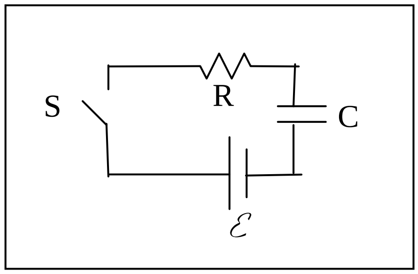
Physics 206b
Homework Assignment IX
SOLUTIONS

1. In the circuit below, the resistance is $37k\Omega$ and the capacitor is 14 nF. After switch S is closed, how long will it be before the voltage difference across the capacitor is ½ $\mathcal{E}$?



First, understand the Physics: The EMF source provides an EMF of $\mathcal{E}$. This causes a current to flow and charges begin to accumulate on the capacitor's plates. As charge accumulates, the capacitor's plates generate a potential difference between them of $V_C = \frac{Q}{C}$ where Q is the accumulated charge and C is the capacitance of the capacitor (which is a constant determined solely by the mechanical construction of the capacitor). Well, the rate of current flow through the resistor is determined by Ohm's law, $V_R = IR$. Here's the crucial point: The potential to be used in this is V_R, the potential difference between one side of the resistor and the other. The potential generated by the capacitor has the opposite sign of the potential of the EMF source. Thus, as the charge on the capacitor grows, the potential difference between the two sides of the resistor will diminish and, by Ohm's law, the current will be reduced. But when the current is reduced, the rate at which charge accumulates on the capacitor will also be reduced. This slows down the rate at which V_C grows and so on.

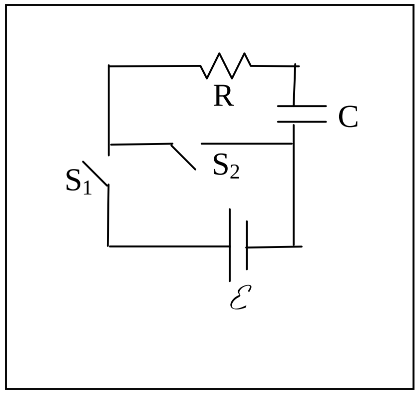
Fortunately, this can be handled mathematically very easily using some tools that you don't have yet (unless you've taken Calculus). But the result is still simple. In this case, we have $V_C = V_0\left(1 - e^{-\frac{t}{\tau}}\right)$. Here, $\tau = RC$, the product of the resistance and the capacitance. (The resistance to use is the total resistance between the EMF source and the capacitor, by the way. It does not have to come before the capacitor. We might see a circuit where there are two resistors, one in series with the capacitor "before" it and one in series with it "after" it. Just add the resistances in the usual way, in this case.) V_0 is the potential difference applied to the capacitor and resistor. It is the potential difference responsible for the current flow which results in the charging of the capacitor. In this case, we have $V_0 = \mathcal{E}$.

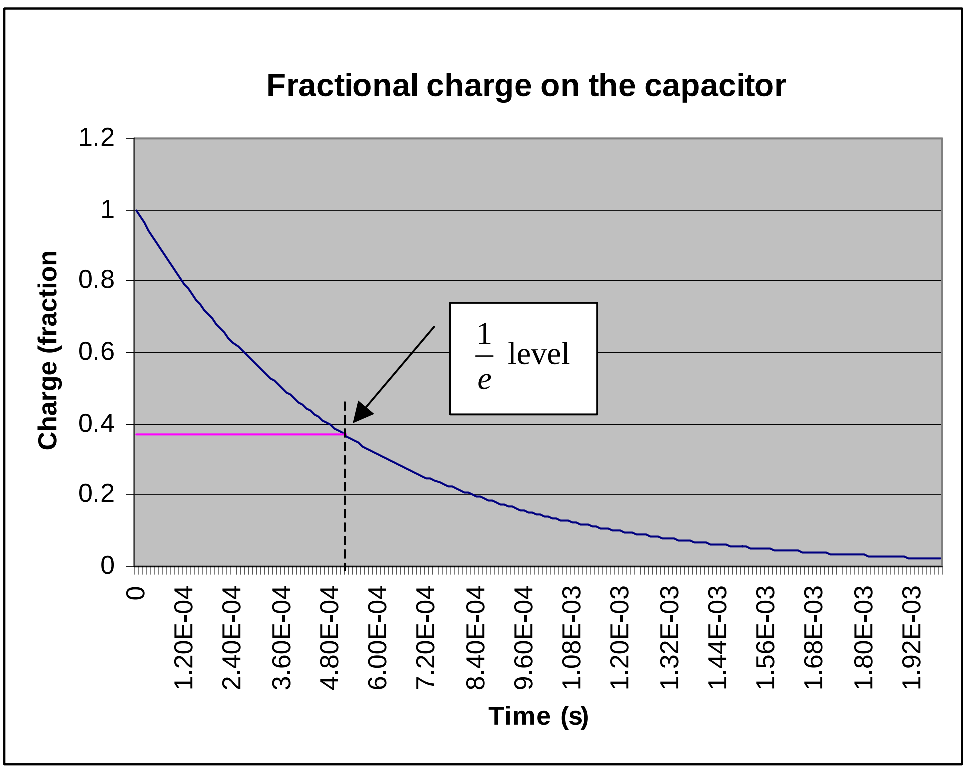
Now, we want to find the time at which $V_C = \frac{\mathcal{E}}{2}$. Substituting this in the expression above, we have $\frac{\mathcal{E}}{2} = \mathcal{E}\left(1 - e^{-\frac{t}{\tau}}\right)$. This gives (divide through by the voltage and subtract appropriately) $e^{-\frac{t}{RC}} = \frac{1}{2}$.

