Uncertainty Analysis

Matthew S. Norton

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In order to deal with uncertainty analysis, we have the annoying equation on page A-9 of your lab book

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\Delta x_2\right)^2} \tag{1}$$

Now, $\frac{\partial f}{\partial x}$ is a partial derivative. A partial derivative is just like a regular derivative, except that you leave everything that is not the variable that you are taking the derivative with respect to, constant. For example, if the function you are interested in is $f(x,y) = 2x^2y^3$, Now

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\left(2x^2y^3\right) = 2 \cdot y^3 \cdot \frac{d}{dx}\left(x^2\right) = 2 \cdot y^3 \cdot 2 \cdot x = 4xy^3 \tag{2}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(2x^2y^3\right) = 2 \cdot x^2 \cdot \frac{d}{dy}\left(y^3\right) = 2 \cdot x^2 \cdot 3 \cdot y^2 = 6x^2y^2 \tag{3}$$

Now that we understand partial derivatives, let's do an example for finding the uncertainty using this method. Let's say, for example. that we are calculating π by measuring the circumference and diameter of a disk. If we measure both with a meter stick, and we assume that, we can measure to one half of the smallest marking on the meter stick. Most meter sticks go to 1mm, so we'll assume that our uncertainty is ± 0.5 mm. So, for our uncertainty equation, $f = \pi = C/d$, and $\Delta x_1 = \Delta x_2 = 0.005$. Taking this, we can calculate the uncertainty in π

$$\Delta\pi = \sqrt{\left(\frac{\partial f}{\partial C}\Delta C\right)^2 + \left(\frac{\partial f}{\partial d}\Delta d\right)^2} \tag{4}$$

$$\Delta\pi = \sqrt{\left[\left(\frac{1}{C}\right)\Delta C\right]^2 + \left[\left(-\frac{C}{d^2}\right)\Delta d\right]^2} \tag{5}$$

$$\Delta\pi = \frac{\sqrt{d^4\left(\Delta C\right)^2 + C^4\left(\Delta d\right)^2}}{Cd^2} \tag{6}$$

This is as far as we can go without using any numbers, so let's say that we measure the circumference to be 31 cm and the diameter to be 10 cm. Now by using these values, we can get

$$\Delta\pi = \frac{\sqrt{(0.10m)^4\left(5\times10^{-4}m\right)^2 + (0.31m)^4\left(5\times10^{-4}m\right)^2}}{(0.31m)\,(0.10m)^2} \tag{7}$$

This leads to a value of $\Delta\pi$ =0.016. So, the value of π that you would report is $\pi = 3.1\pm0.016$. Notice that before I did any calculations, I converted everything into the SI base units. It is important that you do this so you can make sure you are comparing the correct thing. Unless specifically told to do so, you should convert every measurement to the SI base units, meters, kilograms, seconds, etc. before using that measurement.

Fortunately, for most cases you will encounter this semester, you can use a few simple relations to obtain the uncertainty.

Addition or Subtraction

$$f\left(x_1, x_2, x_3\right) = x_1 \pm x_2 \pm x_3 \tag{8}$$

The resulting uncertainty can be found by

$$\Delta f = \sqrt{\left(\Delta x_1\right)^2 + \left(\Delta x_2\right)^2 + \left(\Delta x_3\right)^2} \tag{9}$$

Multiplication or Division

For a function of the form

$$f\left(x_1, x_2, x_3\right) = \frac{x_1 \cdot x_2}{x_3} \tag{10}$$

The resulting uncertainty can be found with

$$\Delta f = f \cdot \sqrt{\left(\frac{\Delta x_1}{x_1}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2 + \left(\frac{\Delta x_3}{x_3}\right)^2} \tag{11}$$

Powers

For a function of the form

$$f(x) = x^k \tag{12}$$

Where k is a positive or negative numbers. The resulting uncertainty can be found with

$$\Delta f = f \cdot \left| k\frac{\Delta x}{x} \right| \tag{13}$$

Multiplication or Division with Powers

For a function of the form

$$f\left(x_1, x_2, x_3\right) = \frac{x_1^k \cdot x_2^m}{x_3^n} \tag{14}$$

Where k, m, and n, are positive or negative numbers. The resulting uncertainty can be found with

$$\Delta f = f \cdot \sqrt{\left(k \cdot \frac{\Delta x_1}{x_1}\right)^2 + \left(m \cdot \frac{\Delta x_2}{x_2}\right)^2 + \left(n \cdot \frac{\Delta x_3}{x_3}\right)^2} \tag{15}$$

Now, we should be able to use these methods to obtain the same result as we did with the partial differential equation earlier. For our uncertainty equation, $f = \pi = C/d$, and $\Delta x_1 = \Delta x_2 = 0.5$. Taking this, we can calculate the uncertainty in π

$$\begin{aligned}
\Delta\pi &= \pi \cdot \sqrt{\left(\frac{\Delta C}{C}\right)^2 + \left(\frac{\Delta d}{d}\right)^2} && (16)\\
\Delta\pi &= \pi \cdot \sqrt{\left(\frac{5 \times 10^{-4}}{0.31}\right)^2 + \left(\frac{5 \times 10^{-4}}{0.1}\right)^2} && (17)\\
\Delta\pi &= \pi \cdot \sqrt{(0.0016129)^2 + (0.005)^2} && (18)\\
\Delta\pi &= \pi \cdot \sqrt{2.760 \times 10^{-5}} && (19)\\
\Delta\pi &= 0.016 && (20)
\end{aligned}$$

So, we get the same answer both ways, $\pi = 3.1 \pm 0.016$. This shows us that we can use both methods to obtain the same result. You may be asking yourself, "why do we deal with that nasty equation with partial derivatives then?" Well. These simplified versions cannot be used in some cases. The example used in the lab book is where a variable appears in multiple locations in the equation. For example,

$$f\left(x_1, x_2, x_3\right) = \frac{x_1 - x_3}{x_1 + x_2} \tag{21}$$

Cannot be simplified to use one of the above examples since x_1 appears in both the numerator and the denominator. Another example where one of these simplified equations cannot be used is if the function depends on its variables nonlinearly. An example of this occurs in lab 10, where the uncertainty analysis involves a function with a $\log_{10}$ in it. Uncertainty analysis is tricky and if it gives you problems, let me know and I will work it out with you.