, u^α) and corresponding adjoint vector λ^α that are obtained under the action of the group $\mathcal{G}_\alpha$ on the variables, that is by*

$$u^\alpha(t) = u\left(\frac{t}{\alpha}\right), \quad x_1^\alpha(t) = \alpha^2 x_1\left(\frac{t}{\alpha}\right), \quad x_2^\alpha(t) = \alpha x_2\left(\frac{t}{\alpha}\right),$$

and

$$\lambda_1^\alpha(t) = \alpha^3 \lambda_1\left(\frac{t}{\alpha}\right), \quad \lambda_2^\alpha(t) = \alpha^4 \lambda_2\left(\frac{t}{\alpha}\right).$$

Then Γ^α again is an extremal for the Fuller problem. ■

A second symmetry is given by reflecting controlled trajectories and their multipliers at the origin, in mathematical terms by the action of the discrete group S_2. Let $\mathcal{R}$ denote the reflection symmetry defined on $(-\infty, 0] \times [-1, 1] \times \mathbb{R}^2 \times \left(\mathbb{R}^2\right)^*$ by

$$\mathcal{R} : \ (t, u, x_1, x_2, \lambda_1, \lambda_2) \mapsto (t, -u, -x_1, -x_2, -\lambda_1, -\lambda_2).$$

Proposition 2 *Given an extremal lift $\Gamma = ((x, u), \lambda)$ for the Fuller problem, define $\check{\Gamma} = ((\check{x}, \check{u}), \check{\lambda})$ as the controlled trajectory $(\check{x}, \check{u})$ and corresponding adjoint vector $\check{\lambda}$ that are obtained under the action of $\mathcal{R}$, that is by $\check{u}(t) = -u(t)$, $\check{x}_1(t) = -x_1(t)$, $\check{x}_2(t) = -x_2(t)$, $\check{\lambda}_1(t) = -\lambda_1(t)$ and $\check{\lambda}_2(t) = -\lambda_2(t)$. Then $\check{\Gamma}$ again is an extremal for the Fuller problem.* ■

Whenever a mathematical problem exhibits symmetries, it is a good strategy to seek solutions that obey these symmetries. In fact, there is one extremal that is invariant under the action of all symmetries $\mathcal{G}_\alpha$ for all $\alpha > 0$ and $\mathcal{R}$, namely the trivial solution for $u \equiv 0$ with $x \equiv 0$ and $\lambda \equiv 0$. (The nontriviality condition is satisfied by $\lambda_0 = 1$.) In some sense, this is responsible for the special properties of trajectories that need to steer the system into the origin. But there also exists a specific value α for which *all* extremals are invariant (as individual curves, not just as the whole family) under the actions of $\mathcal{R}$ and $\mathcal{G}_\alpha$. These are the optimal controlled trajectories for the Fuller problem. Let $\Gamma = ((x, u), \lambda)$ be an extremal for the Fuller problem and suppose $t_0 < 0$ is a switching time where the control switches from $u = +1$ to $u = -1$. Since the switchings are isolated, but must accumulate for $T = 0$, there exists a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of switching times that converge to 0 as $n \to \infty$ and the control switches from $u = +1$ to $u = -1$ at even indices and from $u = -1$ to $u = +1$ at odd indices. Let $\check{\Gamma}_\alpha = ((\check{x}^\alpha, \check{u}^\alpha), \check{\lambda}^\alpha)$ denote the image of the extremal Γ under the combined action $\mathcal{A}_\alpha$ of the reflection $\mathcal{R}$ and the group $\mathcal{G}_\alpha$ for a fixed $\alpha > 0$, i.e., for all $t \leq 0$ we have that

$$\check{u}^\alpha(t) = -u\left(\frac{t}{\alpha}\right), \quad \check{x}_1^\alpha(t) = -\alpha^2 x_1\left(\frac{t}{\alpha}\right), \quad \check{x}_2^\alpha(t) = -\alpha x_2\left(\frac{t}{\alpha}\right),$$

and

$$\check{\lambda}_1^\alpha(t) = -\alpha^3\lambda_1\left(\frac{t}{\alpha}\right), \quad \check{\lambda}_2^\alpha(t) = -\alpha^4\lambda_2\left(\frac{t}{\alpha}\right).$$