To get rid of the exponentiation, we take the natural logarithm of both sides of this. Caution: Do *not* inadvertently take the common logarithm! Usually, calculators will denote the natural logarithm as "*ln*" and the common logarithm as "*log*". Doing this, we have (mind that you keep track of the signs!) $\frac{t}{RC} = 0.693$.

From here it's just a matter of a smidge of multiplication and substituting in the right numbers. $t = 0.693RC = 0.693 \times 3.7 \times 10^{4}\Omega \times 1.4 \times 10^{-8}F = 3.6 \times 10^{-4}s$.

2. **Now consider the circuit below. Initially, S_1 is closed and S_2 is left open. After a long time, S_1 is opened and S_2 is immediately closed. (This is usually accomplished by a single switch just "shunting" to a different path.) Make a sketch of the charge held by the capacitor as a function of time after S_2 is closed. Use the values for R and C from the previous problem. Be sure to indicate on your sketch the time at which the charge will be at the $\frac{1}{e}$ level. If $\mathcal{E}$=3*V*, after 300 microseconds, what will be the charge remaining on the capacitor (this will be an actual value, not just a fraction)?**





Again, let's discuss some Physics first. The capacitor is charged, as in the previous problem. The charging takes place over a "long time." What is a long time? That's a relative statement. In a problem like this, the timescale is set for us by the RC time constant, τ. Thus, "long" would be some time that is very large compared to $\tau = RC = 5.18\times10^{-4}s$. After this long time, we can consider the charging of the capacitor to have stopped. Has it actually stopped? No! But we can always wait until the rate of charging has gotten smaller than any limit we can think of. So this is an approximation that is infinitely good—as long as we're patient enough. When this limit is reached the charge on the capacitor will be such that the potential across it is equal to the potential which charged it in the first place. This is $Q = CV = C\mathcal{E}$.

Now, once the charging potential is removed and the capacitor sits in a circuit with the resistor only, the charge will begin to flow out of it. The potential difference across the resistor, initially, will just be $\mathcal{E}$, so the current at the instant the switches are thrown will be that determined by Ohm's law using this as the potential. However, as the charge on the capacitor goes away, the potential will drop and the rate of current flow will diminish. The charge remaining on the capacitor at any time (taking the zero of time to be the instant the switches are thrown) will be given by $Q = Q_0 e^{-\frac{t}{RC}}$. In this case, $Q_0 = C\mathcal{E}$. However, it is important to realize that we can pick *any* instant after the switches are thrown and the same equation will hold. All that has to change is the Q_0, which will be whatever charge the capacitor happens to have at that instant. This is a remarkable property of the exponential function!

After 1 ms, the charge will be $Q = Q_0 e^{-\frac{t}{RC}} = C\mathcal{E}e^{-\frac{1\times10^{-3}s}{3.7\times10^4\Omega\times1.4\times10^{-8}F}} = C\mathcal{E}e^{-1.93} = .145C\mathcal{E}$. Since $C = 1.4\times10^{-8}F$ and $\mathcal{E} = 3V$, this gives $Q = 0.145\times4.35\times10^{-8} = 6.3\times10^{-9}C$. The $\frac{1}{e}$ point will be reached after 0.518 ms, as shown.

