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PHY 220 Computational Assignment 2
2-D Projectile Motion with Drag
November 15, 2006

Abstract
This assignment, 2-D projectile motion with drag, taught how to use the Runge-Kutta method of solving differential equations with four coupled differential equations. We studied the range of the projectile with both no drag and with linear drag. In doing this experiment, I found for an initial velocity of 100m/s and with no drag, the maximum range was found when the projectile was fired at an angle of º. I then added linear drag with a value of the drag coefficient, κ, being 0.1, 0.01, and 0.001. I found that for κ = 0.1, the maximum range was obtained when the projectile was fired at an angle of º, for κ=0.01, the angle was º, and for κ=0.001, the angle was º.

Introduction
This experiment was designed to further the class discussion on two-dimensional motion of a projectile with no drag and linear drag. For the case with no drag, the differential equations for the motion of the projectile can be given by

$$\frac{dx}{dt} = v_x \quad ; \quad \frac{dv_x}{dt} = 0, \qquad \frac{dy}{dt} = v_y \quad ; \quad \frac{dv_y}{dt} = -g. \tag{1}$$

Once we taken into account linear drag, these equations become,

$$\frac{dx}{dt} = v_x \quad ; \quad \frac{dv_x}{dt} = -\kappa \cdot v_x, \qquad \frac{dy}{dt} = v_y \quad ; \quad \frac{dv_y}{dt} = -g - \kappa \cdot v_y. \tag{2}$$

Finally, we used the weak drag approximation to find a numerical solution for the range of a projectile. The range is given by

$$R = \frac{v_0^2}{g} \cdot \sin(2 \cdot \alpha) - \frac{4 \cdot \kappa \cdot v_0^3}{3 \cdot g^2} \sin(\alpha) \cdot \sin(2 \cdot \alpha) \tag{3}$$

Results and Discussion
Table 1 shows the data for range of a launch angle of 15°, 30°, 45°, 60°, and 75°.

Angle (in degrees)	Range (in m)
15	511.941
30	892.007
45	1025.300
60	885.000
75	512.462

Table 1. Data for range of projectile for selected angles with no drag

Figure 1 shows the trajectory for an initial velocity of 100m/s with a launch angle of 15°, 30°, 45°, 60°, and 75°. Notice how the maximum range for these selected angles is obtained when the projectile is launched from an angle of 45°. Notice how there appears to be a mirroring effect with launch angles equidistant from 45° producing almost the same range. Notice in Table 1, how the range for a launch angle of 15° is almost identical to the range of the launch angle of 75°. They are both about 512 m.

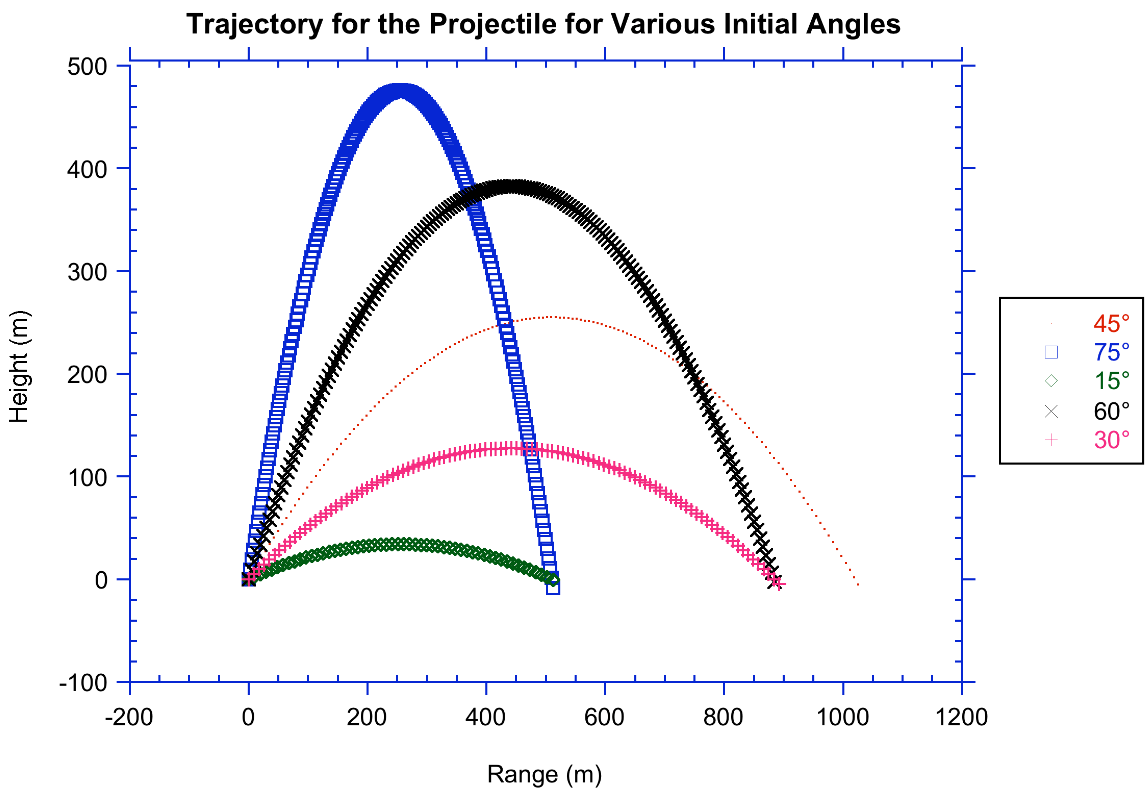

Fig. 1. Trajectory for certain initial angles for an initial velocity of 100m/s.

Table 2 shows data for angles between 1° and 89° with the angle being incremented by 1° each time. Notice how the maximum range occurs when the launch angle is 45°. In general, the range increased as the angle increased until the angle reached 45°. Then they tended to decrease after 45°. Figure 2 shows the graph of the launch angle. Notice how the plot is a parabola with its maximum value at approximately 45°. However, there was a slight increase from 46° to 47°. In an attempt to find out why this occurred, I changed the amount the angle incremented between 46° and 48° to 0.01° instead of 1°. Figure 3 shows the graph of the range versus the notice how the value for the range does fluctuate, but has a definite downward trend. This fluctuation in the values for the range is caused by the Runge-Kutta approximation.