By Propositions 1 and 2, $\check{\Gamma}_\alpha = \mathcal{A}_\alpha(\Gamma)$ again is an extremal, but generally it will be different from Γ. If $\Gamma(t) = \check{\Gamma}_\alpha(t)$ for all $t \leq 0$, i.e., if the extremal is a fixed point under this transformation, then we say it is invariant under this action. Note that if Γ is $\mathcal{A}_\alpha$-invariant, then it is also invariant under the action of any odd power α^{2k+1} for all $k \in \mathbb{Z}$. But there always exists a smallest $\alpha > 1$ and this number will be called the generator.

Proposition 3 [14] *let $\Gamma = ((x,u),\lambda)$ be an extremal for the Fuller problem defined over the semi-infinite interval $(-\infty, 0]$ with switching times $\{t_n\}_{n\in\mathbb{Z}}$ and suppose the control switches from $u = -1$ to $u = +1$ for even indices. If the extremal Γ is invariant under the combined action $\mathcal{A}_\alpha$ of the reflection $\mathcal{R}$ and the group $\mathcal{G}_\alpha$ with generator α, i.e., if $\Gamma(t) = \check{\Gamma}_\alpha(t)$ for all $t \leq 0$, then*

$$\alpha = \sqrt{\frac{1+2\zeta}{1-2\zeta}} \qquad \text{with} \qquad \zeta = \sqrt{\frac{\sqrt{33}-1}{24}}.$$

The switching points lie on the curves

$$\Gamma_+ = \{(x_1,x_2) \in \mathbb{R}^2 : \ x_1 = \zeta x_2^2, \ x_2 < 0\}$$

and

$$\Gamma_- = \{(x_1,x_2) \in \mathbb{R}^2 : \ x_1 = -\zeta x_2^2, \ x_2 > 0\}$$

and switchings are from $u = -1$ to $u = +1$ at points on Γ_+ and vice versa at points on Γ_-. ■

Corollary 2 *The synthesis $\mathcal{F}$ defined in Theorem 2 generates a family of $\mathcal{A}_\alpha$-invariant extremals.* ■

The synthesis $\mathcal{F}$ of extremal controlled trajectories is simple, but this does not prove its optimality. Note that this construction does not preclude the existence of other extremals. For the Fuller problem, indeed there are no others [1, 2]. But in general it is not possible to simply envoke existence results to prove the optimality of a synthesis that was constructed using special properties like the symmetries here. The synthesis for the Fuller problem does not satisfy the classical conditions of Boltyansky for a regular synthesis [3] since the controls are not piecewise continuous. Chattering clearly causes technical difficulties in any approach that aims at establishing the differentiability of the cost-to-go function by integrating the trajectories. However, while the structure of optimal controls is unpleasant in not being piecewise continuous, there are just *two* flows involved and all that really is needed is that these flows can be matched in a C^1 way. We now show that *this in fact is automatic* using a geometric argument that more generally is applicable to chattering syntheses.

The switching curves Γ_+ and Γ_- form the source and target for two real analytic families $\mathcal{E}_\pm$ of normal extremals given by the trajectories for the constant controls $u = \pm 1$. Parameterize the switching curves as

$$\Gamma_+ : P = (-\infty, 0) \to \mathbb{R}^2, \qquad p \mapsto \left(\zeta p^2, p\right)^T$$

and

$$\Gamma_- : P = (0, \infty) \to \mathbb{R}^2, \qquad p \mapsto \left(-\zeta p^2, p\right)^T$$

with $\zeta = \sqrt{\frac{\sqrt{33}-1}{24}}$. We again normalize the parameterizations so that all trajectories reach the origin at the final time $T = 0$. If a trajectory starts at the point $(\zeta p^2, p)^T \in \Gamma_+$ at time t_0, then it can be shown that that the next switching is at time $t_1 = \frac{t_0}{\alpha}$ and that $x_2(t_0) + \alpha x_2(t_1) = 0$. Starting at Γ_+, the control is given by $u \equiv +1$ on $[t_0, t_1] \subset (-\infty, 0)$ and we therefore also get that

$$x_2(t_1) - x_2(t_0) = t_1 - t_0 = \left(\frac{1}{\alpha} - 1\right) t_0.$$

Hence

$$t_0 = \frac{x_2(t_1) - x_2(t_0)}{\frac{1}{\alpha} - 1} = \frac{-\frac{1}{\alpha} - 1}{\frac{1}{\alpha} - 1} x_2(t_0) = \frac{\alpha + 1}{\alpha - 1} p.$$

