Multi-constrained path problems: an algebraic approach

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Abstract

Classical approaches to routing problems invariably require construction of trees and the use of heuristics to prevent combinatorial explosion. The operator calculus approach presented herein, however, allows such explicit tree constructions to be avoided. Introduced here is the notion of generalized zeon algebras and their associated operator calculus. The inherent combinatorial properties of generalized zeons make them useful for routing problems by implicitly pruning the underlying tree structures. Moreover, through the use of generalized idempotent algebras, max-min operators can be implemented for non-additive weights. As an application, an operator calculus approach to multi-constrained path problems is described.

Keywords: shortest paths; message routing; operator calculus; semigroup algebras.

1 Introduction

Operator calculus (OC) methods on graphs have been developed in a number of earlier works by Schott and Staples [10, 11, 12, 13]. The principal idea underlying the approach is the association of graphs with algebraic structures whose properties reveal information about the associated graphs. By constructing the “nilpotent adjacency matrix” associated with a finite graph, information about self-avoiding structures (paths, cycles, trails, etc.) in the graph are revealed by computing powers of the matrix.

In the operator calculus approach, graded semigroup algebras are generated by “null-square” elements such that properties of the algebra “sieve out” paths. In other words, cycles are removed from consideration automatically.

Consider a directed graph \( G = (V, E) \) on \( n \) vertices such that associated with each edge \((v_i, v_j) \in E\) is a vector weight \(\mathbf{w}_{ij} = (w_{ij1}, \ldots, w_{ijm}) \in \mathbb{R}^m\).
The point \( w^* = (w^*_1, \ldots, w^*_m) \in X \subset \mathbb{R}^m \) is referred to as a *Pareto minimum* of \( X \) if there does not exist \( w = (w_1, \ldots, w_m) \in X \) such that

\[
(\forall i) \ [w_i \leq w^*_i], \text{ and } \\
(\exists j) \ [w_j < w^*_j].
\]

Equivalently, one says that \( w^* \) is *nondominated from below*.

Defining the weight of a path in an edge-weighted graph as the sum of vector weights of arcs contained in the path, a *Pareto path* is then a path whose weight is a Pareto minimum.

For the case \( m = 1 \), Dijkstra’s algorithm finds all single source minimum paths in a directed graph on \( n \) vertices with nonnegative edge weights in \( O(n^2) \) time [4]. The Bellman-Ford algorithm finds single source minimal paths in digraphs with arbitrary edge weights and runs in \( O(n |E|) \) time [1, 7].

In the more general case \( m \geq 1 \), Corley and Moon [2] presented an algorithm for finding all Pareto paths with computational complexity \( O(mn^{2n^3} + mn^v) \).

The aim of the current work is to find Pareto paths satisfying multiple constraints. Given a vector \( c = (c_1, \ldots, c_m) \in \mathbb{R}^m \), a path is deemed *feasible* if its vector weight \( w = (w_1, \ldots, w_m) \) satisfies

\[
(\forall i) \ [w_i \leq c_i].
\]

Letting \( P_f \) denote the collection of feasible paths having fixed source \( v_0 \) and fixed target \( v_\infty \), the goal is to find a path in \( P_f \) whose weight is a Pareto minimum. The operator calculus approach described herein can be applied to sieve out the collection of feasible paths and recover all single-source Pareto paths remaining.

# Theory: Generalized zeon algebras

Zeon algebras are commutative algebras generated by collections of null-squares, \( \{\zeta_i : 1 \leq i \leq n\} \) with \( \zeta_i^2 = 0 \) for each \( i \). Their combinatorial properties make them useful for a variety of counting properties, as seen in a number of previous works by the current authors ([10], [11], [13]). By choosing sufficiently large sets of generators, they can be generalized to algebras whose generators are nilpotent of arbitrary index. The resulting *generalized zeon algebras* are suitable for a number of combinatorial applications, including multi-constrained routing problems.

Combinatorial properties of zeons have also been utilized by the authors to develop a “zeon-Berezin” operator calculus having applications in free probability theory [9]. Other zeon-related work includes the papers of Feinsilver [5] and Feinsilver and McSorley [6].

**Definition 2.1.** The *n-particle zeon algebra*, denoted by \( \mathcal{C}l_n^{\text{nil}} \), is defined as the real abelian algebra generated by the collection \( \{\zeta_i\} \ (1 \leq i \leq n) \) along with
the scalar $1 = \zeta_0$ subject to the following multiplication rules:

$$
\zeta_i \zeta_j = \zeta_j \zeta_i \quad \text{for } i \neq j, \quad \text{and} \quad (2.1)
$$

$$
\zeta_i^2 = 0 \quad \text{for } 1 \leq i \leq n. \quad (2.2)
$$

A general element $u \in \mathcal{C}_n^\text{nil}$ can be expanded as

$$
u = \sum_{I \in 2^n} u_I \zeta_I, \quad (2.3)
$$

where $I \in 2^n$ is a subset of $[n] = \{1, 2, \ldots, n\}$ used as a multi-index, $u_I \in \mathbb{R}$, and $\zeta_I = \prod_{i \in I} \zeta_i$.

**Remark 2.2.** The zeon algebra $\mathcal{C}_n^\text{nil}$ can be realized as a commutative subalgebra of the Grassmann algebra $\bigwedge V$ over a $2n$-dimensional vector space $V$ with orthonormal basis $\{\gamma_i\}$ by defining $\zeta_i = \gamma_i \gamma_{n+i}$ for each $1 \leq i \leq n$.

A canonical basis element $\zeta_I$ is referred to as a **blade**. The number of elements in the multi-index $I$ is referred to as the **grade** of the blade $\zeta_I$.

The next lemma shows that it is possible to construct elements with arbitrary index of nilpotency within a zeon algebra of sufficiently high dimension.

**Lemma 2.3.** Let $\{\zeta_i : 1 \leq i \leq n\}$ be the null-square generators of $\mathcal{C}_n^\text{nil}$. Then, for any permutation $\sigma \in S_n$ and positive integers $\ell \leq k \leq n$,

$$
\left( \sum_{j=1}^{k} \zeta_{\sigma(j)} \right)^{\ell} = \ell! \sum_{I \subseteq \{\sigma(1), \ldots, \sigma(k)\}} |I| = \ell \zeta_I. \quad (2.4)
$$

Moreover, if $\ell > k$, then

$$
\left( \sum_{j=1}^{k} \zeta_{\sigma(j)} \right)^{\ell} = 0. \quad (2.5)
$$

**Proof.** Since the generators commute, the multinomial theorem applies, with only square-free terms surviving.

**Definition 2.4.** For positive integer $n$, let $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ be an $n$-tuple of positive integers. Then, the zeon algebra of signature $s$ (or $s$-zeon algebra), denoted $\mathcal{C}_n^s$, is the real abelian algebra generated by the collection $\{\nu_i\}$ ($1 \leq i \leq n$) along with the scalar $1 = \nu_0$ subject to the following multiplication rules:

$$
\nu_i \nu_j = \nu_j \nu_i \quad \text{for } i \neq j, \quad \text{and} \quad (2.6)
$$

$$
\nu_i^{s_i} = 0 \quad \text{for } 1 \leq i \leq n. \quad (2.7)
$$

For convenience, the following **multi-exponent notation** is adopted:

$$
\nu^s : = \prod_{i=1}^{n} \nu_i^{s_i} := \nu_1^{s_1} \cdots \nu_n^{s_n}. \quad (2.8)
$$
Letting $\mathcal{S} = \{(x_1, \ldots, x_n) : 0 \leq x_i \leq s_i \} \subset \mathbb{N}_0^n$, a general element $u \in \mathcal{C}_s^{\text{nil}}$ can be expanded as

$$u = \sum_{\mathbf{x} \in \mathcal{S}} u_{\mathbf{x}} \nu^\mathbf{x},$$

(2.9)

where $u_{\mathbf{x}} \in \mathbb{R}$ for each multi-exponent $\mathbf{x}$.