Angle (°)	Range (m)	Angle (°)	Range (m)	Angle (°)	Range (m)
1	39.994	31	908.6	61	867.81
2	79.951	32	924.37	62	849.74
3	109.85	33	939.31	63	826.26
4	149.63	34	953.39	64	806.6
5	179.32	35	966.6	65	781.85
6	218.79	36	970.82	66	760.6
7	248.14	37	982.32	67	734.57
8	287.18	38	992.89	68	711.75
9	316.06	39	1002.5	69	684.48
10	354.53	40	1011.2	70	656.68
11	382.83	41	1011.3	71	628.35
12	420.6	42	1018.1	72	602.58
13	448.21	43	1023.9	73	573.05
14	485.15	44	1021.5	74	543.01
15	511.94	45	1025.3	75	512.46
16	547.92	46	1021.1	76	481.42
17	573.78	47	1023	77	447.65
18	608.68	48	1017.1	78	415.82
19	633.5	49	1016.9	79	383.53
20	657.78	50	1009.2	80	349.03
21	690.85	51	1000.6	81	316
22	713.93	52	991.22	82	282.52
23	736.4	53	980.96	83	247.39
24	767.38	54	975.72	84	212.19
25	788.49	55	963.61	85	177.8
26	808.91	56	950.63	86	142.3
27	828.64	57	936.78	87	106.77
28	847.63	58	922.06	88	71.195
29	865.87	59	901.32	89	35.777
30	892.01	60	885		

Table. 2. Range vs. Launch angle with no drag for launch angles between 1° and 89°