We therefore define a family $\mathcal{E}_+$ of normal extremals that correspond to the control $u(t, p) \equiv +1$ over the domain

$$D_+ = \{(t, p) : \; p < 0, \; t_-(p) \leq t \leq t_+(p)\}$$

with the functions t_+ and t_- given by

$$t_-(p) = \frac{\alpha + 1}{\alpha - 1} p \qquad \text{and} \qquad t_+(p) = \frac{1}{\alpha} \cdot \frac{\alpha + 1}{\alpha - 1} p.$$

The corresponding controlled trajectories $x(t, p)$ start at the point $\xi_-(p) = \left(\zeta p^2, p\right)^T \in \Gamma_+$ at time $t_-(p)$ and then at time $t_+(p)$ reach the point $\xi_+(p) = \left(-\zeta \left(\frac{p}{\alpha}\right)^2, -\frac{p}{\alpha}\right)^T \in \Gamma_-$. An explicit integration of the trajectories is not required, but, of course, could easily be done here. The multipliers $\lambda(t,p)$ are the solutions to the initial value problems

$$\dot{\lambda}_1(t,p) = -x_1(t,p), \qquad \lambda_1(t_-(p),p) = -\frac{1}{2}\zeta^2 p^3$$

and

$$\dot{\lambda}_2(t,p) = -\lambda_1(t,p), \qquad \lambda_2(t_-(p),p) = 0.$$

As for the double integrator considered earlier, we have $\lambda_2(t_-(p),p) = 0$ since the initial condition lies on the switching curve Γ_+ and λ_2 again is the switching function of the problem. The value for $\lambda_1(t_-(p),p)$ follows from the condition that the Hamiltonian vanishes,

$$0 = H = \frac{1}{2}x_1^2 + \lambda_1 x_2 + \lambda_2 = \frac{1}{2}\zeta^2 p^4 + \lambda_1 p,$$

which gives $\lambda_1(t_-(p),p) = -\frac{1}{2}\zeta^2 p^3$. Also the values of the multipliers at the target directly follow from these relations.

It remains to define the cost γ_- at the source Γ_+ and this term is not obvious. An evaluation of the objective along the trajectory is laborious and this would be an argument that only can be done for this example, but has little general value. In fact, such a calculation is not necessary. The two flows of trajectories for $u = +1$ and $u = -1$ cross the switching surfaces transversally and this in fact implies that the combined value function defined by these flows will remain continuously differentiable along the switching surfaces. This follows from results in [21] where parameterized families of broken extremals are defined and where it is shown that the value of a parameterized family of broken extremals remains continuously differentiable at a switching curve Γ if the two respective flows of trajectories cross Γ transversally. Thus this simple and easily verified geometric property ensures differentiability of the value at switching surfaces. Anticipating this property, we can simply reverse the reasoning and define the value of the cost for the source γ_- on the switching curve Γ_+, $\gamma_-(p) = V(\zeta p^2, p)$, by integrating the differential dV of V along the curve

$$Z : [p, 0] \to \Gamma_+, \quad s \mapsto (\zeta s^2, s).$$

The point obviously is that *we know from the Shadow-Price lemma that this gradient is given by the multiplier* λ and that *we know the value of* λ *on the switching curves.* Therefore, and simply postulating that

$$\nabla V(\zeta s^2, s) = \left(-\frac{1}{2}\zeta^2 s^3, 0\right), \tag{9}$$

we calculate the cost at the source as

$$\begin{aligned}\gamma_-(p) &= V(\zeta p^2, p) - V(0,0) = -\int_Z dV \\ &= \int_0^p \frac{\partial V}{\partial x_1}(\zeta s^2, s)dx_1(s) + \frac{\partial V}{\partial x_2}(\zeta s^2, s)dx_2(s) \\ &= \int_0^p \left(-\frac{1}{2}\zeta^2 s^3 \cdot 2\zeta s + 0\right) ds = -\frac{1}{5}\zeta^3 p^5. \end{aligned} \tag{10}$$