Since the components of signature vectors and arbitrary multi-exponents are nonnegative integers, the 1-norm of such a vector $\mathbf{x}$ is simply the sum of the components; that is,

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n x_i.$$

(2.10)

**Lemma 2.5.** The algebra $\mathcal{C}_s^{\text{nil}}$ is isomorphic to a subalgebra of the zeon algebra $\mathcal{C}_{\|s\|_1}^{\text{nil}}$.

**Proof.** For $k \in \{1, \ldots, n\}$, let $p(k)$ denote the $k$th partial sum

$$p(k) = \sum_{j=1}^k s_j,$$

(2.11)

and define $p(0) = 0$. In light of Lemma 2.3, the desired isomorphism $\mathcal{C}_s^{\text{nil}} \rightarrow \mathcal{C}_{\|s\|_1}^{\text{nil}}$ is obtained from the mapping

$$\nu_i \mapsto \sum_{j=1}^{s_i} \zeta_{j+p(i-1)}.$$

(2.12)

Finally, note that the $s$-zeon algebra is naturally graded according to

$$\mathcal{C}_s^{\text{nil}} = \bigoplus_{k=0}^{\|s\|_1} \left\langle \mathcal{C}_s^{\text{nil}} \right\rangle_k,$$

(2.13)

where the grade-$k$ part of the algebra is defined by

$$\left\langle \mathcal{C}_s^{\text{nil}} \right\rangle_k \text{ span} (\{\nu^\mathbf{x} : \|\mathbf{x}\|_1 = k\}).$$

(2.14)

The notation $\langle \cdot \rangle_k$ extends naturally to elements of $\mathcal{C}_s^{\text{nil}}$.

**3 Operator calculus in generalized zeon algebras**

The motivation for development of $s$-zeon operator calculus is based on polynomial operator calculus. To begin, raising and lowering operators are defined
naturally in terms of polynomial differentiation and integration operators on basis zeons regarded as polynomials in commuting variables. In this formulation, the generators \( \{\nu_i\} \) of \( \mathcal{C}_s^{\text{nil}} \) are regarded as variables in the polynomial sense.

For any generalized zeon algebra with \( n \) generators, let \( \{e_i : 1 \leq i \leq n\} \) denote standard unit vectors of the form \( e_i := (0, \ldots, 1_{i^{\text{th}} \text{ pos.}}, \ldots, 0) \). Arbitrary multi-exponents are then expressed in the form \( x = (x_1, \ldots, x_n) = \sum_{i=1}^n x_i e_i \).

**Definition 3.1.** Let \( s \in \mathbb{N}_0^n \) be an arbitrary zeon signature. For \( 1 \leq j \leq n \), define the \( j \text{-th \( s \)-zeon differentiation operator} \) \( \partial / \partial \nu_j \) on \( \mathcal{C}_s^{\text{nil}} \) by linear extension of

\[
\frac{\partial}{\partial \nu_j} \nu^x = \begin{cases} 
\nu^{x-e_j} & \text{if } x_j \geq 1, \\
0 & \text{otherwise}. 
\end{cases}
\]  

(3.1)

**Definition 3.2.** The \( s \)-zeon integrals are defined by

\[
\int \nu^x d\nu_j = \begin{cases} 
\nu^{x+e_j} & \text{if } x_j < s_j - 1, \\
0 & \text{otherwise}. 
\end{cases}
\]  

(3.2)

These polynomial operators induce combinatorial raising and lowering operators by which \( s \)-zeon monomials (blades) are raised from grade \( k \) to grade \( k+1 \) or lowered from grade \( k \) to grade \( k-1 \). These raising and lowering operators can also be regarded as creation and annihilation operators in the sense of quantum mechanics.

**Definition 3.3.** For each \( 1 \leq j \leq n \), define the \( j \text{-th raising operator} \) \( R_j \) by linear extension of

\[
R_j \nu^x = \int \nu^x d\nu_j = \nu^x \nu_j.
\]  

(3.3)

Define the \( j \text{-th lowering operator} \) \( D_j \) by linear extension of

\[
D_j \nu^x = \frac{\partial}{\partial \nu_j} \nu^x.
\]  

(3.4)

**Definition 3.4.** The \( j \text{-th zeon number operator} \) \( \Lambda_j \) is defined on the generalized zeon algebra \( \mathcal{C}_s^{\text{nil}} \) by linear extension of

\[
\Lambda_j(\nu^x) := x_j \nu^x.
\]  

(3.5)

In particular, for arbitrary multi-exponents \( x, y \) and scalars \( \alpha, \beta \),

\[
\Lambda_j(\alpha \nu^x + \beta \nu^y) = \alpha x_j \nu^x + \beta \|y\|_1 \nu^y.
\]  

(3.6)

**Definition 3.5.** The \( dual \) of the \( j \text{-th zeon number operator} \), denoted \( \Lambda_j^* \), is defined on the generalized zeon algebra \( \mathcal{C}_s^{\text{nil}} \) by linear extension of

\[
\Lambda_j^*(\nu^x) := \begin{cases} 
(1/x_j) \nu^x & \text{if } x_j > 0, \\
0 & \text{otherwise}. 
\end{cases}
\]  

(3.7)
For arbitrary multi-exponents $x, y$ and scalars $\alpha, \beta$,
\[
\Lambda^*_j (\alpha x + \beta y) = \frac{\alpha}{x_j} x^j + \frac{\beta}{y_j} y^j.
\] (3.8)

An element $u \in \mathcal{C}_s^{\text{nil}}$ is said to be scalar-free if its canonical expansion is of the form
\[
\sum_{x \neq 0} u_x x^x.
\] (3.9)

Let $\mathcal{C}_s^{\text{nil}*}$ denote the scalar-free subalgebra of $\mathcal{C}_s^{\text{nil}}$; that is,
\[
\mathcal{C}_s^{\text{nil}*} := \{ u \in \mathcal{C}_s^{\text{nil}} : u \text{ is scalar free} \}. \quad (3.10)
\]

The zeon occupancy operator $\Lambda$ and its dual $\Lambda^*$ are $\Lambda = \bigoplus_{j=1}^n \Lambda_j$ and $\Lambda^* = \bigoplus_{j=1}^n \Lambda_j^*$, respectively.

**Lemma 3.6.** Letting $\mathcal{C}_s^{\text{nil}*}$ denote the scalar-free subspace of $\mathcal{C}_s^{\text{nil}}$,
\[
\Lambda \Lambda^* \big|_{\mathcal{C}_s^{\text{nil}*}} = \Lambda^* \Lambda \big|_{\mathcal{C}_s^{\text{nil}*}} = I.
\] (3.11)

More specifically, $\Lambda \Lambda^* : \mathcal{C}_s^{\text{nil}} \to \mathcal{C}_s^{\text{nil}*}$ is an orthogonal projection.

**Proof.** Since the components of multi-exponents are nonnegative integers, writing $x = (x_1, \ldots, x_n)$ leads to the component sum as the 1-norm; i.e., $\|x\|_1 = \sum_{j=1}^n x_j$. Let $u = \sum_{x \neq 0} u_x x^x \in \mathcal{C}_s^{\text{nil}}$ and consider
\[
\Lambda (\Lambda^* u) = \Lambda \left( \sum_{x \neq 0} \frac{u_x}{\|x\|_1} x^x \right) = \sum_{x \neq 0} \frac{u_x \|x\|_1}{\|x\|_1} x^x = u.
\] (3.12)

A nearly identical argument shows $\Lambda^* (\Lambda u) = u$. From the definitions of $\Lambda$ and $\Lambda^*$, it is apparent that for nonzero scalar $\alpha$, $\Lambda (\alpha x^x) = 0$ if and only if $x = 0$; i.e., $\alpha x^x = 0$. The same can be said of $\Lambda^*$; i.e., $\ker \Lambda = \ker \Lambda^* = \mathbb{R}$. As a result, $\ker \Lambda \Lambda^* = \ker \Lambda^* \Lambda = \mathbb{R}$.