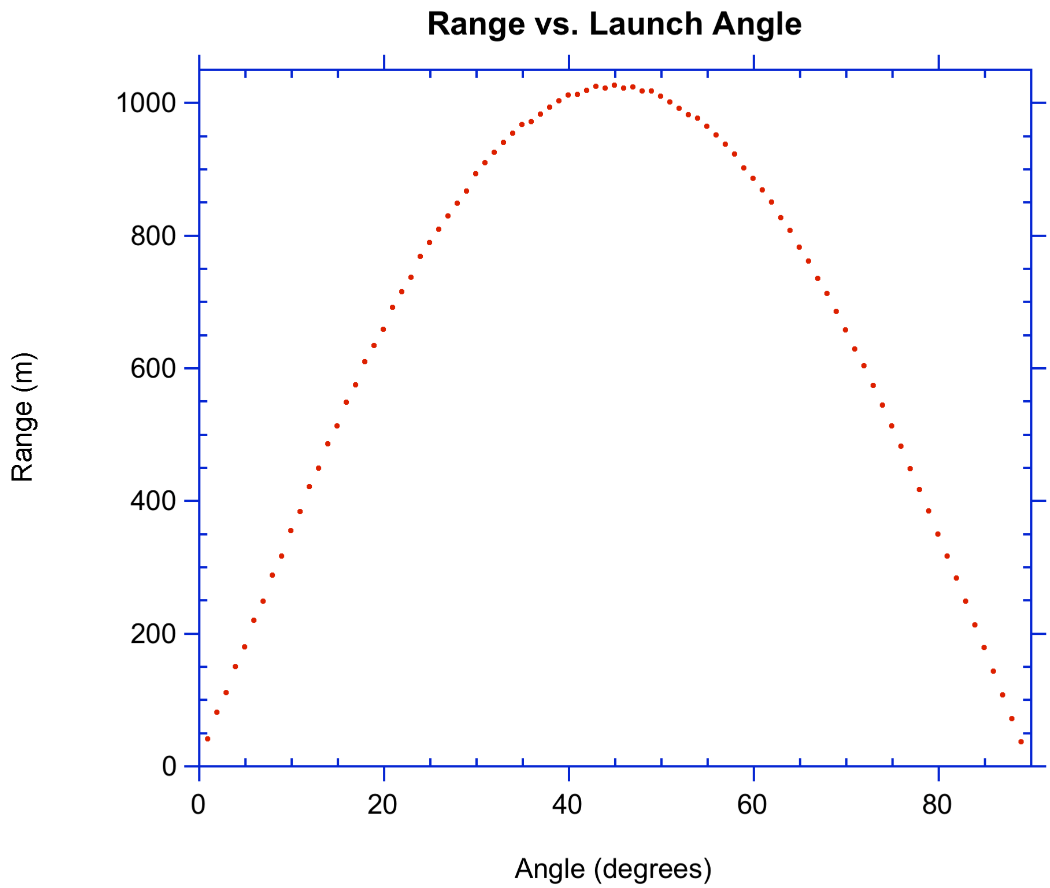

Fig. 2. Range and launch angle with no drag for angles between 1° and 89°

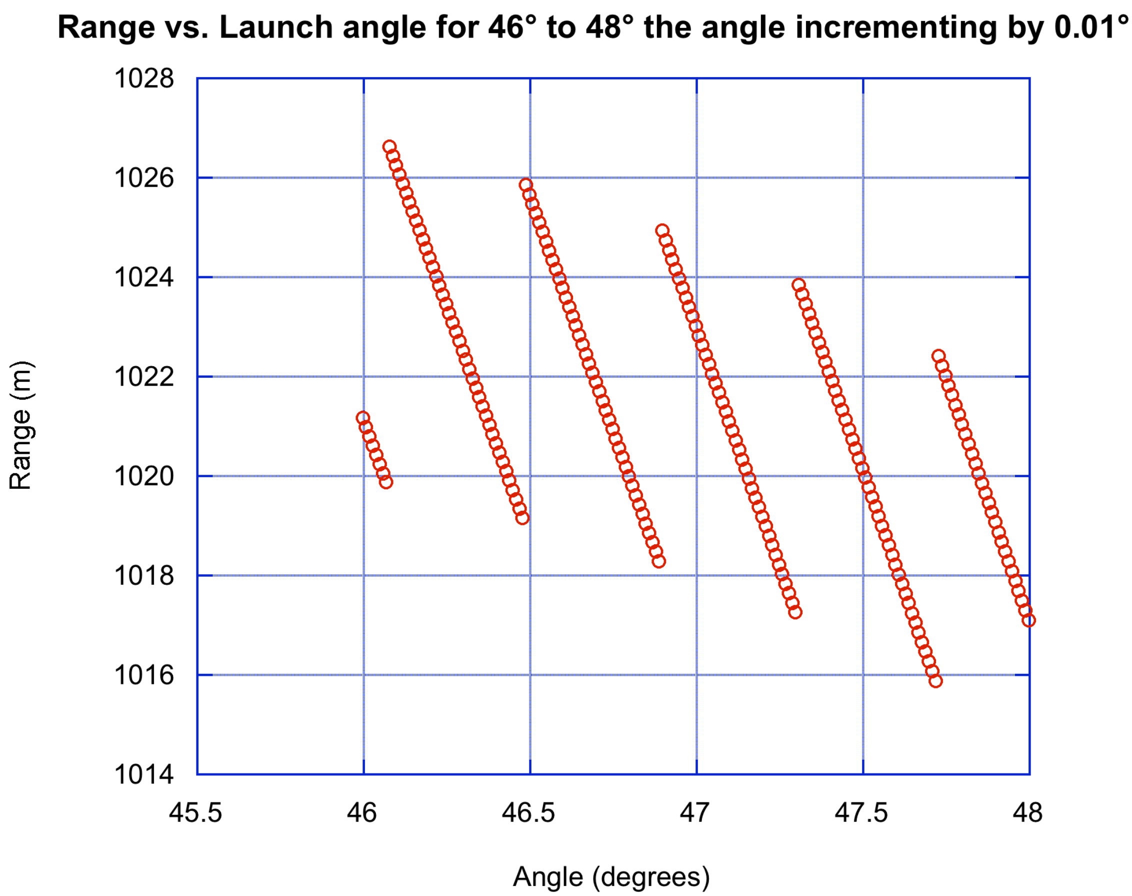

Fig. 3. Zoomed in on the range between a launch angle of 46° and 48° with an increment of 0.01°

At this point in the experiment, I added linear drag to my equations. Table 3 shows the data for range of a projectile for a launch angle of 15°, 30°, 45°, 60°, and 75° with a drag coefficient, κ, of 0.01.

Launch Angle (°)	Range (m)
15	498.611
30	831.965
45	929.921
60	790.104
75	452.139

Table 3. Data for range of projectile for selected angles with κ=0.01

Figure 4 shows the graph for the range vs. the launch angle with κ=0.01. Notice how of the selected angles, 45° still produces the longest range, however, there is no longer a mirroring of the ranges, with angles that are equidistant from 45° no longer produce the same range.

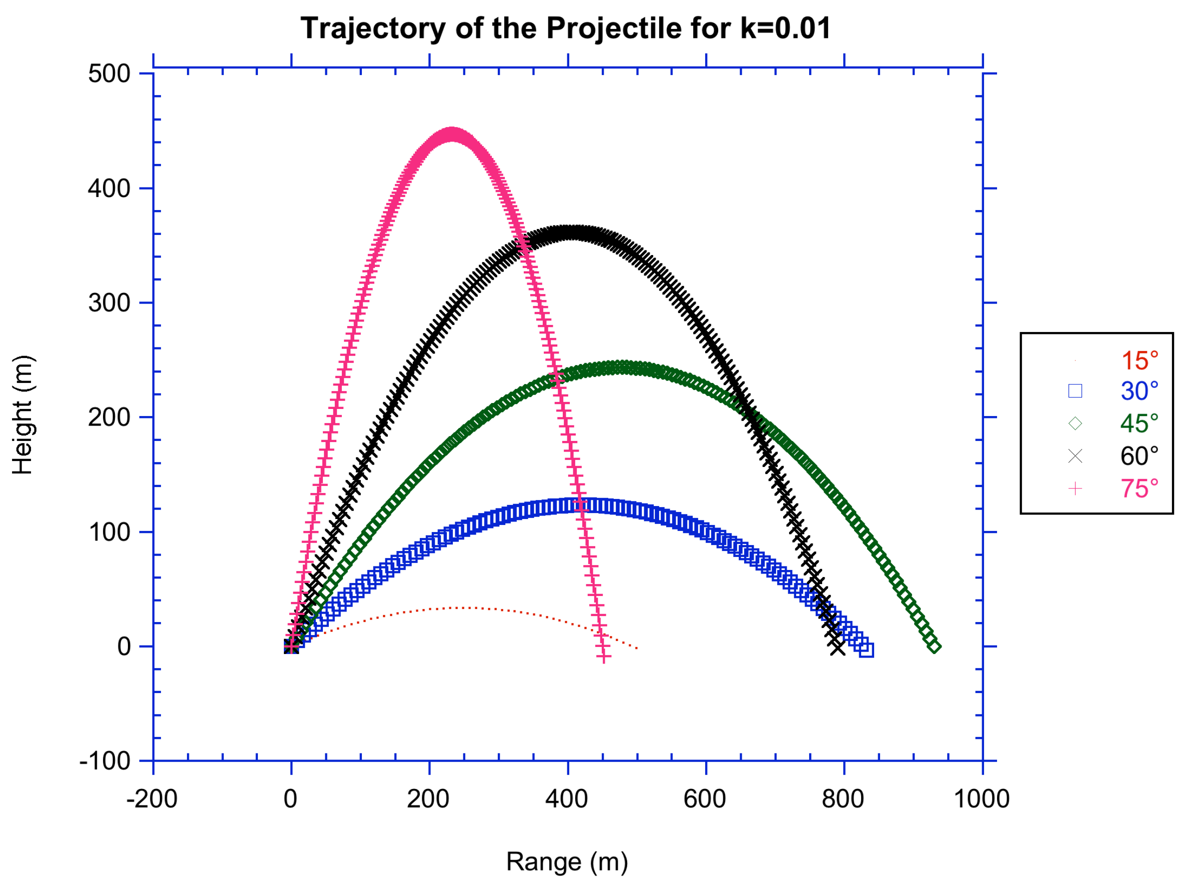

Figure 4. Trajectory for certain initial angles for κ=0.01

Table 4 shows data for angles between 1° and 89° and κ=0.01 with the angle being incremented by 1° each time. Notice how the maximum range occurs when the launch angle is 43°. In general, the range increased as the angle increased until the angle reached 43°. Then they tended to decrease after 43°. Figure 5 shows the graph of the launch angle. Notice how the plot is a parabola with its maximum value at approximately 43°.

