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COMPACTIFICATION OF INFINITE GRAPHS AND SAMPLING

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Compactification of Infinite Graphs and Sampling

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Abstract

In the paper, we consider Hilbert spaces of functions on infinite graphs, and their compactifications.

We arrive at a sampling formula in the spirit of Shannon; the idea is that we allow for sampling of functions f defined on a continuum completion of an infinite graph G, sampling the continuum by values of f at points in the graph G.

Rather than the more traditional frequency analysis of band-limited functions from Shannon, our analysis is instead based on reproducing kernel Hilbert spaces built from a prescribed infinite system of resistors on G.

Key words and phrases : Hilbert space, spectrum, encoding, transfer operator, infinite graphs, wavelets, fractals, stochastic process, path measures, transition probability

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1 Introduction

We consider weighted graphs as electric networks, or in statistical mechanics, where weights are assigned to each of the $2d$ edges, and for each vertex x. This way, a weight of the graph G becomes a positive function defined on the edges of the specified graph G. Below we develop the general theory, illustrate its applications; and we obtain Shannon's result as a special case. An especially attractive statistical mechanics application is [21], [18] and [6].

Now Shannon's view is motivated by signal processing, i.e., engineering of signals, see e.g., [7]: interpolation of functions (signals) on a continuum, determining band-limited functions defined on a continuum from their discrete samples. But in some cases outside the study of lattice graphs, one might only have available the particular given (countably discrete!) graph G; no ambient continuum; and indeed there might well be a variety of choices for an ambient continuum. This is the viewpoint taken in the present paper.

In more detail, starting with a fixed infinite graph G, it will be convenient for us to denote the set of vertices G^0, and the edges G^1. And we will study functions on both sets; more precisely, Hilbert spaces obtained by completion in certain quadratic forms; as well as the interconnections between spaces of functions on G^0, and on G^1.

1.1 Electrical Networks

In an electrical network consider configurations of voltages and of currents. Interconnections between spaces may be understood with the use of the two rules for electrical networks, Ohm's law, and Kirchhoff's sum-rules for current flow. In this case, hence functions on G^0 represent voltages, i.e., voltage potentials, while functions on G^1 can be a configuration of Kirchhoff-current (measured in Amps).

In a variety of frameworks, this analysis on weighted graphs will be developed; and these graphs in turn arise in a host of applications, the network theory being just one of them [23].

In order to visualize our initial summary, the concepts with reference to "grids of resistors," i.e., networks and systems of resistors will be illustrated. Indeed, there are a number of separate applications of the same theory already [14], [10]. Nevertheless we are not aware of theorems that allow for realistic ambient spaces X for a fixed infinite weighted graph G; as well as an interpolation in function spaces on X, from sampling points in the vertices of G. One reason for this is that, starting with G, the choice of such an ambient continuum X is not obvious from the given graph G. This problem will be solved with the use of ideas from stochastic integration.

Now the study of weighted graphs is also a part of the wider subject of random walk analysis. On the other hand, in this wider context we are really dealing with an analysis of weighted graphs. We are looking at a positive function c (the notation c is for conductance, and $c = 1/resistance$). We define a weighted graph to be a pair (G, c) where c is a fixed positive function defined on the set of (unordered) edges G^1.

The network point of view is developed in recent papers by one of the present authors and Erin Pearse [15], [14], motivated by electrical engineering, probability, and statistical mechanics.

Insert an amp at one fixed vertex x, and extract it at another vertex y. Then a current and a voltage distribution in the entire graph will be generated. Now, the current is directed; and this results in an induced directed graph. However, with the direction on the edges depending on choices: A different amp-in amp-out experiment, a different current configuration on G will be induced.

We might even extract the one amp at a point at infinity, so this then involves a subtle analysis of boundaries of infinite graphs.

When G is an infinite tree, then one can easily see that the boundary of G, bd$G$ should be a Cantor set; but to make this precise, a Laplace operator on G (this will depend on the choice of conductance function c) must be introduced; an appropriate Hilbert space in such a way that bd$G$ attains properties in the present discrete context, otherwise familiar from classical (continuous) harmonic analysis, for instance the harmonic functions on G must have an integral representation, with an integral kernel on the Cartesian product $G \times \text{bd}G$, much analogous to the more familiar Poisson kernel from the study of harmonic functions on an open domain in the complex plane.

We then move to the development of formula for the Laplace operator of such a given weighted graph (G, c), and we introduce a Hilbert space $\mathcal{H}(G, c)$. The Laplace operator will be Hermitian as a densely defined linear operator in $\mathcal{H}(G, c)$; it could be bounded or unbounded depending on the context. It might even have non-zero deficiency indices in the sense of von Neumann.

Now this Hilbert space $\mathcal{H}(G, c)$ must be relied on for the essential ingredients in a harmonic analysis. While it might be natural choice to try l^2 sequence space that is more simple looking, $l^2(G^0)$ as a candidate for Hilbert space, this will not work. The papers by Jorgensen and Pearse make the case that “the right” Hilbert space is $\mathcal{H}(G, c)$ which is called the energy Hilbert space. The study of these Hilbert spaces also entails notions of resistance distance, see [13].

This $\mathcal{H}(G, c)$ turns out to be a reproducing kernel Hilbert space (RKHS), however not in the traditional sense of the RKHS notion, see [3, 2]. As a matter of fact, the reproducing kernel in $\mathcal{H}(G, c)$ is a function depending on a pair of points in G^0; not just one point, which is the traditional notion of RKHS; see e.g., [12].