With this specification, the transversality condition (3) is satisfied:

$$\lambda(t_-(p), p)\frac{\partial \xi_-}{\partial p}(p) = \left(-\frac{1}{2}\zeta^2 p^3, 0\right) \cdot \begin{pmatrix} 2\zeta p \\ 1 \end{pmatrix} = -\zeta^3 p^4 = \frac{\partial \gamma_-}{\partial p}(p).$$

Hence $\mathcal{E}_+$ is a real-analytic parameterized family of normal extremals for the Fuller problem. Furthermore, since the control is constant, it simply follows from the uniqueness of solutions to ordinary differential equations that the corresponding trajectories do not intersect and thus the corresponding flow map $\digamma : D_+ \to G_+ \cup \Gamma_+$, $(t,p) \mapsto x(t,p)$, is a diffeomorphism. Since the flow crosses the switching curves Γ_+ and Γ_- transversally, this even holds in an open neighborhood of these two switching curves. Note that the switching curves do not include the origin. Thus $\mathcal{E}_+$ is a real–analytic field of normal extremals. We want to stress that *all these parameterizations are by means of simple calculations that directly follow from the analysis of extremals.*

Analogously, a real-analytic field $\mathcal{E}_-$ of extremal trajectories is constructed that corresponds to the control $u(t,p) \equiv -1$ and starts on the switching curve Γ_-. In this case, the domain is described as $D_- = \{(t,p) : \ p > 0, \ t_-(p) \leq t \leq t_+(p)\}$ with the functions t_+ and t_- now given by

$$t_-(p) = \frac{1+\alpha}{1-\alpha}p \qquad \text{and} \qquad t_+(p) = \frac{1}{\alpha} \cdot \frac{1+\alpha}{1-\alpha}p.$$

The initial conditions for the trajectories $x(t,p)$ on Γ_- are $\xi_-(p) = \left(-\zeta p^2, p\right)^T$, for the multipliers we have $\lambda(t_-(p), p) = \left(-\frac{1}{2}\zeta^2 p^3, 0\right)$, and the cost γ_- at the source is $\gamma_-(p) = \frac{1}{5}\zeta^3 p^5$.

If we denote the flows induced by these two parameterized families of extremal trajectories by $\digamma_+$ and $\digamma_-$, respectively, then it follows from Theorem 1 that the values $V_+ = C \circ \digamma_+^{-1}$ and $V_- = C \circ \digamma_-^{-1}$ are continuously differentiable solutions to the Hamilton-Jacobi-Bellman equation for the Fuller problem on the images $G_+ = \digamma_+(D_+)$ and $G_- = \digamma_-(D_-)$, respectively. We now prove that these two functions V_+ and V_- combine to a continuous function on the switching curves Γ_+ and Γ_-. This is a direct consequence of the construction.

Proposition 4 *The values V_+ and V_- are continuous on the switching curves Γ_+ and Γ_-.*

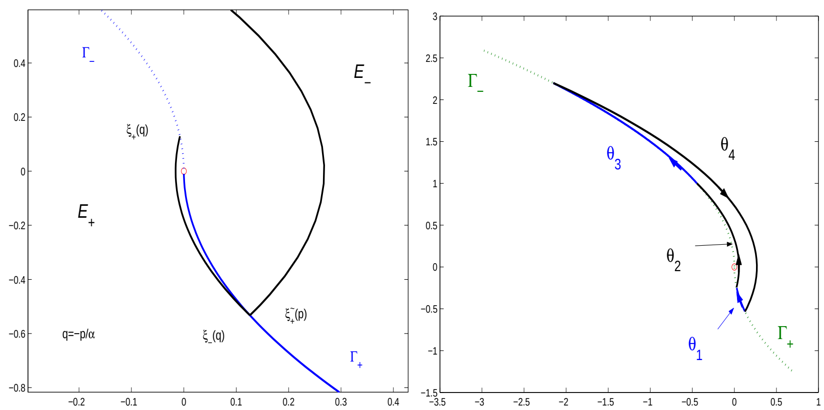

Figure 5: (a, left) The two parameterizations for a junction point on Γ_+ and (b,right) the closed curve Θ

