Physics 206b

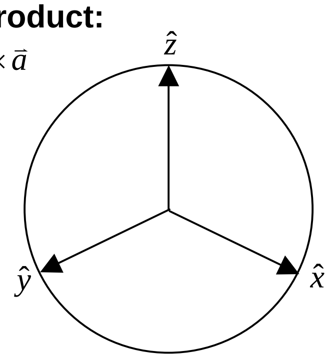
Assignment #9
Cross product practice

These problems will not be collected or graded.

SOLUTIONS

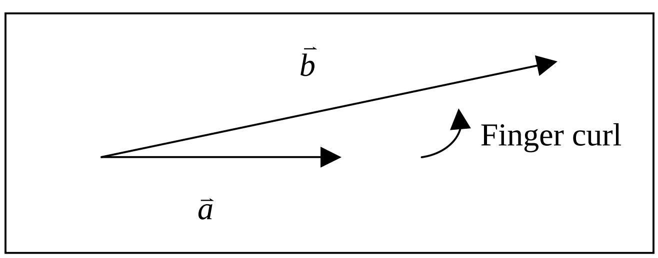
Some important properties of the cross product:

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$
$$\vec{a}\times\vec{a} = 0$$
$$\hat{x}\times\hat{y} = \hat{z}$$
$$\hat{y}\times\hat{z} = \hat{x}$$
$$\hat{z}\times\hat{x} = \hat{y}$$



7. **Using the right hand rule (RHR), what is size and direction of $\vec{a}\times\vec{b}$ if $\vec{a}$ is a vector of length 2 pointed in the $\hat{x}$ direction and $\vec{b}$ is a vector of length 5 making an angle of 20° with $\vec{a}$ in the *x-y* plane?**

Well, I don't have the graphical ability to draw a right hand with the fingers curling, but I can indicate it a bit. The first step is to render the vectors described into some useful graphical form. This gives:



The direction of the finger curl is indicated. The right hand rule says that the resultant, let's call it $\vec{c}$, has a direction found by curling the fingers of the right hand as though to rotate $\vec{a}$ into $\vec{b}$. The thumb of the right hand then points in the direction of $\vec{c}$. Following this, we get that $\vec{c}$ points *out* of the page. The size is $c = ab\sin(\theta) = (2)(5)\sin(20°) = (10)(.34) = 3.4$. The out of page direction is the $+z$ direction.

8. **Now, use the two vectors above to perform the same calculation algebraically.**

Here, we'll first need to render the two vectors in component form. Since $\vec{a}$ points in the $\hat{x}$ direction, this one is easy. For $\vec{b}$ we'll just do the usual vector decomposition to get $\vec{b} = 5\cos(20°)\hat{x} + 5\sin(20°)\hat{y} = 4.7\hat{x} + 1.7\hat{y}$. Now we can perform the cross product easily. $\vec{a}\times\vec{b} = 2\hat{x}\times\left(4.7\hat{x}+1.7\hat{y}\right)$. Now, before going too far, recognize that an immediate repercussion of the property $\vec{a}\times\vec{a} = 0$ is that the cross product of any pair of vectors which are parallel, even if they are of different length, will also be zero. The reason for this is clear: Since all vectors can be thought of as scalar quantities (size only) times unit vectors (direction only), the things which are being multiplied in the cross product operation are the unit vectors. If the sizes are different, it doesn't matter: The unit vectors are all the same size. So the product will vanish if the two things being multiplied are parallel. (This is why we use the cross product in situations, such as torque, in which we want to express the statement "multiply a thing by the component of something else that is perpendicular to it.")

In this case, this means that the $\hat{x}$ component of $\vec{b}$ will not contribute to the product and we are left with $\vec{a}\times\vec{b} = 2\hat{x}\times\left(4.7\hat{x}+1.7\hat{y}\right) = 3.4\left(\hat{x}\times\hat{y}\right) = 3.4\hat{z}$, as found previously using the RHR.

