MATH567 Project 4: due on Monday, July 30, 2018

Problem 1 (10 pts): heat equation with discontinuous data.

(a) Modify `heat_CN.m` to solve the heat equation for $-1 \leq x \leq 1$ with step function initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases} \tag{Ex0.0a}$$

With appropriate Dirichlet boundary conditions, the exact solution is

$$u(x,t) = \frac{1}{2}\,\text{erfc}\left(x/\sqrt{4\kappa t}\right), \tag{Ex0.0b}$$

where erfc is the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\,dz.$$

(i) Test this routine $m = 39$ and $k = 4h$. Note that there is an initial rapid transient decay of the high wave numbers that is not captured well with this size time step.

(ii) How small do you need to take the time step to get reasonable results? For a suitably small time step, explain why you get much better results by using $m = 38$ than $m = 39$. What is the observed order of accuracy as $k \to 0$ when $k = \alpha h$ with α suitably small and m even?

(b) Modify `heat_trbdf2.m` (from Project 3) to solve the heat equation for $-1 \leq x \leq 1$ with step function initial data as above. Test this routine using $k = 4h$ and estimate the order of accuracy as $k \to 0$ with m even. Why does the TR-BDF2 method work better than Crank-Nicolson?

Problem 2 (10 pts): Lax-Wendroff and upwind schemes.
The m-file `advection_LW_pbc.m` implements the Lax-Wendroff method for the advection equation on $0 \le x \le 1$ with periodic boundary conditions.

(a) Observe how this behaves with $m+1 = 50,\ 100,\ 200$ grid points. Change the final time to `tfinal = 0.1` and use the m-files `error_table.m` and `error_loglog.m` to verify second order accuracy.

(b) Modify the m-file to create a version `advection_up_pbc.m` implementing the upwind method and verify that this is first order accurate.

(c) Keep m fixed and observe what happens with `advection_up_pbc.m` if the time step k is reduced, e.g. try $k = 0.4h$, $k = 0.2h$, $k = 0.1h$. When a convergent method is applied to an ODE we expect better accuracy as the time step is reduced and we can view the upwind method as an ODE solver applied to an MOL system. However, you should observe decreased accuracy as $k \to 0$ with h fixed. Explain this apparent paradox.