Angle (°)	Range (m)	Angle (°)	Range (m)	Angle (°)	Range (m)
1	39.914024	31	846.6648	61	774.256
2	79.63232	32	860.5511	62	757.64282
3	109.2473	33	873.60852	63	736.46411
4	148.51791	34	885.8212	64	718.46472
5	177.71083	35	889.8766	65	696.1781
6	216.40562	36	900.4681	66	673.4122
7	245.06054	37	910.17352	67	653.42724
8	283.05361	38	918.97973	68	629.57983
9	311.0569	39	920.025	69	605.268
10	348.22512	40	927.11254	70	583.3367
11	375.46551	41	933.2666	71	557.9761
12	411.6888	42	931.97772	72	532.16894
13	438.05764	43	936.3504	73	505.9226
14	463.9844	44	933.5014	74	479.2438
15	498.6109	45	929.9206	75	452.1393
16	523.51141	46	931.62243	76	422.6209
17	556.9086	47	926.44213	77	394.8297
18	580.682	48	926.28332	78	366.6359
19	603.8946	49	919.48193	79	338.0476
20	635.2903	50	911.9246	80	307.64443
21	657.233	51	903.60821	81	278.43423
22	678.5483	52	894.53082	82	247.71113
23	699.2162	53	884.6903	83	217.9137
24	719.21661	54	874.08312	84	186.90661
25	746.8501	55	862.70941	85	155.8428
26	765.36273	56	850.5675	86	125.30374
27	783.1489	57	837.65704	87	94.01121
28	800.1896	58	823.9774	88	62.69002
29	816.4672	59	809.52844	89	31.34975
30	831.96453	60	790.1043		

Table 4. Range and launch angle for κ=0.01 for angles between 1° and 89°

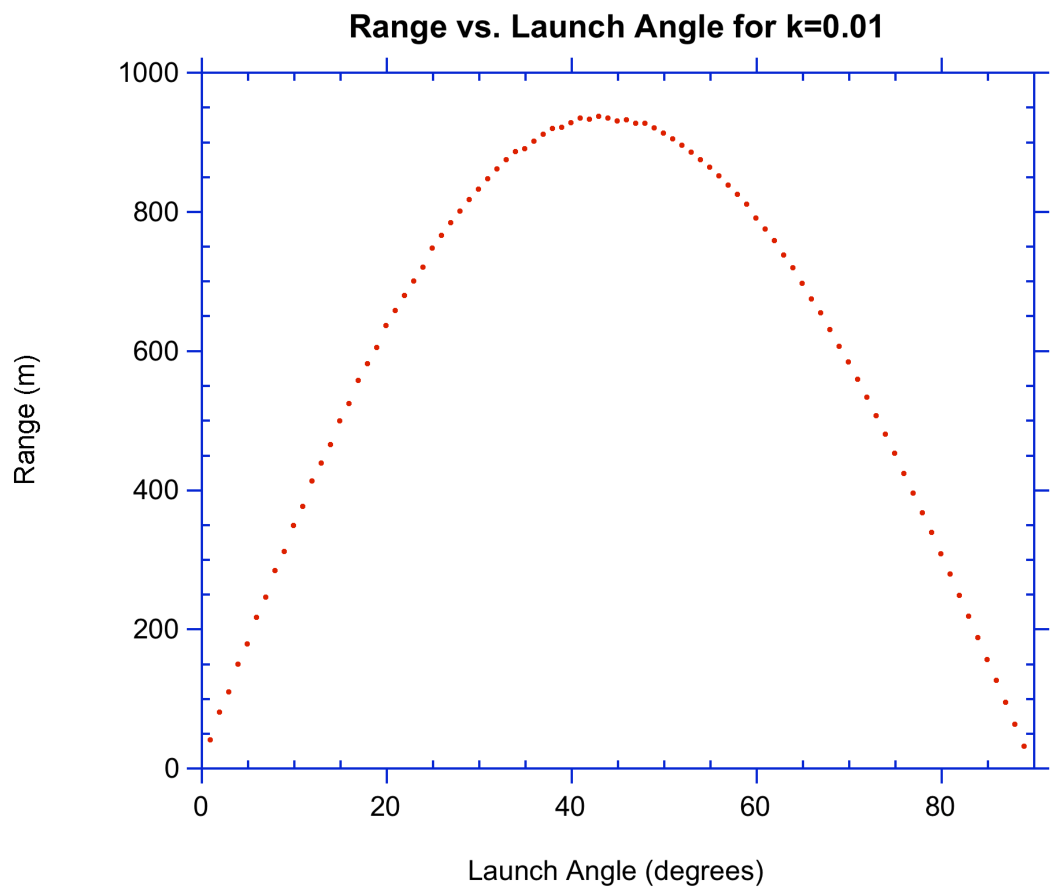

Fig. 5. Range and launch angle for κ=0.01 for angles between 1° and 89°

Now, I switched to higher value of the drag coefficient to see how that affected the range. Table 5 shows the data for range of a projectile for a launch angle of 15°, 30°, 45°, 60°, and 75° with a drag coefficient, κ, of 0.1.

Launch Angle (°)	Range (m)
15	374.174
30	513.926
45	496.25
60	381.536
75	205.509

Table 5. Data for range of projectile for selected angles with κ=0.1

Figure 6 shows the graph for the range vs. the launch angle with κ=0.1. Notice how of the selected angles, 45° still produces the longest range, however, there is no longer a mirroring of the ranges, with angles that are equidistant from 45° no longer produce the same range