Given a $c$-dimensional constraint vector $s$, a total ordering is induced on the set of multi-exponents $x \in \mathbb{N}_0^m$ by defining $x \preceq y$ if and only if $\exists k \geq 1$ such that $x_i \leq y_i \forall i \leq k$. When any such total ordering is assigned to the multi-exponents, one is able to define minimal elements of $\mathcal{C}_s^{\text{nil}}$. This will be useful in subsequent applications in which minimal elements will be associated with optimal solutions.
Definition 3.7. Fixing a total ordering \( \preceq \) of the multi-exponents, define a minimal term of \( u \in \mathcal{C}_\ell^{\text{nil}} \) by

\[
\tilde{u} := u_{x'} x',
\]

(3.13)

where \( x' \preceq x \) for all nonzero multi-exponents in the canonical expansion of \( u \).

The \( s \)-zeon algebra will be applied in later sections to sieve out paths satisfying multiple constraints. In order to retain identifying information about the paths themselves, another generalization of zeon algebras is considered.

3.1 The path algebra \( \mathbb{R} \Omega_n \)

For fixed positive integer \( n \), consider the alphabet \( \Sigma_n := \{ \omega_i : 1 \leq i \leq n \} \). For convenience, we adopt the following ordered multi-index notation. In particular, letting \( u = (u_1, \ldots, u_k) \) for some \( k \), the notation \( \omega_u \) will be used to denote a sequence (or word) of distinct symbols of the form

\[
\omega_u := \omega_{u_1} \omega_{u_2} \cdots \omega_{u_k}.
\]

(3.14)

Appending 0 to the set \( \Sigma_n \), multiplication is defined on the words constructed from elements of \( \Sigma_n \) by

\[
\omega_u \omega_v = \begin{cases} 
\omega_u \cdot v & \text{if } u \cap v = \emptyset, \\
0 & \text{otherwise,}
\end{cases}
\]

(3.15)

where \( u \cdot v \) denotes sequence concatenation.

One thereby obtains the noncommutative semigroup \( \Omega_n \), whose elements are the symbol 0 along with all finite words on distinct generators (i.e., finite sequences of distinct symbols from the alphabet \( \Sigma_n \)). Since there are only \( n \) generators, it is clear that the maximum multi-index size of semigroup elements is \( n \). Moreover, these symbols can appear in any order so that the order of the semigroup is

\[
\sum_{k=0}^{n} \binom{n}{k} k! = \sum_{k=0}^{n} (n)_k.
\]

Defining (vector) addition and real scalar multiplication on the semigroup yields the semigroup algebra \( \mathbb{R} \Omega_n \) of dimension \( |\Omega_n| \). This semigroup algebra will be referred to as a path algebra.

Consider the collection of ordered pairs \( P = \{ (\omega_i, \omega_j) : \Omega_n \times \Omega_n : i \neq j \} \), and note that \( |P| = n^2 - n \). Imposing an ordering on \( P \), a bijection \( f : P \to [n^2 - n] \) is obtained. Any \( k \)-subset of \( [n^2 - n] \) thereby determines a unique finite word of \( \Omega_n \):

\[
\omega_u := \omega_{u_1} \cdots \omega_{u_{k+1}} \leftrightarrow \{ f((u_1, u_2)), \ldots, f((u_k, u_{k+1})) \}.
\]

(3.16)

In this way, one obtains a semigroup homomorphism \( \phi \) from the canonical basis blades of \( \mathcal{C}_\ell^{\text{nil}} \) onto the words of length two or more in \( \Omega_n \). In this sense, the semigroup algebra \( \mathbb{R} \Omega_n \) can be regarded as an extension of a zeon algebra.
Remark 3.8. When the pairs of $P$ are unordered, each $k$-subset of $[n(n-1)/2]$ determines two finite words of $\Omega_n$: $\omega_u$, and its reversion $\overline{\omega}_u$; i.e.,
\[
\omega_u = \omega_{u_1} \cdots \omega_{u_{k+1}} \leftrightarrow \{f((u_1, u_2)), \ldots, f((u_k, u_{k+1}))\} \leftrightarrow \overline{\omega}_{u_{k+1}} \cdots \overline{\omega}_{u_1} = \overline{\omega}_u.
\] (3.17)

3.2 Operator calculus on graphs

As discussed in previous work, the nilpotent adjacency matrix of a finite graph can be used to sieve out the graph’s paths and cycles. The entries of this matrix are the null-square generators of a zeon algebra of appropriate dimension for the graph. The null-square properties of the algebra naturally remove entries corresponding to self-intersecting walks from powers of the nilpotent adjacency matrix.

Of interest in the current work is a method of sieving out paths with multi-dimensional weights (or costs) simultaneously satisfying a number of constraints. Generalized s-zeon generators will be associated with the graph’s edges in such a way that paths whose weights exceed the constraints are zeroed out by the algebra’s nilpotent properties.

In particular, extending this matrix construction to $\mathcal{C}^{\text{nil}}_s \otimes \mathbb{R}\Omega_n$ allows one to enumerate (list) all paths and cycles satisfying multiple constraints in a finite graph by considering powers of the matrix. The associated tree structure underlying the cycle/path enumeration problem is automatically “pruned” by the inherent properties of the algebra.

The first step is defining a nilpotent adjacency matrix that preserves path-identifying information.

Definition 3.9. Let $G = (V, E)$ be a graph on $n$ vertices, either simple or directed with no multiple edges. Let $\{\omega_i\}$, $1 \leq i \leq n$ denote the null-square, noncommutative generators of $\mathbb{R}\Omega_n$. Define the path-identifying nilpotent adjacency matrix $A$ associated with $G$ as the $n \times n$ matrix
\[
A_{ij} = \begin{cases} 
\omega_j & \text{if } (v_i, v_j) \in E, \\
0 & \text{otherwise.}
\end{cases}
\] (3.18)

Recalling Dirac notation, the $i$th row of $A$ is conveniently denoted by $\langle v_i | A$ while the $j$th column is denoted by $A | v_j \rangle$. In this way, $A$ is completely determined by
\[
\langle v_i | A | v_j \rangle = \begin{cases} 
\omega_j & \text{if there exists a directed edge } v_i \to v_j \text{ in } G, \\
0 & \text{otherwise,}
\end{cases}
\] (3.19)
for all vertex pairs $(v_i, v_j) \in E$.

Theorem 3.10. Let $A$ be the path-identifying nilpotent adjacency matrix of an $n$-vertex graph $G$. For any $k > 1$ and $1 \leq i \neq j \leq n$,
\[
\omega_i \langle v_i | A^k | v_j \rangle = \sum_{k \text{-paths } w: v_i \to v_j} \omega_w.
\] (3.20)
Moreover,
\[ \langle v_i | A^k | v_j \rangle = \sum_{k\text{-cycles } \varwedge \text{ based at } v_i} \omega_{\varwedge}. \]  
(3.21)

More specifically, when \( i \neq j \), the product of \( \omega_i \) with the entry in row \( i \), column \( j \) of \( A^k \) is a sum of basis blades indexed by \( k \)-step paths \( v_i \rightarrow v_j \) in \( G \). Moreover, entries along the main diagonal of \( A^k \) are sums of basis blades indexed by the graph’s \( k \)-cycles.

**Proof.** The result follows from straightforward mathematical induction on \( k \) using properties of the multiplication in \( R\Omega_n \) with the observation that the initial vertex of the walk, \( v_i \), is unaccounted for in \( \langle v_i | A^k | v_j \rangle \), as seen in (3.18) of the matrix definition. Hence, each term of \( \langle v_i | A^k | v_j \rangle \) is indexed by the vertex sequence of a \( k \)-walk from \( v_i \) to \( v_j \) with no repeated vertices, except possibly \( v_i \) at some intermediate step. Left multiplication by \( \omega_i \) thus sieves out the \( k \)-paths.

Considering entries along the main diagonal of \( A^k \), note that the final step of a \( k \)-cycle based at \( v_i \) returns to \( v_i \) so that left multiplication by \( \omega_i \) is not required for cycle enumeration.

\[ \chi((v_i, v_j)) \in E = \begin{cases} 1 & \text{if } (v_\ell, v_j) \in E, \\ 0 & \text{otherwise}. \end{cases} \]  
(3.22)

Letting \( \lambda \) denote the number of multiplications involved in this computation,
\[ \lambda = \sum_{\ell=1}^n \#\{k\text{-paths } v_i \rightarrow v_\ell \} \chi((v_i, v_j)) \in E \leq |\{k\text{-paths with source } v_i\}|. \]  
(3.23)

It follows immediately that the number of multiplications performed in determining \( \omega_i \langle v_i | A^k | v_j \rangle \) is bounded above by the number of paths of length at most \( k - 1 \) having initial vertex \( v_i \). Hence, the next corollary is obtained.