Two different approaches to analysis on graphs have been used in a number of recent papers. While separate teams of authors, ask closely related questions about the graphs under study, the tools are different.

There has been a recent revived interest in sampling techniques, motivated by quantization requirements in digital signal processing in part, and by contrast to our present approach, much of this is based on harmonic analysis tools, see e.g., [1, 5, 4, 8, 9], and other investigations on Markov processes [19, 11, 20, 22, 16, 17, 13].

2 Graphs, Signals, Shannon's Interpolation

In this section we briefly review the standard formulation of Shannon interpolation of band-limited signals, and we outline an adaptation to the analogous interpolation on infinite graphs G. In Theorem 2.1 we state an interpolation theorem in this context, offering an interpolation formula for functions defined on "the" boundary $\mathrm{bd}(G)$ of a given graph G of G: we interpolate a certain class of functions on $\mathrm{bd}(G)$ expanded from its values on the vertices in G.

In signal processing Shannon's example

$$\sum(\|signal\|^2) = \int_{\mathbb{R}} \|f(t)\|^2 dt < \infty \tag{1}$$

but not all funtions on $\mathbb{R}$ are signals. Here, signal is referred as observable and band limited.

$$L^2_{band} := \{f \in L^2(\mathbb{R}) : \widehat{f}(\cdot)\text{is supported in } [-\pi, \pi], \text{frequency band}\}. \tag{2}$$

$\uparrow$ band-limited analogue-signals

2.1 Shannon's Theorem:

$f \in L^2_{band}$ satisfies interpolation

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}. \tag{3}$$

Since f is band-limited,

$$\widehat{f}(\theta) = \sum_{n\in\mathbb{Z}} a(n) e^{-in\theta} \quad \text{Fourier transform and Fourier series} \tag{4}$$

$$\begin{aligned}
a(n) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\widehat{f}(\theta)d\theta && (5)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\int_{\mathbb{R}} e^{-i\theta t} f(t)dt && (6)\\
&= \int \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta(n-t)}d\theta f(t)dt && (7)\\
&= \int \delta(n-t)f(t)dt = f(n), \quad \forall n \in \mathbb{Z}. && (8)
\end{aligned}$$

Now substitute (2.8) into (4), and we get

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\theta)e^{i\theta t}d\theta \\
&= \sum_{n} f(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta(t-n)}d\theta \\
&= \sum_{n\in\mathbb{Z}} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}, \quad \forall t\in\mathbb{R}.
\end{aligned}$$

Our plan is to carry on Shannon's idea to the realm of graphs. We need an embedding of G^0B into some counting space, called S^1 below, such that $\exists l_a : G^a \times B$ such that

$$\tilde{f}(\xi) = \sum_{x\in G^0} l_a(x,\xi)f(x) \tag{9}$$

thus extending Shannon in the version (3).

2.2 The General Theory of Without Groups:

Interpolation for our $(G,c)\mapsto \mathcal{H}_E$ finite energy voltage signals, where G^0 represent vertices, G^1 represent edges, and $c : G^1 \mapsto \mathbb{R}_+$.

$$\sum(\text{voltage signal}) = \frac{1}{2}\sum\sum_{x\sim y} c_{xy}|f(x)-f(y)|^2 < \infty. \tag{10}$$

2.3 Problem

Find natural interpolation spaces B in our $\mathcal{H}_E(G,c)$ (FIX: section 4) in such a way that Shannon's formula

$$\begin{gathered}
G|B \\
(3) \rightsquigarrow \mathbb{Z} \mapsto \mathbb{R} \quad (\text{Shannon}) \\
G \mapsto \text{“}B\text{”}
\end{gathered}$$

so that our case Hilbert space of functions on B such that (9) holds. The object is to get a dense subspace $S \hookrightarrow \mathcal{H}_E$ with a stronger Fréchet topology such that the dual $S' :=$ all continuous linear functionals which satisfy

$$s' \in S' \underset{D}{\Leftrightarrow} s'(s_1+\lambda s_2) = s'(s_1)+\lambda s'(s_2) \quad \forall s_1,s_2\in S, \quad \forall\lambda\in\mathbb{C}$$

is Fréchet continuous. $S \hookrightarrow \mathcal{H}_E \hookrightarrow S'$ Schwartz's tempered distributions.

From the inner-product in $\mathcal{H}_E$

$$(u,v)_E = \frac{1}{2}\sum\sum_{x\sim y} c_{xy}(u(x)-u(y))(v(x)-v(y)).$$

We get that $v \sim \langle \cdot, v \rangle \in S'$, so there is a natural embedding of $\mathcal{H}_E \sim \mathcal{H}'_E$ into S'. The space S' carries a white noise measure from the theory of stochastic processes.

2.4 Remark

Gelfand triple

$$S \hookrightarrow \mathcal{H}_E \hookrightarrow S' \quad (S' \subset B) \tag{11}$$

offer candidates for interpolations.

We build kernels $k : G \times B \mapsto \mathbb{C}$ such that functions on B (some cleverly defined subspace $\mathcal{H}_B$.

$$f(\beta) = \sum_{x \in G} k(x, \beta) f(x) \quad \forall \beta \in B. \tag{12}$$

So we want $\mathcal{H}_B \approx \mathcal{H}_E$ and k on $G \times B$ such that (12) holds $\forall \beta \in B$, $\forall f \in \mathcal{H}_B$, i.e., interpolation.