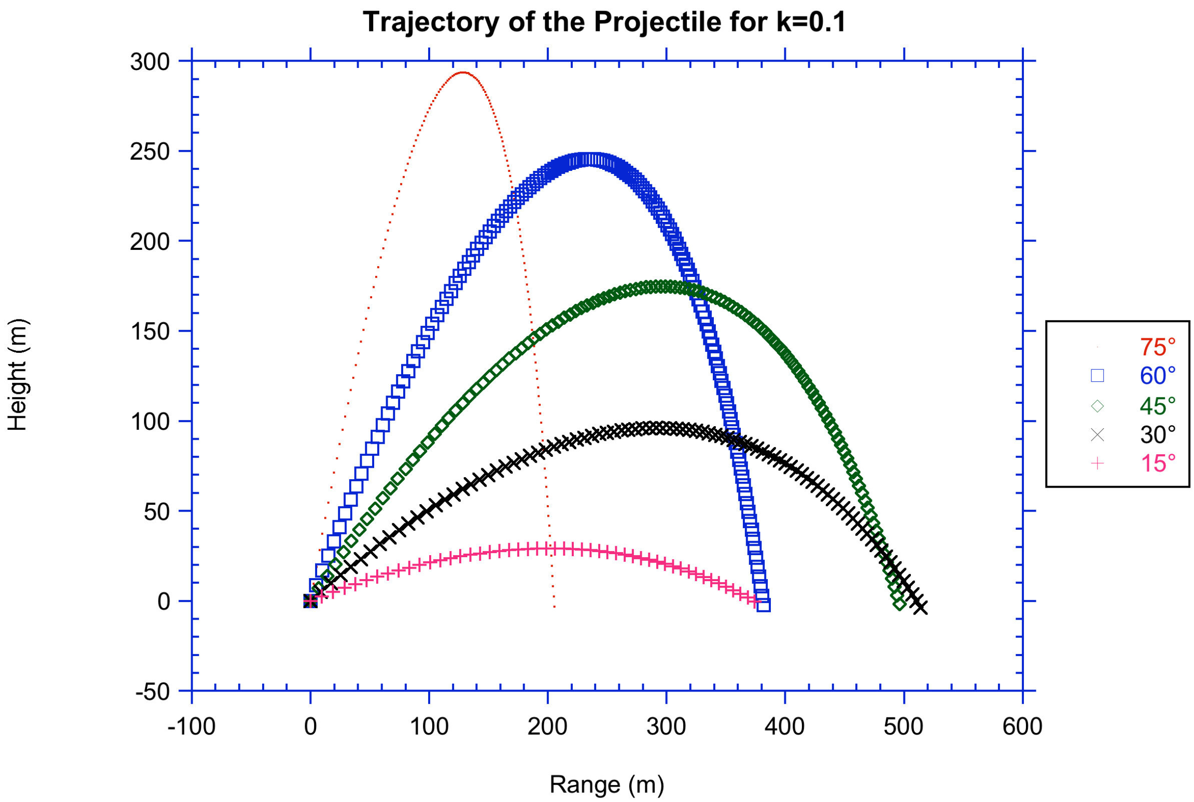

Fig. 6. Range vs. Launch angle for selected angles with κ=0.1

Table 6 shows data for angles between 1° and 89° and κ=0.1 with the angle being incremented by 1° each time. Notice how the maximum range occurs when the launch angle is 34°. In general, the range increased as the angle increased until the angle reached 34°. Then they tended to decrease after 34°. Figure 7 shows the graph of the launch angle. Notice how the plot is a parabola with its maximum value at approximately 34°.

Angle (°)	Range (m)	Angle (°)	Range (m)	Angle (°)	Range (m)
1	39.20459	31	515.56982	61	371.0877
2	76.836822	32	516.7768	62	360.4434
3	104.02312	33	520.7447	63	350.6451
4	130.32354	34	520.98651	64	339.57421
5	164.10292	35	520.8016	65	328.31924
6	188.37812	36	520.1923	66	316.88442
7	211.7817	37	519.16284	67	305.2736
8	241.83953	38	517.7163	68	293.491
9	263.2714	39	518.45654	69	280.7686
10	283.8509	40	516.0996	70	268.69744
11	303.5836	41	513.33923	71	256.46701
12	322.4757	42	507.8377	72	244.0814
13	340.5335	43	504.3656	73	231.5452
14	357.76394	44	500.50131	74	218.2924
15	374.17422	45	496.24951	75	205.50883
16	389.7719	46	491.61471	76	192.58781
17	404.565	47	486.6026	77	179.07774
18	418.56161	48	479.3291	78	165.9351
19	431.7703	49	473.65032	79	152.28541
20	444.1998	50	467.6074	80	138.93814
21	451.058	51	459.5172	81	125.1652
22	462.13061	52	452.8332	82	111.35411
23	472.4471	53	444.2323	83	97.75151
24	477.68014	54	436.9237	84	83.842361
25	486.67544	55	427.8267	85	69.9078
26	494.93991	56	419.9118	86	55.95182
27	498.57894	57	410.3328	87	41.978752
28	505.56253	58	400.5431	88	28.061643
29	508.19653	59	391.78482	89	14.03295
30	513.9258	60	381.53601		

Table 6. Range and launch angle for $\kappa=0.1$ for angles between 1° and 89°

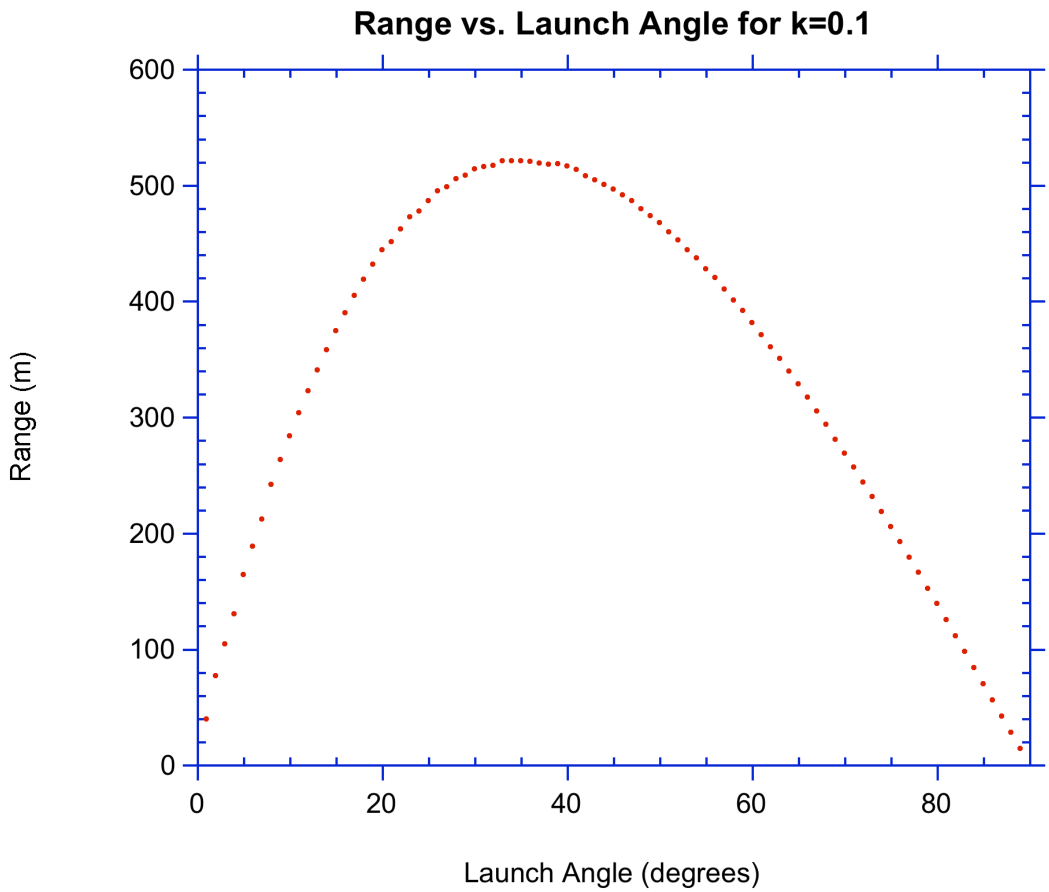

Fig. 7. Range and launch angle for κ=0.1 for angles between 1° and 89°

Now, I switched to lower value of the drag coefficient to see how that affected the range. Table 7 shows the data for range of a projectile for a launch angle of 15°, 30°, 45°, 60°, and 75° with a drag coefficient, κ, of 0.001.

