Bang bang control of elliptic and parabolic PDEs

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SPP 1962

Jackson, July 24, 2018

Model problem

$$(P) \qquad \min_{u \in U_{ad}, y \in Y_{ad}} J(y,u) = \frac{1}{2}\int_D |y-z|^2 + \frac{\alpha}{2}\|u\|_U^2 \quad \text{subject to } y = \mathcal{G}(Bu).$$

$$y = \mathcal{G}(Bu) \text{ iff } Ay = Bu \text{ in } D, \text{ plus b.c. (plus i.c.)}$$

Examples

- $Ay := -\sum_{i,j=1}^d \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^d b_i y_{x_i} + cy$ uniformly elliptic operator, $Y = H^1(\Omega)$
- $Ay := y_t - \sum_{i,j=1}^d \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^d b_i y_{x_i} + cy$ parabolic operator, $Y = W(0,T)$.

Constraint sets:

$$U_{ad} = \{a \le u \le b\} \text{ and } Y_{ad} = \{y \le c\}, \text{ or } \{|\nabla y| \le \delta\}.$$

Bang bang control: $\alpha = 0$.

Existence and uniqueness of solutions, optimality conditions

- For every u we have a unique $y(u) = \mathcal{G}(Bu)$. So we may minimize the reduced functional
$$J(u) := J(\mathcal{G}(Bu), u)$$
instead.

- Problem (P) admits a unique solution u with corresponding state $y = \mathcal{G}(Bu)$.

- $\langle J'(u), v - u \rangle_{U^*,U} \geq 0$ for all
$$v \in F_{ad} := \{v \in U_{ad}, \mathcal{G}(Bv) \in Y_{ad}\}.$$

Necessary optimality conditions ($\alpha > 0$), $Y_{ad} = \{y \le c\}$

Constraint qualification (Slater condition)

$$\exists \tilde{u} \in U_{ad} \text{ such that } \mathcal{G}(B\tilde{u}) < c \text{ in } \bar{D}$$

Then there exist $\mu \in \mathcal{M}(\bar{D})$ and some p such that with y and u there holds

$$Ay = Bu$$
$$A^* p = y - z + \mu$$
$$u = P_{U_{ad}}\left(-\frac{1}{\alpha} RB^* p\right),$$
$$\mu \ge 0,\ \ y \le c \text{ in } D \text{ and } \int_{\bar{D}} (c - y)\, d\mu = 0,$$

Regularity

- elliptic case: $p \in W^{1,s}(\Omega)$ for all $s < d/(d-1)$.
- parabolic case: $p \in L^s(W^{1,\sigma})$ for all $s, \sigma \in [1, 2)$ with $2/s + d/\sigma > d + 1$.

Low regularity of adjoint states introduces difficulties in the numerical analysis

Only state constraints ($\alpha > 0$), $U_{ad} \equiv U$

Optimality conditions:

$$
\begin{aligned}
Ay &= Bu \\
A^* p &= y - z + \mu \\
u &= -\frac{1}{\alpha} R B^* p, \\
\mu \geq 0,\ \ y \leq c \text{ in } D &\text{ and } \int_{\bar{D}} (c - y)\, d\mu = 0,
\end{aligned}
$$

- The state y in general is smoother than the adjoint p.
- The optimal control and the adjoint state p are coupled via an algebraic relation $\Rightarrow$ a discretization of p induces a discretization of u.
- A discretization of y ideally should deliver feasible discrete states.

Only control constraints ($\alpha > 0$), $Y_{ad} \equiv Y$

Optimality conditions:

$$
\begin{aligned}
Ay &= Bu \\
A^* p &= y - z \\
u &= P_{U_{ad}}\left(-\frac{1}{\alpha} RB^* p\right),
\end{aligned}
$$

- The adjoint p in general is smoother than the state y.
- The optimal control and the adjoint state p are coupled via an algebraic relation $\Rightarrow$ a discretization of p induces a discretization of u.
- Reduction to p, say delivers

$$AA^* p - BP_{U_{ad}}\left(-\frac{1}{\alpha} RB^* p\right) = Az,$$

which in the parabolic case is a bvp for p in space-time.