From the theory of energy Hilbert space from [13] we have two systems as follows:

$$\delta_x : G^0 \mapsto \mathbb{R} \quad \delta/G_x(y) = \begin{cases} 1 & y = x \\ 0 & \text{else} \end{cases}$$

$$v_x : G^0 \mapsto \mathbb{R} \quad \Delta v_x = \delta_x - \delta_0. \tag{13}$$

We select a base-point $o \in G^0$. By Riesz $\exists!\ v_x \in \mathcal{H}_E$ such that

$$\langle v_x, f \rangle_E = f(x) - f(o), \quad \forall f \in \mathcal{H}_E; \tag{14}$$

and v_x satisfies (13).

We have

$$\begin{aligned} \langle \delta_x, f \rangle_E &= (\Delta f)(x) \\ &= \sum_{y \sim x} c_{xy} (f(x) - f(y)). \end{aligned}$$

Theorem 2.1.

$$\langle f, \xi \rangle_E = \sum_{x \in G^0 \setminus \{o\}} f(x) (\Delta \xi)(x), \quad \forall f \in S, \forall \xi \in S'. \tag{15}$$

Proof. The topologies on the spaces S and S' are as follows: Given a weighted graph (G, c), then select a fixed vertex o, and let the system $\{v_x\}_{x \in G^0 \setminus \{o\}}$ (called dipoles) be as in equation (14). Since the vertex set G^o is countable, we may apply Gram-Schmidt to the system of dipoles, obtaining thereby an ONB in the Hilbert space $\mathcal{H}_E$, say $\epsilon_1, \epsilon_2, \cdots$, corresponding to $v_{x_1}, v_{x_2}, \cdots \in \mathcal{H}_E$. Using again (14), one checks that

$$v_{x_n} = \sum_{k=1}^{n} \epsilon_{x_k}(x_n)\epsilon_{x_k}, \tag{16}$$

and

$$\epsilon_{x_n} = \sum_{k=1}^{n} \langle \delta_{x_k}, \epsilon_{x_n} \rangle_E v_{x_k} \tag{17}$$

$$= \sum_{k=1}^{n} \left(\Delta\epsilon_{x_n}(x_k) \right) v_{x_k}. \tag{18}$$

In these computations, we used that $\Delta v_x = \delta_x - \delta_0$, for all $x \in G^0 \setminus \{o\}$. We now introduce the following Gelfand-triple:

$$\underset{\text{Fréchet space}}{S} \hookrightarrow \underset{\text{Hilbert space}}{\mathcal{H}_E} \hookrightarrow \underset{\text{dual to } S}{S'} \tag{19}$$

where:

(i)

$$u \in \mathcal{H}_E \underset{\text{Parseval}}{\Longleftrightarrow} \exists (c_j) \in l^2 \quad \text{such that } u = \sum_{j=1}^{\infty} c_j \epsilon_{x_j}, \quad \text{and } \|u\|_E^2 = \sum_{j=1}^{\infty} \|c_j\|^2;$$

(ii) $u \in S \underset{Def}{\Longleftrightarrow}$ the expansion (c_j) in (i) satisfies:

$$\forall p \in \mathbb{N} \quad \exists K < \infty \quad \text{such that } |c_j| \leq K j^{-p}, \forall j \in \mathbb{N};$$

(iii) $\xi \in S' \underset{Def}{\Longleftrightarrow}$ the expansion (c_j) in (i) satisfies:

$$\exists q \in \mathbb{N} \quad \exists K < \infty \quad \text{such that } |c_j| \leq K j^{q}, \forall j \in \mathbb{N};$$

The conditions in (ii) and (iii) then define dual topologies on the pair S, S' turning S into a Fréchet space.

As a result, we conclude that elements $\xi \in S'$ are also functions on G^0 (= the vertex set of G). Therefore the $c-$Laplace operator Δ (defined from the conductance function $c : G^1 \to \mathbb{R}_+$) is also acting on S', i.e., via

$$\left(\Delta\xi\right)(x) = \sum_{y \sim x} c_{xy} \left(\xi(x) - \xi(y)\right).$$

Note that the representations (16) and (17) depend on the ordering $x_1, x_2, \cdots$ of points in $G^0 \setminus \{o\}$, while the conclusion in (15) is independent of this choice.

Also note that since the sum in (16) is finite for each x_n, it follows that the functions $\{v_x\}_{x\in G^0\setminus\{o\}}$ are in S.

We may therefore, for all $f \in S$, take $\tilde{f}(\xi) = \langle f, \xi\rangle_E$ as a function on S'. As a result, $\tilde{f}$ is an extension from G^0 to S' via this identification, i.e., $G^0 \setminus \{o\} \hookrightarrow S'$. The choice of base-point o in G^0 enters as follows: If $x \in G^0$, then

$$\tilde{f}(x) = \langle f, v\rangle_E = f(x) - f(o). \tag{20}$$

Since o is fixed, we may restrict attention to $f \in S$ satisfying $f(o) = 0$.

As a result, the conclusion in (15) takes the form

$$\tilde{f}(\xi) = \sum_{x\in G^0\setminus\{o\}} (\Delta\xi)(x)\tilde{f}(x). \tag{21}$$

For the proof of (21), or equivalently (15), we need:

Lemma 2.2. *The Dirac functions $\{\delta_x\}$ are in S. Indeed, for every $x \in G^0 \setminus \{o\}$, we have:*

$$\delta_x = c(x)v_x - \sum_{y\sim x} c_{xy}v_y \tag{22}$$

when the summation on the RHS in (22) is finite.