Proof. We consider Γ_+ and will show that the cost γ_- at the source of the parameterized field $\mathcal{E}_+$ agrees with the cost $\tilde{\gamma}_+$ at the target of the parameterized field $\mathcal{E}_-$. (In this proof we use a tilde to distinguish the source and target costs for $\mathcal{E}_-$ from those of $\mathcal{E}_+$). In the calculations we need to take into account that the same trajectory is described by different parameters p in these two parameterizations (see Fig. 5(a)). We fix the parameter p in the field $\mathcal{E}_-$. Then the source cost for $\mathcal{E}_-$ on the switching curve Γ_- is given by the value of the objective with initial point $\tilde{\xi}_-(p) = \left(-\zeta p^2, p\right)^T \in \Gamma_-$, that is $\tilde{\gamma}_-(p) = \frac{1}{5}\zeta^3 p^5$. The target cost for $\mathcal{E}_-$ on Γ_+ is denoted by $\tilde{\gamma}_+(p)$ and this is the value of the objective when the initial point is given by the corresponding target

$$\tilde{\xi}_+(p) = \begin{pmatrix} \zeta \left(\frac{p}{\alpha}\right)^2 \\ -\frac{p}{\alpha} \end{pmatrix} \in \Gamma_+.$$

This point becomes the source for the second field $\mathcal{E}_+$, but now is parameterized in the form $\begin{pmatrix} \zeta q^2 \\ q \end{pmatrix} \in \Gamma_+$ for some $q < 0$ and has associated cost $\gamma_-(q) = -\frac{1}{5}\zeta^3 q^5$. The parameters therefore relate as $q = -\frac{p}{\alpha}$ and continuity at the junction is equivalent to

$$\tilde{\gamma}_+(p) = \gamma_-\left(-\frac{p}{\alpha}\right). \tag{11}$$

In principle, this relation can be verified by an explicit computation. But a more elegant and also more general argument can be made using the theory developed so far: We work with the field $\mathcal{E}_-$, but in order not to clutter the notation, will not use a subscript for the associated objects. Thus V denotes the value associated with the field $\mathcal{E}_-$, $V = C \circ F^{-1}$, and so on.

Since the corresponding parameterized trajectories $x(t,p)$ and thus also the parameterized cost $C(t,p)$ extend into an open neighborhood of the domain D, the associated value V extends as a continuously differentiable function into a simply connected open neighborhood $\tilde{G}$ of $G = \digamma(D)$. For small $\epsilon > 0$ consider the closed curve $\Theta = \Theta(\epsilon)$ that lies in $\tilde{G}$ and is obtained by concatenating the following four smooth curves, $\Theta = \theta_1 * \theta_2 * \theta_3 * \theta_4$, (see Fig. 5(b))

$$\begin{aligned}
\theta_1 :&\quad [-p,-\epsilon] \to \Gamma_+, && s \mapsto \theta_1(s) = \tilde{\xi}_+(-s),\\
\theta_2 :&\quad [-\tilde{t}_+(\epsilon), -\tilde{t}_-(\epsilon)] \to G_-, && s \mapsto \theta_2(s) = x(-s,\epsilon),\\
\theta_3 :&\quad [\epsilon,p] \to \Gamma_-, && s \mapsto \theta_3(s) = \tilde{\xi}_-(s),\\
\theta_4 :&\quad [\tilde{t}_-(p), \tilde{t}_+(p)] \to G_-, && s \mapsto \theta_4(s) = x(s,p).
\end{aligned}$$