Control constraints, $\alpha = 0$

The function $u \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state p such that $y = \mathcal{G}(u)$, $p = \mathcal{G}(y - z)$ and

$$(\alpha u + RB^* p, v - u) \geq 0 \text{ for all } v \in U_{ad}.$$

Recall: $u = P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} RB^* p\right)$ for $\alpha > 0$, i.e.

$$u = \begin{cases} a, & \alpha u + RB^* p > 0, \\ -\frac{1}{\alpha} RB^* p, & \alpha u + RB^* p = 0, \\ b, & \alpha u + RB^* p < 0, \end{cases}$$

If $\alpha = 0$:

$$u \begin{cases} = a, & RB^* p > 0, \\ \in [a, b], & RB^* p = 0, \\ = b, & RB^* p < 0. \end{cases}$$

- Control is determined through the sign of $RB^* p$.
- Control in the generic case is either at upper or lower bound.
- Use homothopy $\alpha \to 0$ to compute bang bang control.

Tailored discretization

Discrete counterpart to (P):

$$(P_h) \qquad \min_{u_h \in U_{\text{ad}}^h, y_h \in Y_{\text{ad}}^h} J_h(y_h, u_h) \text{ s.t. } PDE_h(y_h) = B_h(u_h)$$

Questions:

- **Appropriate choice of U_{ad}^h and Ansatz for u_h?**
- **Appropriate choice of Y_{ad}^h and Ansatz for y_h?**

Aim: Capture as much structure as possible of (P) on the discrete level.

Discretization – a variational concept[1]

Discrete optimal control problem:

$$(P_h) \qquad \begin{cases} \min\limits_{u \in U_{ad}} J_h(u) := \dfrac{1}{2}\displaystyle\int_D |y_h - z|^2 + \dfrac{\alpha}{2}\|u\|_U^2 \\ \text{subject to } \; y_h = \mathcal{G}_h(Bu) \text{ and } y_h \le I_h c. \end{cases}$$

This problem is still ∞–dimensional.

Here, $y_h(u) = \mathcal{G}_h(Bu)$ denotes e.g. the

- p.l. and continuous fe approximation to $y(u)$ (elliptic case),
- $dg(0)$ in time and p.l. and continuous fe in space approximation to $y(u)$ (parabolic case), i.e.

$$a(y_h, v_h) = \langle Bu, v_h \rangle \text{ for all } v_h \in X_h.$$

The control is not discretized explicitely!

[1] M. Hinze: A variational discretization concept in control constrained optimization: the linear-quadratic case. J. Comput. Optim. Appl. 30, 45-63 (2005)

Discrete necessary optimality conditions ($\alpha > 0$)

Problem (P_h) admits a unique solution $u_h \in U_{ad}$. Furthermore, uniform convergence of discrete states implies discrete Slater condition

$$\mathcal{G}_h(B\tilde{u}) < I_h b \text{ in } \bar{D} \text{ for } h \text{ small enough}$$

Then there exist $\mu_h \in \mathcal{M}(\bar{D})$ and some $p_h \in X_h$ such that with y_h and u_h there holds

$$\begin{aligned}
a(y_h, v_h) &= \langle Bu_h, v_h \rangle & \forall v_h \in X_h, \\
a(w_h, p_h) &= \int_D (y_h - z) w_h + \int_{\bar{D}} w_h d\mu_h & \forall w_h \in Y_h, \\
u_h &= P_{U_{ad}}\left(-\frac{1}{\alpha} RB^* p_h\right) & \\
\mu_h \geq 0,\ y_h \leq I_h c,\ \text{and} &\int_{\bar{D}} (I_h c - y_h) d\mu_h = 0. &
\end{aligned}$$

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Post processing[2] exploits the projection formula with a discrete adjoint state associated to a fully discrete control.

[2] C. Meyer, A. Rösch. Superconvergence properties of optimal control problems. SIAM J. Control Optim. 43:970-985 (2004)

$\alpha = 0$: optimality conditions for discrete problem

The variational-discrete optimal control problems admits a solution $u_h \in U_{ad}$, which is unique in the case $\alpha > 0$. The state y_h is unique (also in the case $\alpha = 0$).