Proof. For $u \in \mathcal{H}_E$, we compare as follows

$$\begin{aligned}
\langle c(x)v_x - \sum_{y\sim x} c_{xy}v_y, u\rangle_E &= c(x)\langle v_x, u\rangle_E - \sum_{y\sim x} c_{xy}\langle v_y, u\rangle_E \\
&= c(x)\left(u(x) - u(o)\right) - \sum_{y\sim x} c_{xy}\left(u(y) - u(o)\right) \\
&= \sum_{y\sim x} c_{xy}\left(u(x) - u(y)\right) \\
&= (\Delta u)(x) \\
&= \langle \delta_x, u\rangle_E.
\end{aligned}$$

Since this holds for all $u \in \mathcal{H}_E$, the desired formula (22) follows. □

Proof of Theorem 2.1 continued: In view of the Gelfand-triple (19), and the duality relations (i)-(iii), we conclude that it is enough to verify (15) for functions f on $G^0 \setminus \{o\}$ having the following finite sum representation

$$f = \sum_x f(x)\delta_x. \tag{23}$$

Hence, for all $\xi \in S'$, we have

$$\begin{aligned}\tilde{f}(\xi) &= \sum_x f(x)\langle \delta_x, \xi\rangle_E \\ &= \sum_x f(x)(\Delta\xi)(x)\end{aligned}$$

which is the desired conclusion.

Indeed to extend from finite sums, as in (23), to the general case on $f \in S$, we may use (ii) and (iii), and approximation $f \in S$ with finite representations (23). □

Example 2.3. $(\mathbb{Z}_+, 1)$ $G^0 = \mathbb{Z}_+ \cup \{o\}$, $c(x, x+1) = 1$, $\forall x \in \mathbb{Z}_+$ $\|f\|_E^2 = \frac{1}{2}\sum_x |f(x) - f(x-1)|^2$.

Example 2.4. $(\mathbb{Z}, 1)$ $G^0 = \mathbb{Z}$, $c(x, x+1) = 1$, $\forall x$ $\|f\|_E$ same as in Example 2.3 but with sum $\sum_{x\in\mathbb{Z}}$.

Example 2.5. $(\mathbb{Z}, c)$ As in Examples 2.3 and 2.4, but with $c_{x,x+1} = c^x$ with $c > 1$ is some fixed number.

Remark 2.6. We have other properties in mind in our study of (11) but in a way (details below!) (11) is better satisfied for interpolation than for Poisson kernels.

2.5 Several Reasons

We get a non-trivial solution to (11) even in such cases as $(\mathbb{Z}_+, \mathbf{1} = G)$ when G does not have only bd$G$ in our sense. But S', $(\mathbb{Z}_+, \mathbf{1}) = G$ is still $\infty-$dimensional; and it satisfies (12).

Example 2.7. Let $S \subset \mathcal{H}_E \subset S'$ and $\{\epsilon_x\}$ ONB in $\mathcal{H}_E$ then

$$(v_x) \text{ dipole } v_x(o) = 0, \quad \Delta v_x = \delta_x - \delta_o. \tag{24}$$

(ϵ_x) ONB in $\mathcal{H}_E$, $\epsilon_x = v_x - v_{x-1}$

Definition 2.8.

$$\|f\|_E^2 = \frac{1}{2}\sum_{x\in\mathbb{Z}_+} |f(x) - f(x-1)|^2. \tag{25}$$

Lemma 2.9.

$$f \in S \underset{D}{\iff} \sum_{x\in\mathbb{Z}_+} x^p |f(x) - f(x-1)|^2 < \infty \quad \forall p \in \mathbb{Z}_+. \tag{26}$$

$$\xi \in S' \iff \exists A < \infty \exists p_1 \in \mathbb{Z}_+ \tag{27}$$

such that $|\xi(x) - \xi(x-1)| < Ax^{p_1}$ *for all* $x \in \mathbb{Z}_+ \iff$ *polynomial growth.*

Proof. Set

$$\|f\|_p := \left(\frac{1}{2} \sum_{x \in \mathbb{Z}_+} x^p \left| f(x) - f(x-1) \right|^2 \right)^{1/2}. \tag{28}$$

Claim: If ξ is a function on $\mathbb{Z}$ such that (27) holds, then ($p = p_\xi$), then there exists $C < \infty$ such that

$$|\langle \xi, f \rangle_{\mathcal{H}_E}| \leq C \, \|f\|_{2p_1+2}, \quad \forall f \in S. \tag{29}$$

This special case $(\mathbb{Z}_+, 1)$ works in general.