Angle (°)	Range (m)
15	510.586
30	878.856
45	1010.94
60	877.214
75	504.884

Table 7. Data for range of projectile for selected angles with κ=0.001

Figure 8 shows the graph for the range vs. the launch angle with κ=0.001. Notice how of the selected angles, 45° again produces the longest range, in addition, the mirroring of the ranges, with angles that are equidistant from 45° returns.

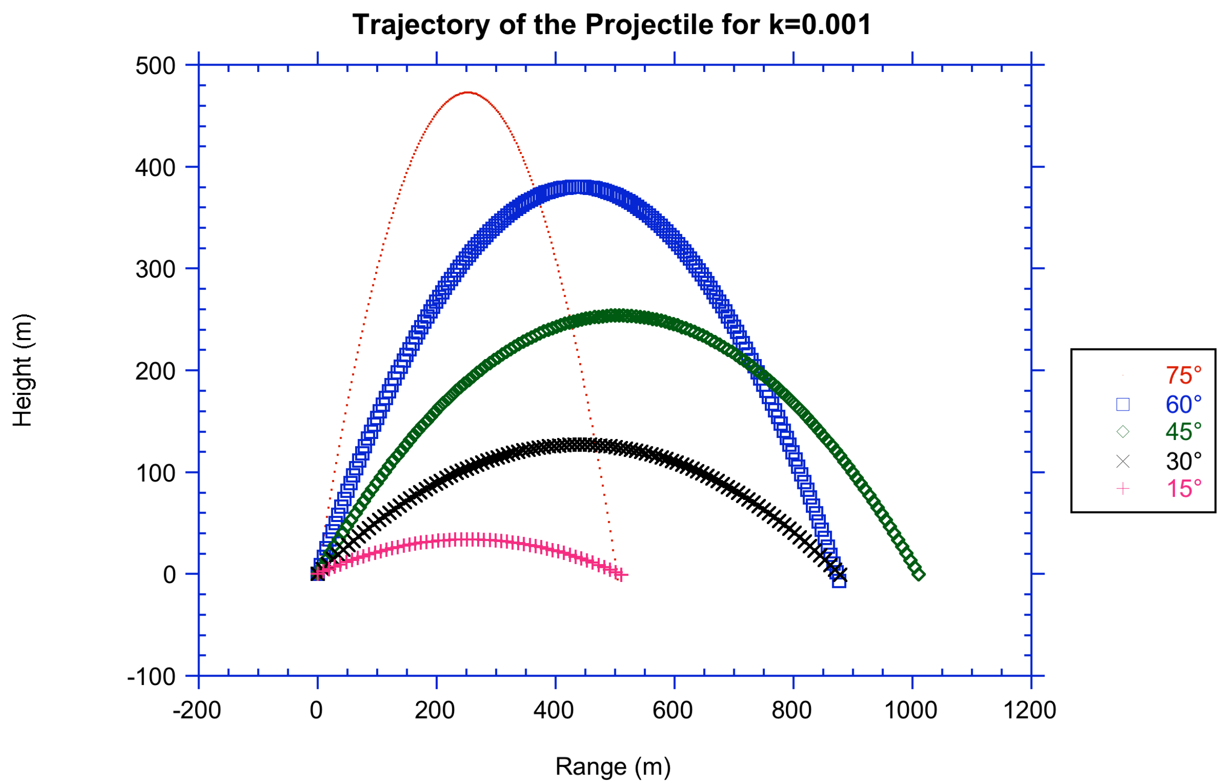

Fig. 8. Range vs. Launch angle for selected angles with κ=0.001

Table 8 shows data for angles between 1° and 89° and κ=0.001 with the angle being incremented by 1° each time. Notice how the maximum range occurs when the launch angle is 44°. In general, the range increased as the angle increased until the angle reached 44°. Then they tended to decrease after 34°. Figure 9 shows the graph of the launch angle. Notice how the plot is a parabola with its maximum value at approximately 44°.

Angle (°)	Range (m)	Angle (°)	Range (m)	Angle (°)	Range (m)
1	39.985912	31	895.3169	61	855.3262
2	79.9193	32	910.9641	62	837.48901
3	109.7889	33	925.7768	63	818.789
4	149.52244	34	939.7364	64	794.9236
5	179.1538	35	952.823	65	774.6561
6	218.55433	36	965.0188	66	749.5378
7	247.8267	37	976.3049	67	727.7127
8	286.76171	38	986.6646	68	701.3578
9	315.5551	39	996.0798	69	674.47161
10	353.89343	40	996.9737	70	650.4147
11	382.08911	41	1004.5651	71	622.322
12	419.70043	42	1011.166	72	593.7155
13	447.1809	43	1009.5491	73	564.60241
14	483.937	44	1014.2441	74	534.9893
15	510.5865	45	1010.938	75	504.8841
16	546.36053	46	1013.6792	76	474.29434
17	572.0649	47	1008.645	77	443.2278
18	597.28204	48	1009.388	78	409.65463
19	631.37994	49	1002.591	79	377.8272
20	655.48803	50	994.96752	80	345.5486
21	688.2997	51	992.70642	81	311.29434
22	711.19	52	983.2781	82	278.3093
23	733.4663	53	973.007	83	243.70631
24	755.1047	54	961.8884	84	210.05342
25	785.0679	55	949.91882	85	175.1426
26	805.2855	56	937.0951	86	140.1782
27	824.7948	57	923.4151	87	105.6838
28	843.57403	58	908.87591	88	70.47363
29	861.6017	59	893.4757	89	35.24218
30	878.8561	60	877.2136		

