MATH567 Project 3: due on Monday, July 23, 2018

Problem 1 (10 pts): 2D Poisson problem.
The MATLAB script `poisson.m` solves the Poisson problem on a square $m \times m$ grid with $\Delta x = \Delta y = h$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x, y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x, y) = 1.25 \exp(x + y/2)$.

(a) Test this script by performing a grid refinement study to verify that it is second order accurate.

(b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on a non-square domain.

(c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script.

Problem 2 (10 pts): BDF2 and FE schemes for heat equation.

(a) The m-file `heat_CN.m` solves the heat equation $u_t = \kappa u_{xx}$ using the Crank-Nicolson method. Run this code, and by changing the number of grid points, confirm that it is second-order accurate. (Observe how the error at some fixed time such as $T = 1$ behaves as k and h go to zero with a fixed relation between k and h, such as $k = 4h$.)

You might want to use the function `error_table.m` to print out this table and estimate the order of accuracy, and `error_loglog.m` to produce a log-log plot of the error vs. h. See `bvp_2.m` for an example of how these are used.

(b) Modify `heat_CN.m` to produce a new m-file `heat_trbdf2.m` that implements the TR-BDF2 method on the same problem. Test it to confirm that it is also second order accurate.

(c) Modify `heat_CN.m` to produce a new m-file `heat_FE.m` that implements the forward Euler explicit method on the same problem. Test it to confirm that it is $\mathcal{O}(h^2)$ accurate as $h \to 0$ provided when $k = 24h^2$ is used, which is within the stability limit for $\kappa = 0.02$. Note how many more time steps are required than with Crank-Nicolson or TR-BDF2, especially on finer grids.

(d) Test `heat_FE.m` with $k = 26h^2$, for which it should be unstable. Note that the instability does not become apparent until about time 1.6 for the parameter values $\kappa = 0.02,\ m = 39,\ \beta = 150$.