**Corollary 3.11.** Given a fixed pair of vertices \( v_0 \) and \( v_\infty \), the complexity of enumerating all \( k \)-paths from \( v_0 \) to \( v_\infty \) with the path-identifying nilpotent adjacency matrix is
\[ O(n |\{\text{paths of length } k - 1 \text{ or less having initial vertex } v_0\}|). \]

**Remark 3.12.** The computational complexity stated above is in terms of basis blade multiplications performed within the algebra. Recall that for disjoint ordered multi-indices \( p, q \), the product \( \omega_p \omega_q = \omega_{p+q} \) is given by sequence concatenation. Hence, some additional polynomial cost is associated with the implementation of the algebra multiplication.
Theorem 3.10 underlies Algorithm 1, which enumerates all hop-minimal paths from $v_0 \rightarrow v_\infty$ in a graph on $n$ vertices having a path-identifying nilpotent adjacency matrix $\Psi$.

Algorithm 1 HopMin[$\Psi$, $v_0$, $v_\infty$]

Require: $\Psi$, $v_0$, $v_\infty$

Ensure: Enumerate all hop-minimal paths from $v_0$ to $v_\infty$

\[ \langle \xi \rangle = \omega_{\{v_0\}} \langle v_0 \rangle \Psi \]

while $\langle \xi \rangle \langle v_\infty \rangle = 0 \land \langle \xi \rangle \neq \langle 0 \rangle$ do

\[ \langle \xi \rangle = \langle \xi \rangle \Psi \]

end while

return $[\langle \xi \rangle \langle v_\infty \rangle]$ 

Example 3.13. As a practical example of the OC approach, we consider the problem of finding hop-minimal paths between a randomly chosen pair of vertices in a randomly generated graph. Here, hop-minimal paths are paths requiring the fewest hops possible. The examples were constructed using Mathematica 8 on a MacBook Pro with a 2.4GHz Intel Core i7 processor and 8 GB of 1333MHz DDR3 memory.

A randomly-generated 120-vertex graph is seen in Figure 3.1. We seek hop-minimal paths from vertex $v_{28}$ to vertex $v_{64}$. For comparison, the ShortestPath function from the Combinatorica\textsuperscript{1} package of Mathematica is first used to locate a shortest path from $v_{28}$ to $v_{64}$. Then, the OC approach of Algorithm 1 is used.

As seen in the Mathematica output appearing in Figure 3.2, ShortestPath successfully locates a two-hop path in roughly 0.075 seconds. On the other hand, the OC approach reveals that there are actually nine two-hop paths in the graph and returns all nine of them in 0.091 seconds.
Paths from 28 to 64
Path: $\{28, 116, 64\}$ found by Combinatorica in 0.075319 seconds.

Paths computed using OC algorithm:
$\hat{w}(28, 1, 64) + \hat{w}(28, 116, 64) + \hat{w}(28, 177, 64) + \hat{w}(28, 28, 64) + \hat{w}(28, 52, 64) + \hat{w}(28, 78, 64) + \hat{w}(28, 109, 64) + \hat{w}(28, 114, 64)$
Computed in 0.091084 seconds.

Figure 3.2: Hop minimal paths from $v_{28}$ to $v_{64}$ in the graph of Figure 3.1.

The process was then repeated using a randomly generated 1000-vertex graph in which each pair of vertices had adjacency probability $p = 0.25$. In this example, the goal was to locate hop-minimal paths from vertex $v_1$ to $v_{1000}$. Because of the density of the graph, its image is not included here.

As seen in Figure 3.3, ShortestPath successfully locates a two-hop path from $v_1$ to $v_{1000}$ in a little less than 45 seconds. By contrast, the OC approach reveals that were 66 two-hop paths from $v_1$ to $v_{1000}$ and returns all 66 of them in less than five seconds.

Paths from 1 to 1000
Path: $\{1, 986, 1000\}$ found by Combinatorica in 44.8745 seconds.

Paths computed using OC algorithm:
$\hat{w}(1, 7, 1000) + \hat{w}(1, 17, 1000) + \hat{w}(1, 25, 1000) + \hat{w}(1, 34, 1000) + \hat{w}(1, 45, 1000) + \hat{w}(1, 47, 1000) + \hat{w}(1, 7, 1000) + \hat{w}(1, 102, 1000) + \hat{w}(1, 110, 1000) + \hat{w}(1, 114, 1000) + \hat{w}(1, 132, 1000) + \hat{w}(1, 156, 1000) + \hat{w}(1, 233, 1000) + \hat{w}(1, 256, 1000) + \hat{w}(1, 284, 1000) + \hat{w}(1, 320, 1000) + \hat{w}(1, 332, 1000) + \hat{w}(1, 354, 1000) + \hat{w}(1, 374, 1000) + \hat{w}(1, 381, 1000) + \hat{w}(1, 404, 1000) + \hat{w}(1, 475, 1000) + \hat{w}(1, 493, 1000) + \hat{w}(1, 495, 1000) + \hat{w}(1, 510, 1000) + \hat{w}(1, 517, 1000) + \hat{w}(1, 542, 1000) + \hat{w}(1, 554, 1000) + \hat{w}(1, 563, 1000) + \hat{w}(1, 613, 1000) + \hat{w}(1, 614, 1000) + \hat{w}(1, 626, 1000) + \hat{w}(1, 648, 1000) + \hat{w}(1, 666, 1000) + \hat{w}(1, 669, 1000) + \hat{w}(1, 689, 1000) + \hat{w}(1, 701, 1000) + \hat{w}(1, 725, 1000) + \hat{w}(1, 743, 1000) + \hat{w}(1, 753, 1000) + \hat{w}(1, 778, 1000) + \hat{w}(1, 810, 1000) + \hat{w}(1, 837, 1000) + \hat{w}(1, 841, 1000) + \hat{w}(1, 888, 1000) + \hat{w}(1, 917, 1000) + \hat{w}(1, 946, 1000) + \hat{w}(1, 957, 1000) + \hat{w}(1, 987, 1000) + \hat{w}(1, 988, 1000) + \hat{w}(1, 989, 1000)$
Computed in 4.82798 seconds.

Figure 3.3: Mathematica output: enumerating hop minimal paths in 1000-vertex graph.

3.3 Operator calculus approach to multi-constrained paths

The goal now is to extend the path-identifying nilpotent adjacency matrix approach to include weighted edges. In particular, each edge of the graph will be weighted by a $c$-tuple of nonnegative integers. In this manner, paths in the graph will have associated $c$-dimensional additive weights.

Given vectors $x = (x_1, \ldots, x_c)$ and $y = (y_1, \ldots, y_c)$, the notation $x \leq y$ is
taken to mean the following:

\[ x \leq y \iff x_i \leq y_i \quad \forall i \in \{1, \ldots, c\}. \tag{3.24} \]

The strict inequality \( x < y \) is analogously defined.

Given a finite graph \( G \) in which each edge is weighted with an \( c \)-tuple of nonnegative integers and a constraint vector \( c = (c_1, \ldots, c_m) \in \mathbb{N}_0^m \), the multi-constrained path problem is defined as follows.

**Definition 3.14.** The MCP (Multi-Constrained Path) problem is to find paths \( p \) from \( v_0 \) to \( v_\infty \) in the graph \( G \) such that

\[ \text{wt}(p) = \sum_{(v_i, v_j) \in p} \text{wt}((v_i, v_j)) \leq (c_1, \ldots, c_m) = c. \tag{3.25} \]

The goal is to find the set of feasible paths \( P_f = \{ p = (v_0, \ldots, v_\infty) : \text{wt}(p) < c \} \), i.e., all paths from \( v_0 \) to \( v_\infty \) that satisfy multiple constraints simultaneously.

An important variant of the MCP problem that is of particular interest is the associated optimization problem.