(I misspoke in discussion with some of you, for which I apologize. I misremembered the numbers I had assigned and thought that I'd assigned a time much larger than tau.)

3. **A 56 mF capacitor is attached to a 3 V battery for a very long time. The battery is then removed and the capacitor is used to power a small lightbulb. It is known that the bulb dissipates 1 W when driven by a 1.5 V battery. How long after the bulb is attached to the capacitor will it be before the power dissipated by the bulb falls to ½ of the power it dissipates immediately after the capacitor is attached to it? Assume that the resistance of the bulb is not affected by its temperature.**

We start out the same as the previous problem: A capacitor is "fully" charged with a battery. (Note that the limit on charging is a property as much of the battery as of the capacitor. If we "fully" charged this capacitor with a 3V battery and then hooked it up to a 12V battery, it would certainly add to its charge. A capacitor in a system is fully charged when the potential due to the charge on the capacitor is equal to the EMF of the source.) The capacitor is then discharged through a lightbulb—just like in the activity you did in class. All we really need to do is find a few simple numbers first.

We are not told the resistance of the lightbulb. We will need that value in order to determine the rate at which the capacitor discharges. But we *are* told that the bulb dissipates 1W when driven by a 1.5V battery. That tells us everything we really need. Since we know that the dissipated power is related to potential change and resistance by $P = \frac{V^2}{R}$ we can solve for the resistance and get $R = \frac{V_{battery}^2}{P} = \frac{2.25V^2}{1W} = 2.25\Omega$.

We are now ready to start working on the problem, *per se*. (Personally, I would have waited until the very end to find the resistance. But it really doesn't matter—solve for it whenever you want and just stick the number aside until you need it.) Notice that this problem *looks* like problem #2 in this assignment except for the fact that we want the time at which the dissipated *power* has dropped to ½ its initial value. We don't have a "formula" for power as a function of time—whatever shall we do? Well, just because you don't have an equation for something *yet* doesn't mean that one can't be found.

We want to find an equation for the power dissipated as a function of time. We have three equations relating power to potential, current, and resistance: $P = VI = \frac{V^2}{R} = I^2R$. Any one of these is equally good. I'll work with the first one, but either of the others will yield the same result with approximately the same level of effort. We know that both V and I vary with time exponentially. It is *tempting* to just slap an $e^{-\frac{t}{\tau}}$ after a *P* and call it a day. But that's not correct. We have the equations $V(t)=V_0e^{-\frac{t}{\tau}}$ and $I(t)=I_0e^{-\frac{t}{\tau}}$. (I encountered some confusion, while speaking with students, over the meaning of things like $V(t)=something$. The time in the parenthesis is just a means of communicating that the potential is a *function* of time. That is, the potential varies with time. You're not supposed to do anything with the *t*, just note that it is one of the independent variables in the problem—in this case, it is the sole independent variable, but there may be several in any particular problem. We want to communicate what the independent variables are to someone reading the equations. That's all.) We substitute these into our equation for power and get $P(t)=V(t)I(t)=V_0e^{-\frac{t}{\tau}}\times I_0e^{-\frac{t}{\tau}}$. Recalling that when we multiply exponentiated quantities with the same base, we add exponents, this allows us to write $P(t)=V_0e^{-\frac{t}{\tau}}\times I_0e^{-\frac{t}{\tau}}=V_0I_0e^{-2\frac{t}{\tau}}$. And since the power dissipated at the very instant the current begins flowing through the bulb is given by $P_0=V_0I_0$, we have $P(t)=P_0e^{-2\frac{t}{\tau}}$. Note that the power decays twice as fast as either the potential or the current!

Using this equation, we now apply the same technique as in problem #2. We have $e^{-2\frac{t}{RC}}=\frac{1}{2}$ in this case, which gives $2\frac{t}{RC}=0.693$ and $t=\frac{0.693RC}{2}=4.37\times10^{-2}s$, where we have now used the resistance found previously.

4. **A 37 mF capacitor is wired in series with a battery and a resistor. It is noted that the current passing through the resistor drops to 1/6 of its initial value in 500 ms. What is the resistance of the resistor?**

This one should be a piece of cake for you by now. I just wanted to give you a bit more practice with exponentials and logarithms. As before, we know that $I(t)=I_0e^{-\frac{t}{\tau}}$. In this case, we know the time and we know $I(t)$ at that time. But we don't know τ. No problem, we just have to do a bit of algebra.