Let $u_h \in U_{ad}$ be a solution of the optimal control problem. Then there exists a unique adjoint state p_h such that $y_h = \mathcal{G}_h(u_h)$, $p_h = \mathcal{G}_h(y_h - z)$ and

$$(\alpha u_h + RB^* p_h, v - u_h) \geq 0 \text{ for all } v \in U_{ad}.$$

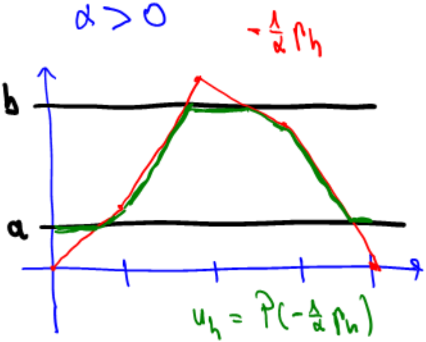
Recall $u_h = P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} RB^* p_h\right)$ for $\alpha > 0$, i.e.

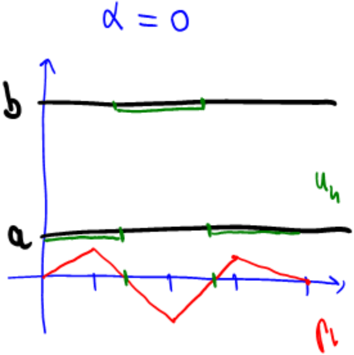
$$u_h = \begin{cases} a, & \alpha u_h + RB^* p_h > 0, \\ -\frac{1}{\alpha} RB^* p_h, & \alpha u_h + RB^* p_h = 0, \\ b, & \alpha u_h + RB^* p_h < 0. \end{cases}$$

If $\alpha = 0$:

$$u_h \begin{cases} = a, & RB^* p_h > 0, \\ \in [a, b] & RB^* p_h = 0, \\ = b, & RB^* p_h < 0. \end{cases}$$

Sketch of concept in 1d



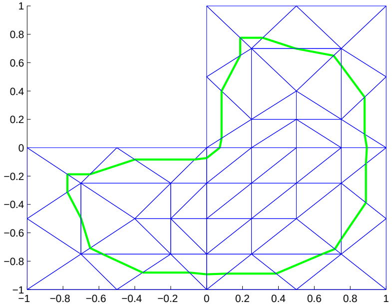


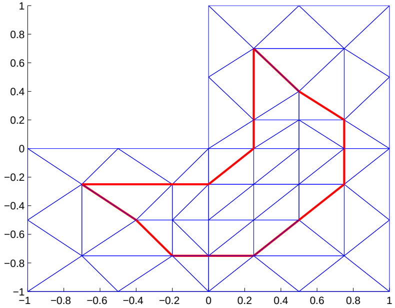
| finite element grid

| boundary of active set

| switching points

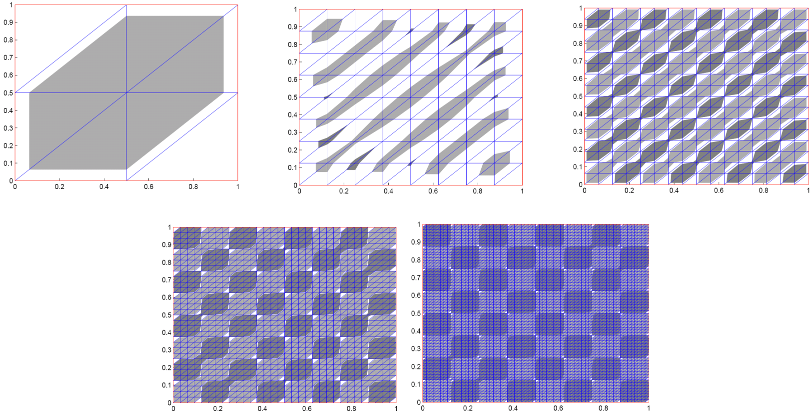
Variational versus conventional discretization ($\alpha > 0$)





Variational discretization in the bang–bang case $\alpha = 0$

$$u(x) = -sign(p(x)), \; p(x) = \frac{1}{128} sin(8\pi x_1) sin(8\pi x_2)$$

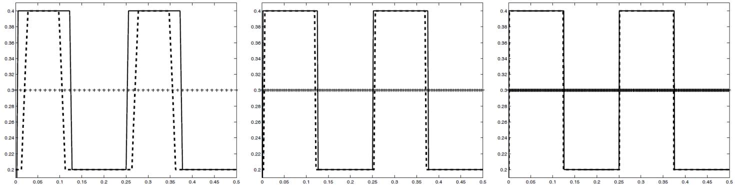


Parabolic bang bang with α–homothopy

$p(t,x) = -\frac{T}{2\pi a}\sin\left(\frac{2\pi a t}{T}\right)\sin 2\pi x_1 \sin 2\pi x_2$, $a = 2$, $a_1 = 0.2$, $b_1 = 0.4$ bounds.

$$u(t) = \begin{cases} a_1, & B^*p > 0, \\ b_1, & B^*p < 0, \end{cases}$$

where $B^*p(t) = \int_\Omega \sin 2\pi x_1 \sin 2\pi x_2 p(t,x)\,dx$.