**Definition 3.15.** Letting \( P \) denote the set of feasible paths from \( v_0 \) to \( v_\infty \) in a weighted graph \( G \), the Multi-Constrained Optimal Path problem (MCOP) is to find a path \( p = (v_0, \ldots, v_\infty) \in P \) from \( v_0 \) to \( v_\infty \) such that

\[ \text{wt}(p) \leq \text{wt}(q) \quad \forall q \in P. \tag{3.26} \]

Note that for fixed \( s \in \mathbb{N}^m \), multi-exponents appearing among basis elements \( \nu^x \in \mathcal{C}_{\ell_s}^{\text{nil}} \) must satisfy \( x < s \). Given a \( c \)-vector \( s \in \mathbb{N}^c \), multiplication of arbitrary \( s \)-zeon blades consequently satisfies

\[ \nu^x \nu^y = \begin{cases} 
\nu^{x+y} & \text{if } x + y < s, \\
0 & \text{otherwise}. 
\end{cases} \tag{3.27} \]

In order to apply the OC approach to problems of identifying paths satisfying multiple constraints (represented by \( s \)), the path-identifying nilpotent adjacency matrix will be extended by allowing entries from the algebra \( \mathcal{C}_{\ell_s}^{\text{nil}} \otimes \mathbb{R}\Omega_n \). In this approach, a path \( u = (u_1, \ldots, u_m) \) of weight \( x \in \mathbb{N}_0^m \) will be represented in \( \mathcal{C}_{\ell_s}^{\text{nil}} \otimes \Omega_n \) by an element of the form \( \nu^x \omega_u \). The concatenation of this path with another path \( v = (v_1, \ldots, v_l) \) of weight \( y \) is then represented by the product

\[ (\nu^x \omega_u)(\nu^y \omega_v) = \begin{cases} 
\nu^{x+y} \omega_{u \cdot v} & \text{if } u \cdot v \text{ is a path of multi-weight less than } s, \\
0 & \text{otherwise.} 
\end{cases} \tag{3.28} \]
3.4 Feasible & optimal paths in \( c \)-weighted graphs

Often in routing problems, there are costs associated with each edge of a graph. Connections between nodes may require time, energy, money, etc. Given an initial vertex \( v_0 \) and terminal vertex \( v_t \) in a weighted graph, the collection of feasible paths from \( v_0 \) to \( v_t \) refers to all paths whose associated total costs satisfy some predefined constraints. Among these feasible paths, an optimal path can then be chosen.

Given a vector of constraints \( c = (c_1, \ldots, c_m) \in \mathbb{N}^m \), properties of the \( c \)-zeon algebra \( \mathcal{C}_c^{\text{nil}} \) can be used to sieve out the feasible paths from the collection of all paths. The feasible paths can then be ranked and an optimal path chosen.

**Definition 3.16.** Let \( c \in \mathbb{N}^m \), and let \( G = (V, E) \) be a graph on \( n \) vertices whose edges \((v_i, v_j)\) are multi-weighted by vectors \( w_{ij} \in \mathbb{N}_0^m \). The \( c \)-constrained path-identifying nilpotent adjacency matrix associated with \( G \) is the \( n \times n \) matrix with entries in \((\mathcal{C}_c^{\text{nil}})^m \otimes \Omega_n\) determined by

\[
\Psi_{ij} = \begin{cases} 
\nu w_{ij} \omega_j & \text{if } (v_i, v_j) \in E, \\
0 & \text{otherwise.} \end{cases} \tag{3.29}
\]

**Theorem 3.17.** Given a multi-weighted graph \( G \) on \( n \) vertices with nilpotent multi-weighted adjacency matrix \( \Psi \), a vector of constraints \( c = (c_1, \ldots, c_k) \), and a pair of distinct vertices \( v_0 \) and \( v_\infty \), the collection of feasible paths \( v_0 \rightarrow v_\infty \) in \( G \) is given by

\[
\nu^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^\ell | v_\infty \rangle = \sum_{\substack{\text{paths } p : v_0 \rightarrow v_\infty \text{ w.t. } \ell \leq c}} \nu^\text{w.t.}(p) \omega_p. \tag{3.30}
\]

More specifically, feasible paths exist if and only if \( \nu^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^\ell | v_\infty \rangle \) is nonzero. For the case \( v_0 = v_\infty \), one has

\[
\langle v_0 | \Psi^\ell | v_0 \rangle = \sum_{\substack{\text{cycles } p : v_0 \rightarrow v_0 \text{ w.t. } \ell \leq c}} \nu^\text{w.t.}(p) \omega_p. \tag{3.31}
\]

**Proof.** The result follows from Theorem 3.10 in consideration of combinatorial properties of \( \mathcal{C}_c^{\text{nil}} \).

**Corollary 3.18.** If \( \nu^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^\ell | v_\infty \rangle \neq 0 \), then the optimal path \( p = (v_0, \ldots, v_\infty) \) exists and is given by

\[
\hat{\Omega} \left( \nu^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^\ell | v_\infty \rangle \right) = \nu^\text{w.t.}(p) \omega_p. \tag{3.32}
\]

**Proof.** By Theorem 3.17, the collection of all feasible paths \( v_0 \rightarrow v_\infty \) is given by \( \nu^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^\ell | v_\infty \rangle \). By the chosen ordering of paths and definition of \( \hat{\Omega} \), the optimal path is as stated.
Figure 3.4: Weighted graph on 70 vertices.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \{4,2,3,3\} & \omega(8) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \{2,3,5,5\} & \omega(8) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \{4,1,2,3\} & \omega(10) & 0 & 0 \\
\{4,5,1,2\} & \omega(4) & 0 & \{2,2,4,3\} & \omega(4) & 0 & 0 & \{3,1,3,3\} & \omega(9) & 0 & 0 \\
0 & 0 & 0 & 0 & \{5,1,1,4\} & \omega(8) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \{3,2,3,1\} & \omega(7) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Figure 3.5: An $8 \times 8$ submatrix of the matrix $\Psi$ associated with the graph of Fig. 3.4.

**Example 3.19** (Static case). Figure 3.4 depicts a weighted graph on 70 vertices. Each edge is assigned a vector of random positive weights in $\mathbb{N}^4$. The graph is not symmetrically weighted; i.e., weights of vertex pairs vary depending on direction. The constraint vector is $\mathbf{c} = (50, 70, 70, 90)$.

A randomly-selected submatrix of the $70 \times 70$ weighted path-identifying adjacency matrix associated with the graph is seen in Fig. 3.5. The collection of admissible 4-paths from $v_{34}$ to $v_{47}$ are then computed using *Mathematica*, as seen in Fig. 3.6.

Let $\mathcal{P}_f^0$ denote the collection of feasible paths with source $v_0$. That is,

\[
\mathcal{P}_f^0 = \{ \mathbf{p} = (v_0, \ldots, v_*) : \text{wt}(\mathbf{p}) < \mathbf{c}, v_* \in V \}.
\]  

Following the approach of Corollary 3.11, the number of multiplications required in computing $\nu^0_{\omega_0}(v_0|\Psi^\ell|v_\infty)$ is seen to depend on the number of paths
Paths from 34 to 47 satisfying \( w \leq (50, 70, 70, 90) \)... 
\[
\begin{align*}
(11, \{8, 7\})_{V} & \omega_{(34, 30, 2, 47)} + (12, \{8\})_{V} \omega_{(34, 30, 50, 47)} + (8, \{10, 8\})_{V} \omega_{(34, 38, 5, 47)} = (7, 12, \{8, 11\})_{V} \omega_{(34, 38, 50, 47)}
\end{align*}
\]

Minimal path term: \((7, 12, \{8, 11\})_{V} \omega_{(34, 38, 50, 47)}\)

Minimal path weight: \((7, 12, 8, 11)\)

Minimal Path: \((34, 38, 50, 47)\)

Figure 3.6: Admissible 4-paths from \(v_{34}\) to \(v_{47}\) in the graph of Fig. 3.4. Minimal weight determined by ordering of multi-exponents.

of length \(\ell - 1\) or less having initial vertex \(v_{0}\) and simultaneously satisfying the constraints represented by \(c\). The following corollary is obtained as an immediate consequence.