9. **Perform the following cross products:**

    a. $\left(27\hat{x}+3\hat{y}\right)\times\left(9\hat{x}+1\hat{y}\right)$

    b. $\left(27\hat{x}+3\hat{y}\right)\times\left(9\hat{x}-1\hat{y}\right)$

    c. $\left(3\hat{x}+5\hat{y}+7\hat{z}\right)\times\left(11\hat{x}-13\hat{y}+6\hat{z}\right)$

In each of these solutions, I'm going to use the general rules of multiplication to come up with a sum of terms, each of which will have a cross product in it. I'm going to use a little shortcut and simply ignore any term where the two vectors being crossed are parallel since these terms will equal zero. For clarity, I'll use an asterisk ($*$) to indicate multiplication of scalars.

a. $\left(27*1\right)\left(\hat{x}\times\hat{y}\right)+\left(3*9\right)\left(\hat{y}\times\hat{x}\right) = 27\hat{z} - 27\hat{z} = 0$

This first deserves some notice: Note that the vector on the left of the cross is just three times the vector on the right of the cross. This drives home an important fact: The cross product of vectors which are parallel is zero! The nice thing about this is that, while we could have saved some work by noting this in the first place, the math took care of things for us. We can just plod through, mechanically, and be confident that we won't miss anything unless we make an actual mistake. When in doubt, just let the algebra do the work for you!

b. $(-27*1)(\hat{x}\times\hat{y})+(3*9)(\hat{y}\times\hat{x})=-27\hat{z}-27\hat{z}=-54\hat{z}$

What a difference that little sign makes!

c.
$$\begin{aligned}&(-3*13)(\hat{x}\times\hat{y})+(3*6)(\hat{x}\times\hat{z})+(5*11)(\hat{y}\times\hat{x})\\&+(5*6)(\hat{y}\times\hat{z})+(7*11)(\hat{z}\times\hat{x})+(-7*13)(\hat{z}\times\hat{y})\\&=-39\hat{z}-18\hat{y}-55\hat{z}+30\hat{x}+77\hat{y}+91\hat{x}\\&=121\hat{x}+59\hat{y}-94\hat{z}\end{aligned}$$

10. By directly performing the following cross products in two different orders, show that, in general $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$

a. $5\hat{x}\times 3\hat{z}\times 9\hat{z}$

b. $(2\hat{x}+5\hat{y}+8\hat{z})\times(9\hat{x}-13\hat{y}+11\hat{z})\times(-6\hat{x}+7\hat{y}+17\hat{z})$

First, we take the parentheses around the first pair and then the second pair. This gives:

a.

i) $(5\hat{x}\times 3\hat{z})\times 9\hat{z}=18(\hat{x}\times\hat{z})\times 9\hat{z}=-18\hat{y}\times 9\hat{z}=-162\hat{x}$

ii) $5\hat{x}\times(3\hat{z}\times 9\hat{z})=0 \quad \text{since } \hat{z}\times\hat{z}=0$

Obviously, i) and ii) are not equal.

b. Now for the tough one! (Not really tough, just very long and tedious.)

i)
$$\begin{aligned}&[(2\hat{x}+5\hat{y}+8\hat{z})\times(9\hat{x}-13\hat{y}+11\hat{z})]\times(-6\hat{x}+7\hat{y}+17\hat{z})\\&=[-26(\hat{x}\times\hat{y})+22(\hat{x}\times\hat{z})+45(\hat{y}\times\hat{x})+55(\hat{y}\times\hat{z})+72(\hat{z}\times\hat{x})-104(\hat{z}\times\hat{y})]\\&\times(-6\hat{x}+7\hat{y}+17\hat{z})\\&=(-26\hat{z}-22\hat{y}-45\hat{z}+55\hat{x}+72\hat{y}+104\hat{x})\times(-6\hat{x}+7\hat{y}+17\hat{z})\\&=(-71\hat{z}+50\hat{y}+159\hat{x})\times(-6\hat{x}+7\hat{y}+17\hat{z})\\&=426\hat{y}+497\hat{x}+300\hat{z}+850\hat{x}+1113\hat{z}-2703\hat{y}\\&=1347\hat{x}-2277\hat{y}+1413\hat{z}\end{aligned}$$