(a) $\ell = 3$ (b) $\ell = 4$ (c) $\ell = 5$

Here, $\alpha = h^2 = k^{4/3}$ with $h = 2^{-l}$. Furthermore, with this coupling[3]

$$\|u - u_{k,h,\alpha}\|_{L^1} \leq ch^2.$$

[3] N. von Daniels, M. Hinze: Variational discretization of a control-constrained parabolic bang-bang optimal control problem. J. Comput. Math., to appear

Algorithms for $\mathbb{P}_h^\alpha$

Define

$$G_h(u) = u - P_{U_{\rm ad}}\left(-\frac{1}{\alpha}p_h(y_h(u))\right).$$

The optimality condition reads $G_h(u) = 0$ and motivates the fix–point iteration

- u given, do until convergence

$$u^+ = P_{U_{\rm ad}}\left(-\frac{1}{\alpha}p_h(y_h(u))\right), \quad u = u^+.$$

1. Is this algorithm numerically implementable?

Yes, whenever for given u it is possible to numerically evaluate the expression

$$P_{U_{\rm ad}}\left(-\frac{1}{\alpha}p_h(y_h(u))\right)$$

in the $i-th$ iteration, with an numerical overhead which is *independent* of the iteration counter of the algorithm.

Semi–smooth Newton algorithm for $\alpha > 0$

2. Does the fix–point algorithm converge?

Yes, if $\alpha > \|RB^* S_h^* S_h B\|_{\mathcal{L}(U)}$, since $P_{U_{ad}}$ is non–expansive.

Condition too restrictive for our purpose → semi–smooth Newton method applied to $G_h(u) = 0$:

- u given, solve until convergence

$$G_h'(u)u^+ = -G_h(u) + G_h'(u)u, \quad u = u^+.$$

1. This algorithm is implementable whenever the fix–point iteration is, since

$$-G_h(u) + G_h'(u)u = -P_{U_{ad}}\left(-\frac{1}{\alpha}p_h(u)\right) - \frac{1}{\alpha}P_{U_{ad}}'\left(-\frac{1}{\alpha}p_h(u)\right)S_h^* S_h u.$$

2. For every $\alpha > 0$ this algorithm is locally fast convergent[4].

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[4] M. Hinze, M. Vierling: Variational discretization and semi-smooth Newton methods; implementation, convergence and globalization in pde constrained optimzation with control constraints. Optim. Meth. Software 27:933-950 (2012)

Error analysis, sketch for elliptic case[5]

It is well known that

$$\|y - y_h\| + \alpha\|u - u_h\| \sim \|y - y_h(u)\| + \|p - p_h(y(u))\|$$

So one expects estimates for $y - y_h$ also in the case $\alpha = 0$.

Estimates for $\|u - u_h\|$?

Estimate for the states ($S := \overline{\{x \in \Omega \,|\, p(x) \neq 0\}} \subset \bar{\Omega}$)

$$\begin{aligned} \|y - y_h\| &\leq C\left(h^2 + (b-a)\|p - R_h p\|_{L^1(\Omega\setminus S)} + \|p - R_h p\|_{L^\infty}\|u - u_h\|_{L^1(S)}\right), \\ \|p - p_h\|_{L^\infty} &\leq C\|y - y_h\| + \|p - R_h p\|_{L^\infty}, \end{aligned}$$

follow from

- $0 \leq (p - p_h, u_h - u) = (R_h p - p_h, u_h - u) + (p - R_h p, u_h - u) \equiv I + II.$
- $I \leq -\frac{1}{2}\|y - y_h\|^2 + \frac{1}{2}\|y - R_h y\|^2$
- $II = \int_{\Omega\setminus S}(p - R_h p)(u_h - u) + \int_S (p - R_h p)(u_h - u).$

[5] K. Deckelnick, M. Hinze: A note on the approximation of elliptic control problems with bang-bang controls. Comput. Optim. Appl. 51:931-939 (2012)