**Corollary 3.20.** Given a fixed pair of vertices \(v_{0}\) and \(v_{\infty}\), the complexity of computing the optimal feasible path from \(v_{0}\) to \(v_{\infty}\) via the operator calculus method is

\[
\mathcal{O}(n |P_{f}^{0}|).
\]

### 3.5 The dynamic multi-constrained path problem

Given a finite set \(V\) and positive integer \(r\), an \(r\)-dimensional *weighting function* \(\varphi\) is defined on ordered pairs of vertices; i.e., \(\varphi : V \times V \rightarrow \mathbb{N}_{0}^{r}\). A weighted graph is then defined by the ordered triple \(G = (V, E, \varphi)\), where the weight of an arbitrary edge \((v_{i}, v_{j}) \in E\) is the \(r\)-vector \(\varphi((v_{i}, v_{j}))\).

Given a fixed collection of vertices \(V\) and fixed weighting function \(\varphi : V \times V \rightarrow \mathbb{N}_{0}^{r}\), a *weighted graph process* on \(V\) is defined as a sequence of weighted finite graphs \((G_{t}) := ((V, E_{t}, \varphi))\). Given a pair of vertices \(v_{0}, v_{\infty} \in V\) and a vector \(c \in \mathbb{N}^{r}\) of constraints, the problem being considered is to recover the set of feasible paths \(v_{0} \rightarrow v_{\infty}\) simultaneously satisfying the constraints \(c\).

Note that any graph sequence \((G_{t})\) naturally induces an associated sequence of weighted path-identifying nilpotent adjacency operators \((\Psi_{t})\), where each \(\Psi_{t}\) has entries in \(\mathbb{C}^{\ell_{t} \times \ell_{t}} \otimes \Omega_{|V|}\).

It is important to note that a feasible path \(p\) from \(v_{0}\) to \(v_{\infty}\) is not necessarily unique in the dynamic case because the sequence \(p\) can be partitioned into steps occurring in different frames of the process. As a consequence, scalar coefficients must be introduced to represent the path multiplicity within the collection \(P\).

**Theorem 3.21.** The collection of feasible paths of length \(k \geq 1\) from \(v_{0} \rightarrow v_{\infty}\) requiring \(f\) or fewer frames is given by

\[
\nu^{0}_{v_{0}} \omega_{v_{0}} \sum_{0 \leq \ell_{1}, \ldots, \ell_{f}} \langle v_{0} | \Psi_{1}^{\ell_{1}} \cdots \Psi_{f}^{\ell_{f}} | v_{\infty} \rangle = \sum_{k\text{-paths } p=(v_{0}, \ldots, v_{\infty}) \in P} \alpha_{p} \nu^{-w(t)}(p) \omega_{p}
\]

(3.34)
where $\alpha_p$ is a scalar coefficient representing the multiplicity of the path $p$ in the collection $\mathcal{P}$.

**Proof.** Note that $k \geq 1$ ensures that at least one of the integers $\ell_i$ is nonzero. For fixed nonnegative integers $\ell_1, \ell_2$ with $\ell_1 + \ell_2 = k$,

$$\nu^0_0 \omega_{v_0} \langle v_0 | \Psi_1^{\ell_1} \Psi_2^{\ell_2} | v_{\infty} \rangle = \nu^0_0 \omega_{v_0} \sum_{v_j \neq v_0} \langle v_0 | \Psi_1^{\ell_1} | v_j \rangle \langle v_j | \Psi_2^{\ell_2} | v_{\infty} \rangle$$

$$= \sum_{v_j \neq v_0} \left( \sum_{(\ell_1, \ell_2): \ell_1 \text{-paths } p = (v_0, \ldots, v_j) \in \mathcal{P}} \alpha_p \nu^{wt(p)}_p \omega_p \right) \left( \sum_{(\ell_2, q): \ell_2 \text{-paths } q = (v_j, \ldots, v_{\infty}) \in \mathcal{P}} \alpha_q \nu^{wt(q)}_q \omega_q \right)$$

$$= \sum_{k \text{-paths } p = (v_0, \ldots, v_{\infty}) \in \mathcal{P}^{\ell_1, \ell_2}} \alpha_p \nu^{wt(p)}_p \omega_p, \quad (3.35)$$

where $\mathcal{P}^{\ell_1, \ell_2}$ denotes the collection of feasible paths from $v_0$ to $v_{\infty}$ in which $\ell_1$ steps occur in frame 1 and $\ell_2$ steps occur in frame 2. Proceeding by induction, the result is established for fixed $m$-tuple of nonnegative integers $(\ell_1, \ldots, \ell_m)$ with $\ell_1 + \cdots + \ell_m = k$. Assuming that for positive integer $m_0$ and fixed nonnegative integers $\ell_1, \ldots, \ell_{m_0}$ with $\ell_1 + \cdots + \ell_{m_0} = k'$, one has

$$\nu^0_0 \omega_{v_0} \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_{m_0}^{\ell_{m_0}} | v_{\infty} \rangle = \sum_{k' \text{-paths } p = (v_0, \ldots, v_{\infty}) \in \mathcal{P}^{\ell_1, \ldots, \ell_{m_0}} \subseteq \mathcal{P}^{\ell_1, \ldots, \ell_{m_0}+1}} \alpha_p \nu^{wt(p)}_p \omega_p, \quad (3.36)$$

it follows that for $\ell_{m_0+1} = k - k'$, one has

$$\nu^0_0 \omega_{v_0} \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_{m_0}^{\ell_{m_0}} \Psi_{m_0+1}^{\ell_{m_0+1}} | v_{\infty} \rangle$$

$$= \nu^0_0 \omega_{v_0} \sum_{v_j \neq v_0} \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_{m_0}^{\ell_{m_0}} | v_j \rangle \langle v_j | \Psi_{m_0+1}^{\ell_{m_0+1}} | v_{\infty} \rangle$$

$$= \sum_{v_j \neq v_0} \left( \sum_{(k': (k' - k)\text{-paths } q = (v_j, \ldots, v_{\infty}) \in \mathcal{P}^{\ell_1, \ldots, \ell_{m_0}} \subseteq \mathcal{P}^{\ell_1, \ldots, \ell_{m_0}+1})} \alpha_p \nu^{wt(p)}_p \omega_p \right) \left( \sum_{(k')': (k')' \text{-paths } q = (v_j, \ldots, v_{\infty}) \in \mathcal{P}^{\ell_1, \ldots, \ell_{m_0}+1}} \alpha_q \nu^{wt(q)}_q \omega_q \right)$$

$$= \sum_{k \text{-paths } p = (v_0, \ldots, v_{\infty}) \subseteq \mathcal{P}^{\ell_1, \ldots, \ell_{m_0}+1}} \alpha_p \nu^{wt(p)}_p \omega_p, \quad (3.37)$$

Hence, the result is established for positive integer $m$:

$$\nu^0_0 \omega_{v_0} \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_m^{\ell_m} | v_{\infty} \rangle = \sum_{k \text{-paths } p = (v_0, \ldots, v_{\infty}) \subseteq \mathcal{P}^{\ell_1, \ldots, \ell_m}} \alpha_p \nu^{wt(p)}_p \omega_p. \quad (3.38)$$

The proof is thus completed by summing over all such $m$-tuples. \[\square\]

**Corollary 3.22.** The collection of feasible paths of all lengths from initial vertex $v_0$ to terminal vertex $v_{\infty} \neq v_0$ requiring $\ell$ or fewer frames is recovered from the
canonical expansion of
\[ \nu^0 \omega v_0 \sum_{0 \leq \ell_1, \ldots, \ell_f \leq n} \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_f^{\ell_f} | v_\infty \rangle = \sum_{\text{paths } p: v_0 \rightarrow v_\infty} \alpha_p \nu^{\text{wt}(p)} \omega_p, \] (3.39)

where \( \alpha_p \) denotes multiplicity of path \( p \in P \).

Proof. First, consider the degenerate case \( \ell_i = 0 \) for \( 1 \leq i \leq m \). In this case, the product of \( \Psi_i \)'s is the identity operator, and \( v_0 \not= v_\infty \) gives 0 on the left-hand side of the equation. Observing that the maximum path length is \( n = |V| \), the rest follows from Theorem 3.21.