$$\langle \xi, f \rangle_{\mathcal{H}_E} \leq \frac{1}{2} \sum_{x \in \mathbb{Z}_+} |(\xi(x) - \xi(x-1))| \, |(f(x) - f(x-1))| \tag{30}$$

$$= \frac{1}{2} \sum_{x \in \mathbb{Z}_+} \frac{1}{x^{p_1+1}} |(\xi(x) - \xi(x-1))| \, x^{p_1+1} \, |(f(x) - f(x-1))| \tag{31}$$

$$\underset{by 27}{\leq} \frac{A}{2} \sum_{x \in \mathbb{Z}_+} \frac{1}{x} x^{p_1+1} \, |(f(x) - f(x-1))| \tag{32}$$

$$\underset{\text{by Schwarz}}{\leq} \frac{A}{2} \left(\sum_{x \in \mathbb{Z}_+} \frac{1}{x^2} \right)^{1/2} \left(\sum_{x \in \mathbb{Z}_+} x^{2p_1+2} \left| f(x) - f(x-1) \right|^2 \right)^{1/2} \tag{33}$$

$$= \frac{A}{2} \left(\frac{\pi^2}{6} \right)^{1/2} \|f\|_{2p_1+2} \tag{34}$$

proves (29) where $C = \frac{A}{2} \frac{\pi^2}{\sqrt{6}}$, $\zeta(2) = \frac{\pi^2}{6}$ and (28).

□

This proves that (27) makes $f \rightsquigarrow \langle \xi, f \rangle_{\mathcal{H}_E}$ into a distribution, i.e., $\xi \in S'$. The converse are from standard functional analysis.

As in the given case, given S is Fréchet topology from the seminorms (28), we have

$$\{v_x\} \subset S \hookrightarrow \mathcal{H}_E \subset S'. \tag{35}$$

Note that (35) implies that S is dense in $\mathcal{H}_E$.

Comments:

$$p_1 = 0 : |\xi(x) - \xi(x-1)| \leq A \Rightarrow \text{linear growth}$$
$$|\xi(x)| \leq \sum_{y=1}^{x} |\xi(y) - \xi(y-1)| \leq xA$$
$$p_1 = 1 \Rightarrow |\xi(x)| \leq A \sum_{y=1}^{x} y = A \frac{x(x+1)}{2} = \mathcal{O}(x^2)$$
$$p_1 = q \Rightarrow |\xi(x)| \leq A \sum_{y=1}^{x} y^q = \mathcal{O}(x^{q+1})$$

If $p_1 = 0$, then for all $f \in S$, we have:

$$\begin{aligned}\langle \xi, f \rangle_{\mathcal{H}_E} &\leq A \sum_{x \in \mathbb{Z}_+} \frac{1}{x} x \,|f(x) - f(x-1)| \\ &\underset{\text{Schwarz}}{\leq} \frac{A}{2} \left(\frac{\pi^2}{6} \right)^{1/2} \left(\sum_{x \in \mathbb{Z}_+} x^2 \,|f(x) - f(x-1)|^2 \right)^{1/2} \\ &= \frac{A\pi}{2\sqrt{6}} \|f\|_2 \quad \text{the 2-seminorms on } S \text{, see (28).}\end{aligned}$$

Corollary 2.10. *Both S and S' are infinite dimensional.*

Proof. We also noted in (35) that $\{v_x\} \subset S$ so S is infinite dimensional, But now use (30) to conclude that

$$\{\epsilon_x\} \subset S' \tag{36}$$

Indeed, $\xi = \epsilon_{x_0}$ satisfies

$$\langle \xi, f \rangle \underset{\text{by(14)}}{=} \xi(x) - \xi(x-1) \tag{37}$$
$$= \epsilon_{x_0}(x) - \epsilon_{x_0}(x-1) \tag{38}$$
$$= 0 \quad \text{if } x > x_0 \tag{39}$$

Hence (27) is satisfied and (36) holds. □

But note that S' is still much bigger than $\mathcal{H}_E$.

2.6 Topologies

Fréchet: stronger than the norm in $\mathcal{H}_E$ nuclear embedding a la Gelfand. Dual of stronger is weaker than the norm topology. Even if Δ is bounded in this exampling, then $S \subsetneq \mathcal{H}_E$, i.e., is strictly smaller.

Recall. Computations in the example, $(\mathbb{Z}_+, \mathbf{1})$ Comment on (27). In the case $(\mathbb{Z}_+, \mathbf{1})$, we have (27) $\iff$ polynomial growth, but not in general (G, c), etc.

Proof. Clearly, (27) implies polynomial growth, and the converse is trivial.

If (27) holds, then $\exists A, p$, s.t.

$$|\xi(x) - \xi(x-1)| \leq Ax^p, \quad \forall x \in \mathbb{Z}_+ \quad (\xi(0) = 0)$$

so

$$|\xi(x)| = |(\xi(1) - \xi(0)) + (\xi(2) - \xi(1)) + \cdots + (\xi(x) - \xi(x-1))| \tag{40}$$

$$\leq A1^{p_1} + 2^{p_1} + \cdots + x^{p_1} \tag{41}$$

$$= A\sum_{y=1}^{x} y^{p_1} \simeq \mathcal{O}(x^{p_1+1}), \tag{42}$$

which is polynomial growth by $p_1 + 1$. □

2.7 Conclusions:

S' = all polynomial growth functions on $\mathbb{Z}_+$. Clearly very infinite dimensional. But the white noise measure ω is supported only on a "small" subset of S'. So in the example $(\mathbb{Z}_+, \mathbf{1})$, we get S' = all polynomially bounded sequences on $\mathbb{Z}_+$; at least a concrete answer.

3 Sampling and Kernels

Our introduction of a suitable reproducing kernel allows us, in a general framework, to offer a representation of irregular sampling; and this approach includes both traditional periodic Shannon-sampling, as well as sampling on infinite graph, i.e., sampling on vertex-points for functions defined on an ambient boundary set for a fixed infinite graph.

