MATH567 Project 2: due on Monday, July 16, 2018

Problem 2 (10 pts): ODE(BVP) symmetric discretization.

Program the central finite difference method for the self-adjoint BVP

$$\begin{aligned}(\beta(x)u')' - \gamma(x)u(x) &= f(x), \quad 0 < x < 1, \\ u(0) = u_a, \qquad & au(1) + bu'(1) = c,\end{aligned}$$

using a uniform grid and the central difference scheme

$$\frac{\beta_{i+\frac{1}{2}}(U_{i+1} - U_i)/h - \beta_{i-\frac{1}{2}}(U_i - U_{i-1})/h}{h} - \gamma(x_i)U_i = f(x_i). \tag{6}$$

Test your code for the case where

$$\beta(x) = 1 + x^2, \quad \gamma(x) = x, \quad a = 2, \quad b = -3, \tag{7}$$

and the rest of functions or parameters are determined from the exact solution

$$u(x) = e^{-x}(x-1)^2. \tag{8}$$

Plot (i) the local truncation error, (ii) computed solution and the exact solution, (iii) and the errors for a particular grid, say $n = 80$. Do the grid refinement analysis to determine the order of accuracy of the global solution. Also try to answer the following questions:

- What happens when $a = 0$ or $b = 0$?
- If we use the central difference scheme for the equivalent differential equation

$$\beta u'' + \beta' u' - \gamma u = f, \tag{9}$$

  what are the advantages or disadvantages?

Write another MATLAB code of a central difference scheme based on (9), and compare the numerical results (solution errors, order of accuracy, CPU time) with the above finite difference scheme (6).
Check [TB] Section 2.15, Example 2.1 for more detail.
Check [TB] Appendix A.3–A.6 for more detail on how to estimate order of accuracy with grid refinement.

Problem 2 (10 pts): ODE(BVP) with boundary layer.

Consider the finite difference scheme for the 1D steady state *convection-diffusion* equation

$$\epsilon u'' - u' = -1, \quad 0 < x < 1 \tag{10}$$

$$u(0) = 1, \qquad u(1) = 3. \tag{11}$$

(a) Verify the exact solution is

$$u(x) = 1 + x + \left(\frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1} \right). \tag{12}$$

(b) Compare the following two finite difference methods for $\epsilon = 0.3$, 0.1, 0.05, and 0.0005,
(1): Central difference scheme:

$$\epsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_{i+1} - U_{i-1}}{2h} = -1. \tag{13}$$

(2): Central-upwind difference scheme:

$$\epsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_i - U_{i-1}}{h} = -1. \tag{14}$$

Do grid refinement analysis for each case to determine the order of accuracy. Plot the computed solution and the exact solution for $h = 0.1$, $h = 1/25$, and $h = 0.01$. You can use Matlab command *subplot* to put several graphs together.

(c) From you observation, give your opinion to see which method is better.

Check [TB] Appendix A.3–A.6 for more detail on how to estimate order of accuracy with grid refinement.