The curve Θ is closed and lies in $\tilde{G}$. Hence we have that

$$0 = \int_\Theta dV = \int_{\theta_1} dV + \int_{\theta_2} dV + \int_{\theta_3} dV + \int_{\theta_4} dV.$$

The curve θ_4 is the trajectory corresponding to the parameter p and the curve θ_2 is the trajectory for the parameter ϵ, but run backwards. Hence we have that

$$\int_{\theta_4} dV = V(\tilde{\xi}_+(p)) - V(\tilde{\xi}_-(p)) = \tilde{\gamma}_+(p) - \tilde{\gamma}_-(p)$$

and

$$\int_{\theta_2} dV = V(\tilde{\xi}_-(\epsilon)) - V(\tilde{\xi}_+(\epsilon)) = \tilde{\gamma}_-(\epsilon) - \tilde{\gamma}_+(\epsilon) = \int_{\tilde{t}_-(\epsilon)}^{\tilde{t}_+(\epsilon)} \frac{1}{2} x_1(s,\epsilon)^2 ds.$$

As $\epsilon \to 0$, the length of the interval $[\tilde{t}_-(\epsilon), \tilde{t}_+(\epsilon)]$ converges to 0,

$$\tilde{t}_+(\epsilon) - \tilde{t}_-(\epsilon) = \left(\frac{1}{\alpha} - 1\right) \cdot \frac{1+\alpha}{1-\alpha}\epsilon = \frac{1+\alpha}{\alpha}\epsilon \to 0,$$

and $x_1(s,\epsilon)$ remains bounded on this interval as $\epsilon \to 0$. Hence $\int_{\theta_2} dV \to 0$ as $\epsilon \to 0$.

The limits of θ_1 and θ_3 as $\epsilon \to 0$ are well-defined and parameterize the sections on the switching curves Γ_+ and Γ_- that connect the origin with the points $\tilde{\xi}_+(p)$ and $\tilde{\xi}_-(p)$, respectively. It follows from Theorem 1 that the gradient of the function V associated with the family $\mathcal{E}_-$ is given by the multiplier λ of the parameterized family $\mathcal{E}_-$. Hence, and consistent with our original definition, it follows for the portion along Γ_- that

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{\theta_3} dV &= \int_0^p \nabla V(-\zeta s^2, s) d\tilde{\xi}_-(s)\\
&= \int_0^p \lambda_1(t_-(s),s) d\tilde{\xi}_{1,-}(s) + \lambda_2(t_-(s),s) d\tilde{\xi}_{2,-}(s) ds\\
&= \int_0^p \left(-\frac{1}{2}\zeta^2 s^3 \cdot (-2\zeta s) + 0\right) ds = \frac{1}{5}\zeta^3 p^5 = \tilde{\gamma}_-(p).
\end{aligned}$$

Analogously, the gradient of the function V along the switching curve Γ_+ is given by the multiplier λ at the terminal time $\tilde{t}_+(p)$. For $s \in [-p, -\epsilon]$ the terminal points in the target are given by $\tilde{\xi}_+(-s) = \left(\zeta\left(\frac{s}{\alpha}\right)^2, \frac{s}{\alpha}\right)^T \in \Gamma_+$ and from the properties of the synthesis we have that

$$\lambda_1(t_+(-s), -s) = -\frac{1}{2}\zeta^2 \tilde{\xi}_{2,+}(t_+(-s), -s)^3 = -\frac{1}{2}\zeta^2 \left(\frac{s}{\alpha}\right)^3$$

and $\lambda_2(t_+(-s), -s) = 0$. Hence

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{\theta_1} dV &= \int_{-p}^{0} \nabla V \left(\zeta\left(-\frac{s}{\alpha}\right)^2, \frac{s}{\alpha}\right) d\tilde{\xi}_+(-s) \\
&= \int_{-p}^{0} \lambda_1(t_+(-s), -s) d\tilde{\xi}_{1,+}(-s) + \lambda_2(t_+(-s), -s) d\tilde{\xi}_{2,+}(-s) \\
&= \int_{-p}^{0} \left(-\frac{1}{2}\zeta^2 \left(\frac{s}{\alpha}\right)^3 \cdot 2\zeta\left(\frac{s}{\alpha}\right)\frac{1}{\alpha} + 0\right) ds \\
&= -\frac{1}{5}\zeta^3 \left(\frac{s}{\alpha}\right)^5 \Bigg|_{-p}^{0} = -\frac{1}{5}\zeta^3 \left(\frac{p}{\alpha}\right)^5 = -\gamma_-\left(-\frac{p}{\alpha}\right).
\end{aligned}$$