Table 8. Range and launch angle for $\kappa=0.001$ for angles between 1° and 89°

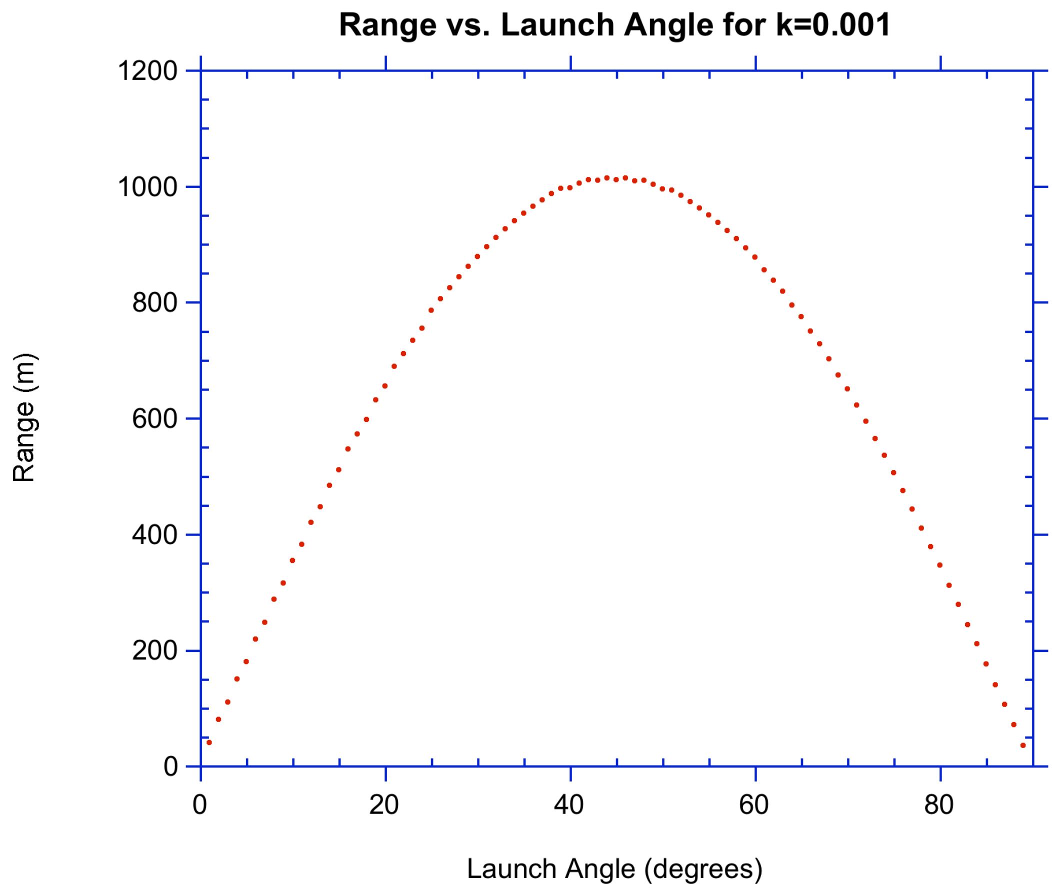

Fig. 9. Range and launch angle for κ=0.001 for angles between 1° and 89°

Now, I switched to trying out a numerical approximation of the drag on the projectile by using equation (3). Figure 10 shows the graph of the numerical approximation for a drag coefficient of κ = 0.01. In an attempt to find how the numerical solution varied with the data found with the Runge-Kutta method, I found the difference between the values each approximation found for each launch angle. Figure 11 shows the graph of the difference between the ranges found by the numerical solution and the Runge-Kutta approximation. Notice how the difference does not remain constant, nor is there a noticeable trend. The least error occurs at a launch angle of 13º with an error of 0.06 and the most error occurs at a launch angle of 81º with an error of 32.39. Figure 12 shows the graph of the percent difference between the numerical solution and the Runge-Kutta in relation to the launch angle. Notice how the percent difference is lowest for angles between 20º and 40º. In addition, the maximum percent difference occurred at a launch angle of 89º, with the numerical solution showing a range of 1.08×10^{-13} m and the Runge-Kutta approximation showing a range of 31.35 m.