Once the collection of feasible paths is obtained, the optimal path can be selected based on an ordering of the multi-exponents.

Corollary 3.23. Given a preferential ordering of multi-exponents, the optimal path \( p = (v_0, \ldots, v_\infty) \) from \( v_0 \) to \( v_\infty \) in the first \( f \) frames of the graph sequence \( (G_t) \) is given by

\[ \bigcup \left(\nu^0 \omega v_0 \sum_{0 \leq \ell_1, \ldots, \ell_f \leq n} \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_f^{\ell_f} | v_\infty \rangle \right) = \alpha_p \nu^{\text{wt}(p)} \omega_p, \] (3.40)

provided \( \nu^0 \omega v_0 \langle v_0 | \Psi_1^{\ell_1} \cdots \Psi_f^{\ell_f} | v_\infty \rangle \not= 0 \) for some \( f \)-tuple of nonnegative integers \( \ell_1, \ldots, \ell_f \), not all of which are zero. Here, \( \alpha_p \) denotes the multiplicity of path \( p \in P \).

4 Generalized idempotents and max-min operators

Sometimes additive weights are not suitable for the problem being considered. For example, in one version of a store-and-forward satellite constellation used for communications, the “sum” of two multi-weight vectors \((v_1, v_2, v_3)\) and \((w_1, w_2, w_3)\) on a coincident pair of links may be defined as

\[ (v_1, v_2, v_3) \boxplus (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, \min\{v_3, w_3\}), \] (4.1)

while also being subject to some constraint. Here, the third component might represent the transmission capacity of a link, such that a minimum capacity is required for all links appearing in a feasible path.

Definition 4.1. For positive integer \( n \), let \( \mathcal{C}_{\ell_n}^{\text{idem}} \) denote the abelian algebra generated by the collection \( \{\varepsilon_i : 1 \leq i \leq n\} \) along with the scalar \( 1 = \varepsilon_0 \) subject to the following multiplication rules:

\[ \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \text{ for } i \not= j, \text{ and } \] (4.2)

\[ \varepsilon_i^2 = \varepsilon_i \text{ for } 1 \leq i \leq n. \] (4.3)
Remark 4.2. The algebra $C \ell_n^{\text{idem}}$ is constructed within a $2n(n - 1)$-particle fermion algebra. Fix $n > 0$ and consider elements of the form

$$
\varepsilon_i = \frac{1}{2} \left( 1 + \left( \frac{f_i + f_i^+}{2} \right) \left( \frac{f_{n^2-n+i} + f_{n^2-n+i}^+}{2} \right) \right),
$$

where $f_i$ and $f_i^+$ denote the $i$th fermion annihilation and creation operators, respectively.

The idem-Clifford algebra was first used in [10] for purposes of computing higher moments of cycle numbers in homogeneous random graphs. As illustrated in the next lemma, it is also useful for computing the maximum and minimum of a pair of integers. First, we consider the extension to the infinite-dimensional version of the idem-Clifford algebra.

**Definition 4.3.** The generalized idempotent algebra $C \ell^{\text{idem}}$ is the infinite-dimensional Abelian algebra defined by the direct sum

$$
C \ell^{\text{idem}} := \bigoplus_{n=1}^{\infty} C \ell_n^{\text{idem}}.
$$

The generators $\{\varepsilon_j : 1 \leq j \}$ are pairwise commutative and satisfy $\varepsilon_j^2 = \varepsilon_j$ for all $j \geq 1$.

Recalling multi-index notation and the $n$-set notation, $[n] := \{1, \ldots, n\}$, the max-min operators are now written using generalized idempotents.

**Lemma 4.4.** Let $\{\varepsilon_i : 1 \leq i \leq n\}$ denote the idempotent generators of $C \ell_n^{\text{idem}}$. For nonnegative integers $s, t \leq n$,

$$
\varepsilon[s] \vee \varepsilon[t] := \varepsilon[s] \varepsilon[t] = \varepsilon[\max\{s, t\}].
$$

Whence, one can define

$$
\varepsilon[s] \wedge \varepsilon[t] := \varepsilon[s] + \varepsilon[t] - \varepsilon[s] \varepsilon[t] = \varepsilon[\min\{s, t\}].
$$

**Proof.** Proof is by direct computation. \hfill $\blacksquare$

Identifying $\varepsilon_0 \mapsto \varepsilon_0 := 1$ to obtain the multiplicative identity of the algebra, one also obtains the identity with respect to the maximum operator.

Relative to the minimum operator (4.7), one formally defines the element $\varepsilon[\infty]$ such that

$$
\varepsilon[\infty] \wedge \varepsilon[j] = \varepsilon[j]
$$

for all $j \in \mathbb{N}$.

Note that in the finite case, this identity element is given by

$$
\varepsilon[\infty] := \varepsilon[n] \in C \ell_n^{\text{idem}}.
$$
Remark 4.5. In $C^\ell_n^{\text{idem}}$, an alternative formulation of the minimum operator is obtained by

$$\varepsilon[s] \land \varepsilon[t] := (\varepsilon[s]^* \varepsilon[t]^*)^*,$$

where the notation $\ast$ indicates the involution defined by the multi-index set-complement

$$\varepsilon[m]^* := \varepsilon[n] \setminus [m]$$

for $m \leq n$. This extends formally to $C^\ell^{\text{idem}}$ by

$$\varepsilon[m]^* := \varepsilon\{m+1, \ldots\},$$

for $m \in \mathbb{N}_0$.

Letting $m$ denote the number of nonadditive weights assessed by min-max operators, it is now possible to consider multi-constrained paths by constructing a nilpotent adjacency matrix with entries from $C^\ell_s^{\text{nil}} \otimes C^\ell_m^{\text{idem}} \otimes \omega_n$. More to the point, one can define a max-min signature $m = (m_1, m_2)$ where $m_1, m_2 \geq 0$ and $m_1 + m_2 = m$ for generalized idempotents and an associative binary operation $\ast$ on $C^\ell_s^{\text{nil}} \otimes C^\ell_m^{\text{idem}}$ satisfying

$$(\varepsilon[\ell_1] \otimes \cdots \otimes \varepsilon[\ell_{m_1+m_2}]) \ast (\varepsilon[g_1] \otimes \cdots \otimes \varepsilon[g_{m_1+m_2}]) = (\varepsilon[\ell_1] \lor \varepsilon[g_1]) \otimes \cdots \otimes X^{j\text{th}} \otimes \cdots \otimes (\varepsilon[g_{m_2}] \land \varepsilon[g_{m_2}]) \quad (4.8)$$

In particular, the $j^{\text{th}}$ factor, $X$, appearing in the tensor product is given by

$$X = \begin{cases} 
\varepsilon[\ell_j] \lor \varepsilon[g_j] & \text{if } 1 \leq j \leq m_1, \\
\varepsilon[\ell_j] \land \varepsilon[g_j] & \text{if } m_1 + 1 \leq j \leq m_1 + m_2.
\end{cases} \quad (4.9)$$

With this in mind, the notation $C^\ell_s^{\text{nil}} \otimes C^\ell_m^{\text{idem}}$ is clear.