Thus, overall we get that

$$\begin{aligned}
\tilde{\gamma}_+(p) = \tilde{\gamma}_-(p) + \int_{\theta_4} dV &= \lim_{\epsilon \to 0}\left(\int_{\theta_3} dV + \int_{\theta_4} dV\right) \\
&= -\lim_{\epsilon \to 0}\left(\int_{\theta_1} dV + \int_{\theta_2} dV\right) = \gamma_-\left(-\frac{p}{\alpha}\right).
\end{aligned}$$

Hence the values join to a continuous function along Γ_+. □

Thus the combined value function V,

$$V(x) = \begin{cases} V_+(x) & \text{if } x \in G_+ \cup \Gamma_+ \\ V_-(x) & \text{if } x \in G_- \cup \Gamma_- \end{cases}$$

is well-defined and is continuous on $\mathbb{R}^2$. In fact, this function is continuously differentiable away from the origin and, as stated earlier, this is an immediate consequence of the fact that the two flows corresponding to $\mathcal{E}_+$ and $\mathcal{E}_-$ cross the switching surfaces transversally. Hence the value V constructed above is a continuous function that is continuously differentiable away from the origin and solves the HJB equation. Analogously to Corollary 1 this proves the global optimality of the synthesis. Obviously, there is no need nor reason to actually calculate the value function V explicitly. *Having the synthesis of extremals, the method of characteristics immediately and with little further ado implies the optimality of the corresponding controlled trajectories.* While we have carried out this argument for the Fuller problem, in fact the reasoning is general and we expand upon this further in the next section.

4 Chattering in Syntheses for Mathematical Models for Tumor Anti-Angiogenesis

Tumor anti-angiogenesis is an indirect cancer treatment approach pioneered by J. Folkman that targets the vasculature of a growing tumor, that is, the blood vessels and capillaries that form the support system the tumor needs for its supply with nutrients and oxygen. It has the advantage that by targeting healthy cells, not the genetically unstable and continuously mutating cancer cells, it is not prone to drug resistance, the major obstacle in conventional chemotherapy. On the other hand, as an indirect approach it does not kill the cancer cells and therefore needs to be combined with other therapies like radio- or chemotherapy. Mathematically, however, because of the great complexity that the overall problem of cancer treatment is, it is prudent to start the analysis with simplified models and consider monotherapies first.

In [13] Hahnfeldt, Panigrahy, Folkman and Hlatky, a group of researchers then at Harvard Medical School, developed and biologically validated a low-dimensional, minimally parameterized, population based mathematical model for tumor anti-angiogenesis with the primary tumor volume, p, and the carrying capacity of the vasculature, q, as its principal state variables. The latter is defined as a measure for the tumor volume sustainable by the vascular network. The dynamics describes the interactions between these variables. In the paper [13], based on an asymptotic analysis of the consumption-diffusion equation that underlies angiogenesis, the following dynamics is proposed:

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right), \tag{12}$$

$$\dot{q} = bp - dp^{\frac{2}{3}}q - \mu q. \tag{13}$$

Equation (12) is a standard Gompertzian growth model on the tumor with ξ a tumor growth parameter and q the carrying capacity. In this model, q is taken as variable and its dynamics modelled in (13) consists of a balance of stimulatory and inhibitory effects. The stimulation asserted by the tumor is taken proportional to the tumor volume with b a constant, mnemonically labelled for 'birth.' The inhibition term consists of a term μq due to natural causes (cell death etc.).and a term $dp^{\frac{2}{3}}q$ that is due to tumor induced inhibition. Here d again is a constant representing 'death' and the exponent $\frac{2}{3}$ arises since in this interaction the tumor volume is scaled down to the tumor surface.