Below we study sampling in the context of reproducing kernel Hilbert space (RKHS.) The framwork includes the case of boundary sets derived from infinite graphs, see (21).

By a positive definite kernel, we mean a function K on a set $X \times X$ where X is fixed, and such that

$$\sum_x \sum_y c_x c_y K(x, y) \geq 0 \tag{43}$$

for all finitely supported functions $c : X \mapsto \mathbb{R}$ on the set X. While the theory works for complex valued functions, the real case suffices in ons present analysis. A discrete subset $\Gamma \subset X$ is said to be a sampling set if the following conditions hold. To state these, we introduce a reproducing kernel Hilbert space, RKHS $\mathcal{H} := \mathcal{H}(K)$ correspond to the fixed kernel K, and we set

$$K_x := K(x, \cdot) \in \mathcal{H}. \tag{44}$$

It is known that $K_x \in \mathcal{H}$, $\forall x \in X$; and that

$$f(x) = \langle f, K_x \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{H}, \quad \forall x \in X. \tag{45}$$

(The last property (45) is called the reproducing property.) Here $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the Hilbert inner-product in $\mathcal{H}$.

For $\Gamma \subset X$ to be a sampling set, we require the following two condition to hold: There exists an orthonormal basis (ONB) $\{e_n\}_{n\in\mathbb{N}}$ and a function

$$A : \mathbb{N} \times \Gamma \mapsto \mathbb{R} \tag{46}$$

such that

$$e_n = \sum_{\gamma\in\Gamma} A(n, \gamma) K_\gamma, \quad \forall n \in \mathbb{N} \tag{47}$$

with the summation in (47) convergent in the norm of $\mathcal{H}$.

Theorem 3.1. *Let K, X, and Γ be as above. Then there is a sampling kernel*

$$S : \Gamma \times X \mapsto \mathbb{R} \tag{48}$$

such that, for all $f \in \mathcal{H}$, and all $x \in X$, the following representation holds:

$$f(x) = \sum_{\gamma\in\Gamma} f(\gamma) S(\gamma, x). \tag{49}$$

So the interpolation is valid for all function $f \in \mathcal{H}$, and all points $x \in X$.

Proof. Since Γ is a sampling set for the R=KHS $\mathcal{H} = \mathcal{H}(K)$, there is an ONB $\{e_n\}_{n\in\mathbb{N}}$ satisfying (47).

Let $f \in \mathcal{H}$; then we have the following representation:

$$f(x) = \sum_{n\in\mathbb{N}} \langle f, e_n \rangle_{\mathcal{H}} e_n \tag{50}$$

with convergence in the norm of $\mathcal{H}$. This is by assuming the ONB property of $\{e_n\}_{n\in\mathbb{N}}$, and the kernel property of $\mathcal{H}$, i.e., $f(x) = \langle f, K_x \rangle_{\mathcal{H}}$.

These facts also allow us to justify the following exchange of summations: We substitute (47) inside the inner-product $\langle f, \cdot \rangle_{\mathcal{H}}$ on the RHS in (50). As a result we get:

$$\begin{aligned} f &= \sum_{n\in\mathbb{N}} \langle f, \sum_{\gamma\in\Gamma} A(n, \gamma) K_\gamma \rangle_{\mathcal{H}} e_n \\ &= \sum_{n\in\mathbb{N}} \sum_{\gamma\in\Gamma} A(n, \gamma) \langle f, K_\gamma \rangle_{\mathcal{H}} e_n \\ &= \sum_{\gamma\in\Gamma} f(\gamma) \sum_{n\in\mathbb{N}} A(n, \gamma) e_n. \end{aligned}$$

Hence, setting $S(\gamma, x) := \sum_{n\in\mathbb{N}} A(n, \gamma) e_n(x)$, the desired interpolation formular (49) follows. ∎

4 Sampling and Stochastic Process

In this section we offer certain frame estimates for our interpolation formulas. Recall that a "frame" is a generalization of expansion formulas which are better known for the case of orthonormal bases (ONBs) in Hilbert space. In short, a frame is an over-complete basis, see below. In Theorem 3.1 we offer a procedure for building frames with the use of certain Gelfand-triples. The idea here is that, if a signal is given as a function in a specific Hilbert space $\mathcal{H}$ of signals on the fixed infinite graph G, then there is an associated Gelfand triple offering an extension S' of $\mathcal{H}$ and an associated Gaussian system built on S'. This in terms allows us to make precise frame estimates for functions in $\mathcal{H}$.

$$\mathcal{H} := L^2_{\mathbb{R}}(\mathbb{R}, dx), \quad dx = \text{ Lesbegue measure on } \mathbb{R}. \tag{51}$$

$$\mathcal{K} \subset \mathcal{H} \quad \text{ a closed subspace.} \tag{52}$$

$$S_T = \{t_k\}_{k=0}^{\infty} \subset \mathbb{R} \quad \text{ some discrete set of sample points.} \tag{53}$$

Assume $(S, \mathcal{K}) : \exists A, B \in \mathbb{R}_+ \; 0 < A < B < \infty$ such that

$$A\|\varphi\|^2 \leq \sum_{k=0}^{\infty} |\varphi(t_k)|^2 \leq B\|\varphi\|^2, \quad \forall \varphi \in K \quad \text{ with } \|\varphi\|^2 = \|\varphi\|^2_{\mathcal{H}} = \int_{\mathbb{R}} |\varphi(x)|^2 dx. \tag{54}$$