Error estimates

If the adjoint state p in the solution with some $\beta \in (0,1]$ satisfies the structural assumption

$$\exists C > 0 \forall \epsilon > 0 : \mathcal{L}(\{x \in \bar{\Omega}; |p(x)| \le \epsilon\}) \le C\epsilon^{\beta},$$

one for h small enough gets a unique variational discrete control u_h, which together with the discrete state y_h and the discrete adjoint p_h satisfies

$$\begin{aligned} \|y - y_h\| + \|p - p_h\|_{L^\infty} &\le C\left(h^2 + \|p - R_h p\|_{L^\infty}^{\frac{1}{2-\beta}}\right), \\ \|u - u_h\|_{L^1} &\le C\left(h^{2\beta} + \|p - R_h p\|_{L^\infty}^{\frac{\beta}{2-\beta}}\right). \end{aligned}$$

Sketch of proof for $\beta = 1$

$$\|u - u_h\|_{L^1}, \|y - y_h\|, \|p - p_h\|_{L^\infty} \le C\left\{h^2 + \|p - R_h p\|_{L^\infty}\right\}$$

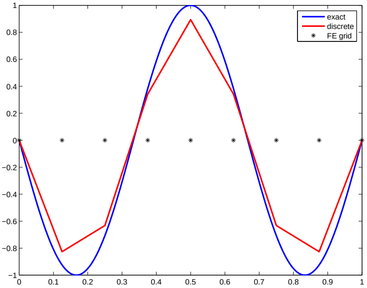
Sketch of proof:

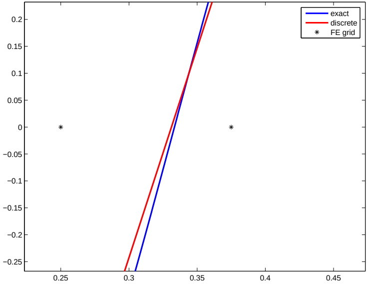
- $0 \le (p - p_h, u_h - u) = (R_h p - p_h, u_h - u) + (p - R_h p, u_h - u) \equiv I + II.$
- $I \le -\frac{1}{2}\|y - y_h\|^2 + \frac{1}{2}\|y - R_h y\|^2$
- $II = \int_S (p - R_h p)(u_h - u)$. Combine now
    - $\|u - u_h\|_{L^1} \le (b - a)\mathcal{L}(\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\})$
    - $\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\} \subseteq \{|p(x)| \le \|p - p_h\|_\infty\} \Rightarrow$
    - $\mathcal{L}(\{|p(x)| \le \|p - p_h\|_\infty\}) \le C\|p - p_h\|_\infty$
    - $\|u - u_h\|_{L^1} \le C\|p - p_h\|_\infty$
    - $\|p - p_h\|_\infty \le \|p - R_h p\|_\infty + \|R_h p - p_h\|_\infty$
    - $\|R_h p - p_h\|_\infty \le C\|y - y_h\|$

  to estimate II.

Numerical example with 2 switching points, fix-point iteration

$u(x) = -sign(-sin(m\pi x))$, $y(x) = sin(m\pi x)$, and $p(x) = -sin(m\pi x)$, m=2.





Experimental order of convergence:

- $||u - u_h||_{L^1}$: **3.00077834**
- **Function values 1.99966106**
- $||p - p_h||_{L^\infty}$: **1.99979367**
- $||y - y_h||_{L^\infty}$: **1.9997965**
- $||p - p_h||_{L^2}$: **1.99945711**

Special cases

1. If the desired state z is reachable with a feasible control, then

$$\|y - y_h\| + \|p - p_h\|_{L^\infty} \le Ch^2.$$

2. If $p \in C^1(\bar{\Omega})$ satisfies

$$\min_{x \in K} |\nabla p(x)| > 0, \qquad \text{where } K = \{x \in \bar{\Omega} \,|\, p(x) = 0\}.$$

Then, the structural assumption is satisfied with $\beta = 1$.

3. If $p \in W^{2,\infty}(\Omega)$ and satisfies the structural assumption, then

$$\|y - y_h\| + \|p - p_h\|_{L^\infty} + \|u - u_h\|_{L^1} \le Ch^2 |\log h|^{\gamma(d)}.$$

Further reading

- M. Hinze: A variational discretization concept in control constrained optimization: the linear-quadratic case. J. Comput. Optim. Appl. 30:45-63 (2005).
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End

Thank you very much for your attention!