ii)
$$\begin{aligned}&(2\hat{x}+5\hat{y}+8\hat{z})\times[(9\hat{x}-13\hat{y}+11\hat{z})\times(-6\hat{x}+7\hat{y}+17\hat{z})]\\&=(2\hat{x}+5\hat{y}+8\hat{z})\times[63\hat{z}-153\hat{y}-78\hat{z}-221\hat{x}-66\hat{y}-77\hat{x}]\\&=(2\hat{x}+5\hat{y}+8\hat{z})\times[-298\hat{x}-219\hat{y}-15\hat{z}]\\&=-438\hat{z}-30\hat{y}+1490\hat{z}-75\hat{x}-2384\hat{y}+1752\hat{x}\\&=1677\hat{x}-2414\hat{y}+1052\hat{z}\end{aligned}$$

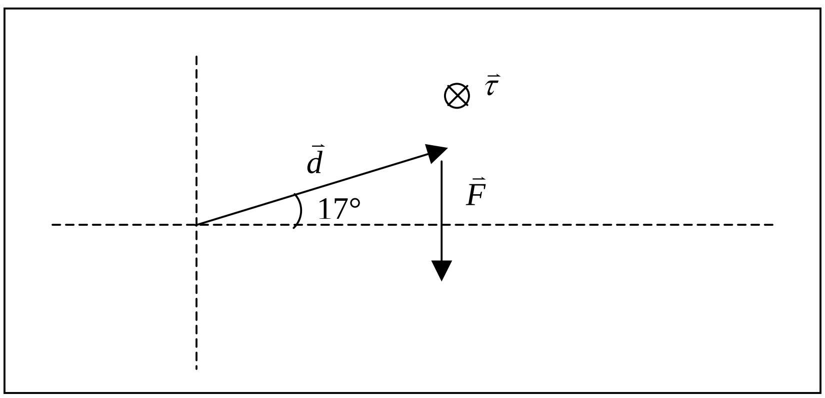
Once again, i) and ii) are clearly not the same. This is what was to be shown.

11. Taking $\vec{\tau} = \vec{d} \times \vec{F}$, find $\vec{\tau}$ when

a. $\vec{F} = 80N$ in the $-\hat{y}$ direction and $\vec{d} = 1m$ at an angle of 17° relative to the *x* axis in the *x-y* plane. Sketch the force, the moment arm (*d*), and the torque ($\vec{\tau}$).

b. $\vec{F} = 12N\,\hat{x} - 3N\,\hat{y} + 5N\,\hat{z}$ and $\vec{d}$ is 4 meters long and makes a 45° angle with all three of the Cartesian axes (i.e., it points out of the corner of the cube formed of *x*, *y*, and *z*.)

a. Ah! Good ol' torque! The sketch appears below. Since the resultant of a cross product is perpendicular to both of the vectors being crossed, the torque in this case must point in the $\hat{z}$ direction, although we'll have to determine the sign. Using the right hand rule (which is tough to draw), I get it to be in the $-\hat{z}$. Let's take this to be into this page. (The symbol"$\otimes$" represents a vector pointed into the page.) The size of the torque will be $\tau = dF\sin(\theta) = 80\,N\cdot m\sin(73°) = 76.5\,N\cdot m$. This gives:



b. The hard part in this question is figuring out $\vec{d}$. In three dimensions, Pythagoras' theorem is $h^2 = x^2 + y^2 + z^2$. Therefore, a vector at 45° relative to all three axes would be $\vec{d} = \frac{4m}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$. Using this, we have

$$\vec{\tau} = \vec{d} \times \vec{F} = \frac{4m}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z}) \times (12N\,\hat{x} - 3N\,\hat{y} + 5N\,\hat{z})$$

$$= \frac{4N\cdot m}{\sqrt{3}}(-3\hat{z} - 5\hat{y} - 12\hat{z} + 5\hat{x} + 12\hat{y} + 3\hat{x}) = \frac{4N\cdot m}{\sqrt{3}}(8\hat{x} + 7\hat{y} - 15\hat{z})$$

Of course, the common factor of $\frac{4}{\sqrt{3}}$ could be multiplied through, if one desired.