Theorem 4.1. *Let (K, S_T) be a pair as specified in equations (52), (53) and (54) above, and let*

$$S \hookrightarrow \mathcal{H} \hookrightarrow S'$$

In the norm Gelfand-space $\mathcal{H} = L^2(\mathbb{R}, dx)$. Let P_K be the projection onto $K \subset \mathcal{H}$. Then

$$E_W(X(\varphi)^2) = \|P_K\varphi\|^2, \forall \varphi \in S. \tag{55}$$

More generally, $E_W(X(\varphi)X(\psi)) = \langle P_K\varphi, \psi\rangle \; \forall \; \varphi, \; \psi \in S$. Indeed, if $E_W(\cdot) := \int_{S'}(\cdots)dP_W$ is the norm expectation with respect to the Wiener measure P_W on S', then

$$E_W(e^{i\langle\varphi,\cdot\rangle}) = \int_{S'} e^{i\langle\varphi,w\rangle} dP_W(w) = e^{-\frac{1}{2}\|P_K\varphi\|^2} \tag{56}$$

Note that (56) as a special case, $K =$ band-limited functions $= \widecheck{L^2(\widetilde{-\frac{1}{2}, \frac{1}{2}}, du)} = \{\varphi \in L^2(\mathbb{R}, dx) | \varphi(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi xu}\hat{\varphi}(u)du\}$. Moreover, the product $X = X_{(\mathcal{K}, S_T)}$ is given by the formula

$$X(\varphi) = \sum_j (P_{\mathcal{K}}\varphi)(t_j)Z_j(\cdot) \tag{57}$$

where $Z_j \sim N(0, \|v_j\|^2)$ *on* v_j *is a system of vector* $v_j \in K$ *such that* $P_{\mathcal{K}}\varphi = \sum_j (P_{\mathcal{K}}\varphi)(t_j)v_j$ *holds in* $\mathcal{H} := L^2(\mathbb{R}, dx)$.

Example 4.2.

$$K = L^2(\widetilde{[-\frac{1}{2}, \frac{1}{2}]}, du) = \text{ the usual band-limited } L^2- \text{ function on } \mathbb{R};\ S = \mathbb{Z}. \tag{58}$$

Fact: In case (58), we have equal signs in both estimates in (54),

$$\sum_{k\in\mathbb{Z}} |\varphi(k)|^2 = \|\varphi\|^2 \tag{59}$$

Proof. If $\varphi \in K$, then by L^2- theory we get

$$\hat{\varphi}(u) = \sum_{k\in\mathbb{Z}} \hat{\varphi}(k)e^{-i2\pi ku} \tag{60}$$

where

$$\hat{\varphi}(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\varphi}(u)e^{i2\pi ku}du = \int_{\mathbb{R}} \varphi(y)\frac{\sin\pi(k-y)}{\pi(k-y)}dy = \langle\varphi, w_k\rangle_H = \varphi(k), k \in \mathbb{Z} \tag{61}$$

where $\{w_k\}$ is the Shannon band, and so if $\varphi \in K$ (see (58)), then

$$\|\varphi\|^2 = \sum_{k\in\mathbb{Z}} |\langle\varphi, w_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\varphi(k)|^2 \tag{62}$$

which is the described two equal signs in equation (54). □

5 Frame-estimates Project

In this and the next section we show how, via the Gelfand-triples from sect 3, functions in $\mathcal{H}$ take the form of Gaussian random variables. In this guise the frame decompositions take the form of Karhunen-Loeve expansions, i.e., expansions in a fundamental system if independent (identically distributed) standard Gaussians. Below we fixed system (Z_k) of i.i.d $N(0,1)$ random variables, and associated random variables, $\sum_k c_k Z_k$ where (c_k) are constants, or systems of functions. The system (Z_k) will refer to a fixed probability space (c', cylinder $\sigma-$algebra, $\mathbb{P}$). (K, S) in (54) is similar to, or different from Example 4.2: $d\sigma(u) = \Xi_{[-\frac{1}{2},\frac{1}{2}]}(u)du$. In Example 4.2,

$$X_\sigma(\varphi) = \sum_{k\in\mathbb{Z}} \varphi_b(k)Z_k(\cdot) \quad \forall\varphi \in S, \tag{63}$$

and

$$X(\psi) = \sum_{k \in \mathbb{Z}} \psi(k) Z_k(\cdot) \tag{64}$$

for all $\psi \in \widehat{L^2(-\frac{1}{2}, \frac{1}{2})} =: K_1 =$ the standard band-limited function. Note, in both (63) and (64), $(Z_k)_{k\in\mathbb{Z}}$ is a system of i.i.d. $N(0,1)$ Gaussian r.v.s.

6 Projection

Last for analogue of (63), expansion with respect to a fixed system (Z_k) of i.i.d. $N(0,1)$ random variables. If $P_{\mathcal{K}} :=$ the projection of $\mathcal{H} = L^2(\mathbb{R}, dx)$ write the subspace $\mathcal{K}$, then a natural extension of (26) to $(\mathcal{K}, S_T)$ as in (52)-(53) can be written in

$$X(\varphi) = \sum_j (P_{\mathcal{K}}\varphi) Z_j(\cdot), \quad \forall \varphi \in S \tag{65}$$

where (Z_j) is as before, be an independent $N(0,1)$ system.