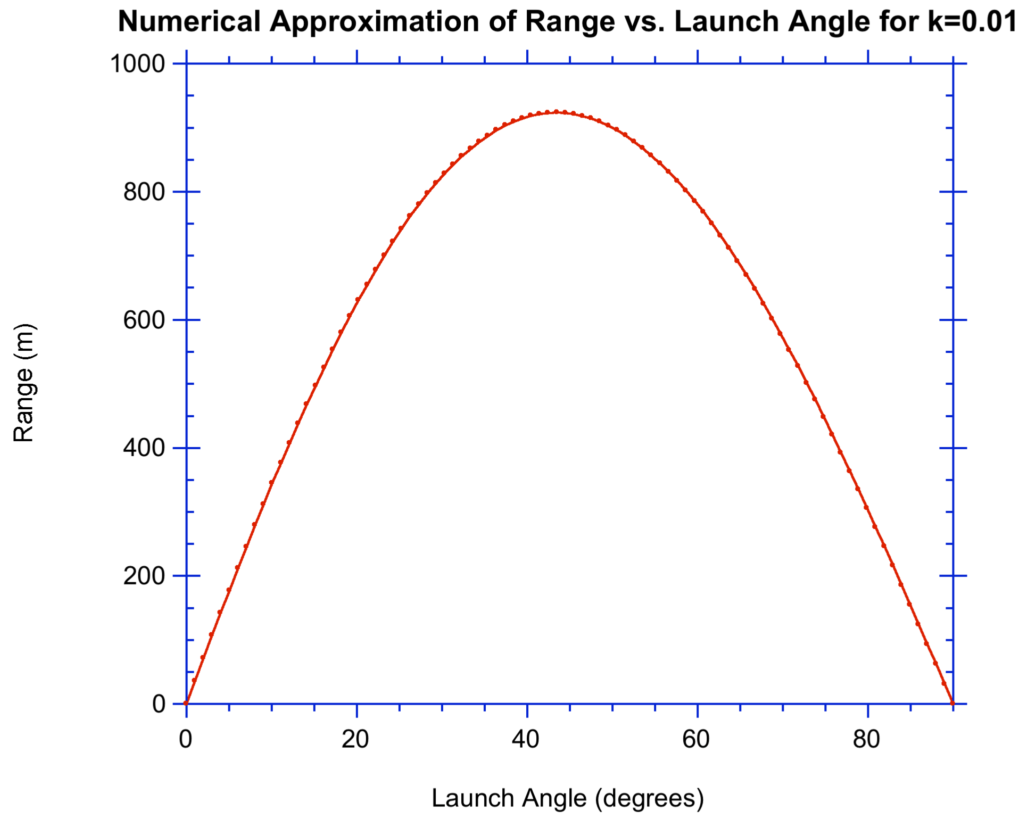

Fig. 10. Range vs. launch angle for the numerical approximation for linear drag

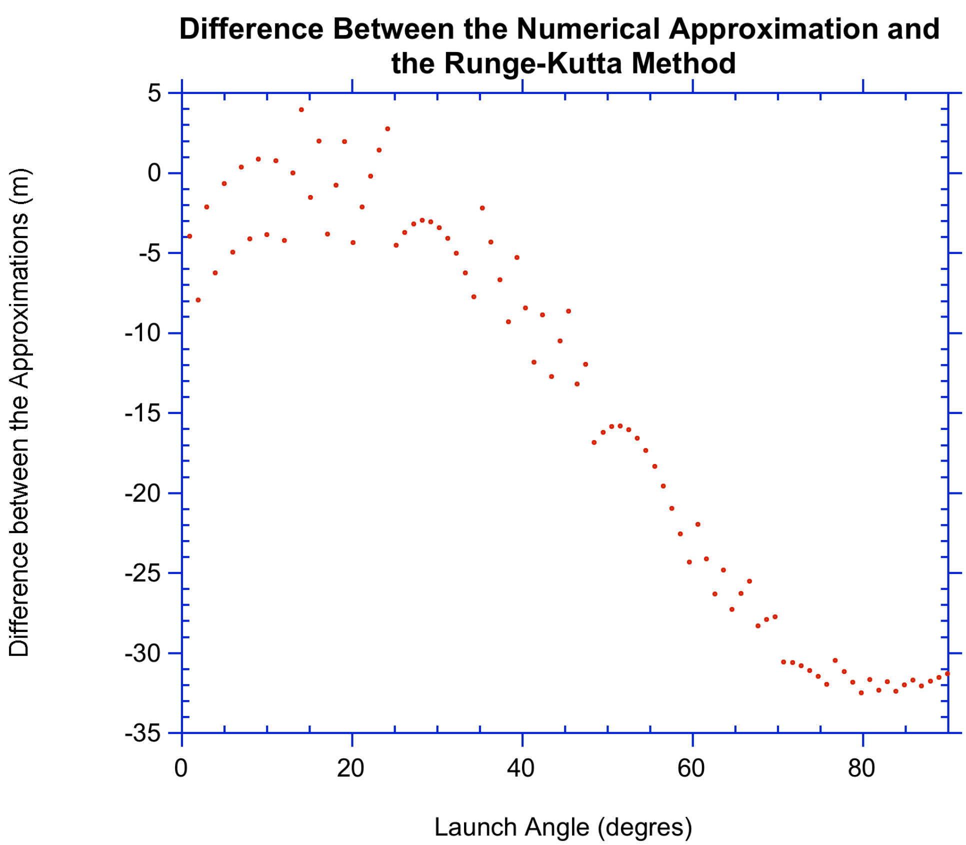

Fig. 11. Difference between the numerical solution and Runge-Kutta approximation for linear drag with κ=0.01

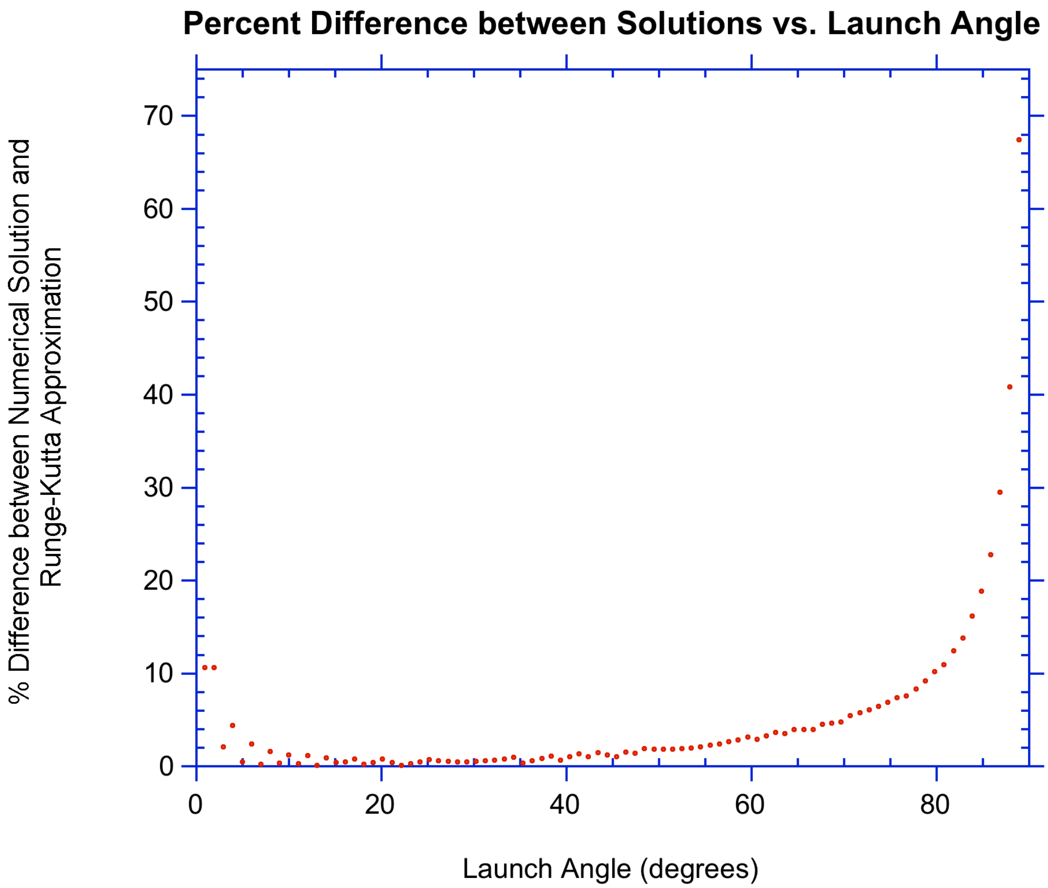

Fig. 12. Percent difference between the numerical solution and the Runge-Kutta approximation vs. launch angle

Conclusion

In doing this experiment, I found that the higher the drag coefficient was, the lower the angle that produced the maximum range. When there was no drag, the angle that produced the maximum range was 45º. When I added linear drag, with a drag coefficient of 0.01, the angle that produced the maximum range was 43º. For a drag coefficient of 0.1, I found that the angle that produces the maximum range was 34º. Then, when I change the drag coefficient to 0.001, I found that the angle that produced the maximum range was 44º. Moreover, I found that the numerical solution for linear drag differed from the Runge-Kutta approximation by a mean value of -14.72, with the mean percent difference being 7.02%.