Anti-angiogenic treatments bring in external agents that reduce the vasculature. These are in limited supply and still very expensive. It is therefore a natural question how to best apply a given amount of anti-angiogenic agents in order to minimize the tumor volume. Using a standard linear pharmacokinetic model to describe the drug concentrations (a simple model of exponential growth and decay), this leads to the following version of the general optimal control problem [OC]:

(OC) for a free terminal time T, minimize $J(u) = p(T)$ over all Lebesgue measurable functions $u : [0, T] \to [0, a]$ subject to

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right), \qquad p(0) = p_0 \tag{14}$$

$$\dot{q} = bp - \left(\mu + dp^{\frac{2}{3}} + \gamma c\right) q, \qquad q(0) = q_0 \tag{15}$$

$$\dot{c} = -kc + u, \qquad c(0) = 0 \tag{16}$$

$$\dot{y} = u, \qquad y(0) = 0 \tag{17}$$

and terminal condition $y(T) \leq A$.

The control u is given by the dosage of the anti-angiogenic agent and the upper limit in the control set a denotes a fixed upper limit. Equation (16) models the agent's concentration in the plasma and the linear term γcq describes its effect on the vasculature with γ a constant that represents the anti-angiogenic killing parameter. The last equation (17) models an isoperimetric constraint and simply keeps track of the amount of anti-angiogenic inhibitors that are being used with A the total available amount.

In [16] we have given a full solution in the form of a regular synthesis of optimal controlled trajectories for a simplified version of this problem when the linear pharmacokinetic model (16) is dropped. An optimal singular arc is the center piece of this synthesis and the corresponding singular control is of order 1 [19, 27]. Consequently optimal controls typically take the forms **as0**, i.e., are concatenations of a full dose segment, followed by a singular piece and ending with a trajectory when no inhibitors are given, but when the tumor volume still decreases due to after effects. If a linear pharmacokinetic model is added, the optimal singular arc is preserved, but the order of the singular control increases to 2 [17] and the transitions onto the singular arc are made by means of chattering controls.

Geometrically, based on the above construction for the Fuller problem and results from [27], the following picture emerges: there exists a locally optimal singular arc $\mathcal{S}$ in (p, q, c)-space that can be determined explicitly through Lie-derivative based computations [18]. For initial conditions that are close enough to $\mathcal{S}$ (essentially, the overall amount A of inhibitors is large enough) and a given point z on the singular arc, optimal trajectories that connect with $\mathcal{S}$ in the point $z \in \mathcal{S}$, lie on a 2-dimensional stratified "surface" Θ_z that is "transversal" to the singular arc $\mathcal{S}$ and is defined in two pieces by the flows corresponding to the constant controls $u = 0$ and $u = a$ which are patched together along switching curves Γ_+^z and Γ_-^z. Optimal controls chatter in onto the singular arc along the stratified surfaces Θ_z and trajectories cross the switching surfaces transversally. Using the same argument as above, it can be shown that the restriction of the value function defined by this family of trajectories to Θ_z remains continuously differentiable at the switching surfaces. Since the singular arc is transversal to the family of surfaces $\{\Theta_z\}_{z \in S}$, it then follows that this value function is continuous on a small tubular neighborhood of the singular arc and continuously differentiable away from this singular arc. This, in connection with Corollary 1 and a definition of the cost on the singular arc, can be used to show the local optimality of these trajectories.

Globally, the synthesis becomes more complicated since optimal trajectories both chatter in and out of this singular arc. These structures are determined by the extra variable y which measures the amount of inhibitors that is left at every point. Thus this becomes a 4-dimensional problem and the singular arc S really is a hypersurface in (p, q, c, y)-space. For example, if the amount A of inhibitors is insufficient to first reach the singular arc and then to reach the diagonal $p = q$ (where the minimum tumor volumes are realized), then optimal strategies switch from chattering "in" to chattering "out" before S is reached and this generates bang-bang trajectories with a finite, but arbitrarily large number of switchings. Overall, the global synthesis in $\mathbb{R}^4$ is quite complicated and has not yet been worked out completely. But its structure can easily be conjectured based on the geometry of the chattering controls.

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