7 Closer Look at Systems

$(\mathcal{K}, S)$ as in (52) and (53) for which (54) is assumed. Note: (54) $\Rightarrow \exists w_1, w_2, \cdots \in K$ (by Riesz) such that

$$\varphi(t_k) = \langle \varphi, w_k \rangle \tag{66}$$

where $\langle \cdot, \cdot \rangle$ is the $H = L^2(\mathbb{R}, dx)$ inner-product. In Example 4.2, (w_k) is the Shannon system, but in general $\{w_k\}$ in (66) is only determined abstractly. Now substitute (66) $\rightarrow$ (54), to get a frame estimate

$$A\|\varphi\|^2 \leq \sum_{k=0}^{\infty} |\langle \varphi, w_k \rangle|^2 \leq B\|\varphi\|^2, \quad \forall \varphi \in \mathcal{K}. \tag{67}$$

Lemma 7.1. *Let $\mathcal{A} : \mathcal{K} \to l^2(\mathbb{N}_0)$ be the operator $\mathcal{A}_\varphi = (\varphi(t_k))_k = (\langle \varphi, w_k \rangle)_k \in l^2(\mathbb{N}_0)$ then $\mathcal{A}^*\mathcal{A}$ is invertible of on*

$$\mathcal{A}^*\mathcal{A}\varphi = \sum_{k=0}^{\infty} \langle \varphi, w_k \rangle w_k \tag{68}$$

Proof. Starting in the theory of frames. A system as in (67) is called a frame. Set

$$v_k = (\mathcal{A}^*\mathcal{A})^{-1} w_k, \tag{69}$$

then

$$\varphi = \sum_{k=0}^{\infty} \langle \varphi, v_k \rangle w_k = \sum_{k=0}^{\infty} \langle \varphi, w_k \rangle v_k \tag{70}$$

and (70) is called a dual frame system for the closure subspace $K \subset \mathcal{H} = L^2(\mathbb{R}, dx)$.

Setting $Z_k(\cdot) = \tilde{v}_k(\cdot) \in L^2(S', P_W)$ via on usual pair $S \leftrightarrow S'$ are its extension to $\mathcal{H} \leftrightarrow S'$ $\tilde{h}(w) = \langle h, w \rangle$, $\forall w \in S'$. Then it can be proven

$$X_{\mathcal{K}}(\varphi) = \sum_{j=0}^{\infty} (P_K\varphi)(t_j) Z_j(\cdot). \tag{71}$$

Consider formulas (66), (69), (70). We use the second expansion on RHS (69).

$$\mathbb{E}_W(\cdot) = \int_{S'} dP_W$$

We get

$$\mathbb{E}_W(Z_j Z_k) = \langle v_j, v_k \rangle_{\mathcal{H}} \quad \forall j, k \in \mathbb{N}_0. \tag{72}$$

To prove lemma, we only need to show that

$$\mathbb{E}_W(X(\varphi)^2) = \|P_K\varphi\|^2, \quad \forall \varphi \in S(\text{the usual Schwartz space}). \tag{73}$$

Proof. of (73) on theory of Lemma 7.1

$$\begin{aligned}
E_W(X(\varphi)^2) &= \int_{S'} X(\varphi)^2 dP_W \\
&\underset{\text{by (40)}}{=} \sum_j \sum_k P_{\mathcal{K}}\varphi(t_j) P_K\varphi(t_k) E_W(Z_j Z_k) \\
&= \sum_j \sum_k P_K\varphi(t_j) P_{\mathcal{K}}\varphi(t_k) E_W(\langle v_j \rangle \langle v_k \rangle) \\
&= \sum_j \sum_k P_K\varphi(t_j) P_{\mathcal{K}}\varphi(t_k) E_W(\tilde{v}_j \tilde{v}_k) \\
&= \sum_j \sum_k P_{\mathcal{K}}\varphi(t_j) P_K\varphi(t_k) E_W(\langle v_j, v_k \rangle) \\
&= \langle \sum_j P_K\varphi(t_j) v_j, \sum_k P_{\mathcal{K}}\varphi(t_k) v_k \rangle \\
&= \left\| \sum_j P_{\mathcal{K}}\varphi(t_j) v_j \right\|^2 \\
&\underset{\text{by (30)}}{=} \left\| \sum_j \langle P_{\mathcal{K}}\varphi, w_j \rangle v_j \right\|^2 \\
&\underset{\text{since } w_j \in \mathcal{K}}{=} \left\| \sum_j \langle \varphi, w_j \rangle v_j \right\|^2 \\
&\underset{\text{by (37)}}{=} \|P_{\mathcal{K}}\varphi\|^2
\end{aligned}$$

which is the desired form (51). Then obtain formula (52) followed by the stated in Minlos property. □

In order to apply Minlos' theorem, we then get $S \ni \varphi \longrightarrow \|P_{\mathcal{K}}\varphi\|^2$ is contained in the Fréchet tolopology of S. But the following from the formular

$$P_{\mathcal{K}}\varphi = \sum_j (P_{\mathcal{K}}\varphi)(t_j)v_j$$

which we determined in (70) proof above.

$$\|P_{\mathcal{K}}\varphi\|^2 \underset{see(54)}{\leq} \frac{1}{A}\sum_j |P_{\mathcal{K}}\varphi(t_j)|^2 \leq \text{ Const.}$$

So continuous in the Fréchet topology on S'

□

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