Letting $a$ denote the number of additive weights, the multi-exponent notation is now extended to $C^\ell_s^{\text{nil}} \otimes C^\ell_{m_1,m_2}^{\text{idem}}$ by

$$\xi(x_1, \ldots, x_a, m_1, \ldots, m_m) = \nu(x_1, \ldots, x_a) \otimes \varepsilon[m_1] \otimes \cdots \otimes \varepsilon[m_m]. \quad (4.10)$$

The identity element of $C^\ell_s^{\text{nil}} \otimes C^\ell_{m_1,m_2}^{\text{idem}}$ is then written as

$$\xi^0 := \nu^0 \otimes \varepsilon^0 \otimes \varepsilon[\infty] \otimes \varepsilon[m_2]. \quad (4.11)$$

Define the binary operation $\oplus$ on $\mathbb{N}_0^{a+m_1+m_2}$ by

$$(u \oplus v)_i := \begin{cases} 
u_i & \text{if } 1 \leq i \leq a, \\
\max\{u_i, v_i\} & \text{if } a + 1 \leq i \leq a + m_1, \\
\min\{u_i, v_i\} & \text{if } a + m_1 + 1 \leq i \leq a + m_1 + m_2. 
\end{cases} \quad (4.12)$$
The commutative multiplication $\ast$ on $C_{\ell}^{s \text{ nil}} \otimes C_{\ell_{m_1}, m_2}^{\text{idem}}$ is defined by linear extension of

$$\xi^u \ast \xi^v = \begin{cases} \xi^{u \boxplus v} & \text{if } u \boxplus v \in C, \\ 0 & \text{otherwise.} \end{cases} \quad (4.13)$$

This extends to $C_{\ell}^{s \text{ nil}} \otimes C_{\ell_{m_1}, m_2}^{\text{idem}} \otimes \Omega_n$ by

$$\xi^v \omega_{v_i} \xi^w \omega_{v_j} = (\xi^v \ast \xi^w) \omega_{v_i} \omega_{v_j}, \quad (4.14)$$

which extends inductively to

$$\xi^v \omega_{p} \xi^w \omega_{v_k} = (\xi^v \ast \xi^w) \omega_{p} \omega_{v_k} = \begin{cases} \xi^{v \boxplus w} \omega_{p,v_k} & \text{if } v \boxplus w \in C \text{ and } p \cap v_k = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (4.15)$$

for any $k$-path $p = (v_0, \ldots, v_k)$ of weight $v$ and vertex $v_k$ adjacent to $v_k$ via an edge of weight $w$.

The path-identifying nilpotent adjacency matrix can now be defined with entries in $C_{\ell}^{s \text{ nil}} \otimes C_{\ell_{m_1}, m_2}^{\text{idem}} \otimes \Omega_n$. Further, extending $\ast$ to matrix multiplication according to

$$\langle v_i | (A \ast B) | v_j \rangle = \sum_{\ell=1}^{n} a_{i\ell} \ast b_{\ell j}, \quad (4.16)$$

the following corollary to Theorem 3.10 is immediately obtained.

**Theorem 4.6.** Let $G$ denote a multi-weighted graph on $n$ vertices, and let $C$ denote a system of constraints corresponding to generalized zeon signature $s$ and max-min signature $(m_1, m_2)$. Let $v_0$ and $v_\infty$ denote a pair of distinct vertices, and let $\Psi$ be the multi-weighted nilpotent adjacency matrix for $G$ having entries in $C_{s \text{ nil}} \otimes C_{\ell_{m_1}, m_2}^{\text{idem}} \otimes \Omega_n$. The collection of feasible paths $v_0 \to v_\infty$ in $G$ is then given by

$$\xi^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^{\ast \ell} | v_\infty \rangle = \sum_{\text{paths } p: v_0 \to v_\infty \text{ wt}(p) \in C} \xi^{\text{wt}(p)} \omega_p. \quad (4.17)$$

More specifically, feasible paths exist if and only if $\xi^0 \omega_0 \sum_{\ell=1}^{n} \langle v_0 | \Psi^{\ast \ell} | v_\infty \rangle$ is nonzero. For the case $v_0 = v_\infty$, one has

$$\langle v_0 | \Psi^{\ast \ell} | v_0 \rangle = \sum_{\text{cycles } p: v_0 \to v_0 \text{ wt}(p) \in C} \xi^{\text{wt}(p)} \omega_p. \quad (4.18)$$

While the *Mathematica* implementations of the max-min operators (4.6) and (4.7) are accomplished by simply applying the appropriate built-in operators, it is important to note that all necessary operations can be accomplished by using the inherent combinatorial properties of the algebras described. Moreover, all of these algebras occur as subalgebras of Clifford algebras–lending them a natural connection to quantum probability and (by extension) to quantum computing.
Example 4.7. Figure 4.1 depicts a weighted graph on 150 vertices. Each edge is assigned a vector of random weights in $\mathbb{N}_0^3$.

The graph is not symmetrically weighted; i.e., weights of vertex pairs vary depending on direction. Defining the set

$$\mathcal{C} := \{(v_1, v_2, v_3) \in \mathbb{N}_0^3 : (v_1 \leq c_1) \land (v_2 \leq c_2) \land (v_3 \geq c_3)\},$$

a weight vector $v = (v_1, v_2, v_3)$ is said to satisfy the constraint $c = (c_1, c_2, c_3)$ if $v \in \mathcal{C}$.

The “sum” of weights taken between two coincident edges is then computed by (4.1), provided $v \sqcup w \in \mathcal{C}$. The constraint vector used in the example is $c = (500, 700, 20)$. That is, the additive sums of the first and second component values are bounded above by 500 and 700, respectively, while the minimum value of the third component taken over all links in a path is bounded below by 20.

In order to define optimal paths, weights are ordered as follows:

$$(v_1, v_2, v_3) \leq (w_1, w_2, w_3)$$

$$\Leftrightarrow (v_1 < w_1) \lor [(v_1 = w_1) \land (v_3 > w_3)] \lor [(v_1 = w_1) \land (v_3 = w_3) \land (v_2 \leq w_2)].$$

A $7 \times 7$ submatrix of the $150 \times 150$ weighted path-identifying adjacency matrix associated with the graph is seen in Fig. 4.2.

The collection of admissible 3-paths from $v_{13}$ to $v_{138}$ are then computed using Mathematica, as seen in Fig. 4.3.

The dynamic case of Theorem 3.21 is extended similarly. Note that when considering the collection of feasible paths, $p \in \mathcal{P}$ only if $\text{wt}(p) \in \mathcal{C}$. 

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Figure 4.2: A $7 \times 7$ submatrix of the matrix $\Psi$ associated with the graph of Fig. 4.1.

$$
\begin{pmatrix}
0 & 0 & 0 & (48, 7, 18) & \xi & \omega(17) & 0 & 0 & 0 \\
0 & 0 & (28, 5, 23) & \xi & \omega(16) & 0 & 0 & 0 & 0 \\
0 & (25, 2, 11) & \xi & \omega(15) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (26, 17, 15) & \xi & \omega(13) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

Minimal Path Term: $(52, 40, 27)$

Minimal path weight vector: $(52, 40, 27)$

Minimal Path: $(50, 108, 79, 99)$

Figure 4.3: Admissible 3-paths from $v_{50}$ to $v_{99}$ in the graph of Fig. 4.1. Minimal weight determined by ordering of multi-exponents.

Theorem 4.8. Let $(G_\ell : 1 \leq \ell)$ denote a sequence of multi-weighted graphs on $n$ vertices, and let $C$ denote a system of constraints corresponding to generalized zeon signature $\mathbf{s}$ and max-min signature $(m_1, m_2)$. Let $(\Psi_\ell : 1 \leq \ell)$ be a sequence of multi-weighted nilpotent adjacency matrices for $(G_\ell)$ having entries in $C_{\mathbf{s}}^{\text{nil}} \otimes C_{m_1, m_2}^{\text{idem}} \otimes \Omega_n$. Let $v_0$ and $v_\infty$ denote a fixed pair of distinct vertices. The collection of feasible paths of length $k \geq 1$ from $v_0 \to v_\infty$ requiring $f$ or fewer frames is then given by

$$
\xi^0 \omega_{v_0} \sum_{0 \leq \ell_1, \ldots, \ell_f \leq \ell} \langle v_0 | \Psi_1^{\ast \ell_1} \cdots \Psi_f^{\ast \ell_f} | v_\infty \rangle = \sum_{k\text{-paths } p=(v_0, \ldots, v_\infty) \in \mathcal{P}} \alpha_p \xi^{\text{wt}(p)} \omega_p
$$

where $\alpha_p$ is a scalar coefficient representing the multiplicity of the path $p$ in the collection $\mathcal{P}$.

5 Conclusion

The operator calculus approach provides convenient symbolic computational tools for a broad range of combinatorial problems and practical applications.
The methods developed here can be applied directly to multi-constrained quality of service (QoS) problems as well as problems related to precomputed routing in store-and-forward satellite constellations [3].

References