We have $I(500ms)=I_0e^{-\frac{t}{\tau}}=\frac{1}{6}I_0$. Thus $e^{-\frac{t}{\tau}}=\frac{1}{6}$ when $t=500ms$. We take the natural logarithm of both sides of the "=" and get (once again, being very careful of the sign and remembering the rules for dealing with logarithms) $\frac{t}{\tau} = \ln(6)$. So $\tau = RC = \frac{t}{\ln(6)}$ which gives, finally, $R = \frac{t}{\ln(6) \times C} = \frac{0.5s}{1.792 \times 3.7 \times 10^{-2} F} = 7.54\Omega$.

5. **A 37 mF capacitor is attached to a 12 V battery for a very long time. The battery is then removed. The capacitor is then discharged through a 10 kΩ resistor for 15 seconds. Then, the 10 kΩ resistor is replaced by a 130 Ω resistor. After two seconds, what is the potential across the resistor? What is the current through it? What is the charge remaining on the capacitor?**

Step by step—that's the only way to deal with something like this. One piece at a time.

We once again are confronted with a capacitor that is initially "fully" charged—this time with a 12V battery. It is then partially discharged through a resistor. While there is still substantial charge left on it (and the fact that the remaining charge is "substantial" really isn't of any concern to us—everything we do next would be the same if it were insignificant, we just would be less likely to be interested in the answer), it is hooked up to a different resistor. No big deal. Let's just go step by step.

The initial potential on the capacitor is the same as that of the battery: 12V. This means that it will have a charge given by the capacitor formula $Q_0 = CV = 37mF \times 12V = 0.444C$.

After 15 seconds hooked up to the first resistor, the potential on the capacitor will be given by $V(t) = V_0 e^{-\frac{t}{\tau}} = 12V \times e^{-\frac{15s}{0.037F \times 10{,}000\Omega}} = 11.5V$ while the charge remaining on it will be given by $Q(t) = Q_0 e^{-\frac{t}{\tau}} = 0.444C \times e^{-\frac{15s}{0.037F \times 10{,}000\Omega}} = 0.426C$.

The capacitor is then hooked up to a different resistor. What's key here is that neither the capacitor nor the resistor knows how the capacitor got into the state that it's in at this point. Everything behaves as thought *this* is the beginning. Remember that there really isn't any such thing as a fully charged capacitor (until the potential causes the internal electric field to exceed the dielectric strength within the gap, of course). "Fully charged" simply has to do with the relationship of the battery and the capacitor. Once the battery is out of the picture, the capacitor simply has on it whatever charge it happens to have. So we approach this step *exactly* the way we did the previous step, just using the new numbers. We have $V(t) = V_1 e^{-\frac{t}{\tau}} = 11.5V \times e^{-\frac{2s}{0.037F \times 130\Omega}} = 7.6V$ and

$Q(t) = Q_1 e^{-\frac{t}{\tau}} = 0.426C \times e^{-\frac{2s}{0.037F \times 130\Omega}} = 0.281C$. To find the current through the resistor, we could use the current-decay equation, $I(t) = I_0 e^{-\frac{t}{\tau}}$, but that would be way more work than is necessary: We've already found the revised potential, so we can simply use Ohm's law with the new potential. This gives $I = \frac{V}{R} = \frac{7.6V}{130\Omega} = 5.8 \times 10^{-2} A.$ (Be sure you understand why this can be done. If necessary, use the current-decay formula to convince yourself that it is true.)

6. **Write down, in words, Ampere's Law.**

Qualitatively first: Ampere's law states that the sum of the magnetic field around *any* closed path times the length of the path, considering only the component of the field in the direction parallel to the path at each point, is equal to the total current (times a constant) passing through a surface of any size or shape that has the closed path as its edge.

Stated mathematically: $\sum B_{||} \Delta L = \mu_0 I$.

(Notice how simple it is to say this in an equation and how hard it is to say it in English! Some concepts require just the right language to be stated efficiently.)

Stated operationally: If you have any combination of currents in a region of space, create a closed path (a "loop" of any shape you like, as long as it's closed) and a "membrane" with that loop as its edge. Add up the currents passing through the "membrane," being careful to include positive signs for currents flowing "into" the membrane and negative signs for currents flowing "out of" the membrane. This will be a scalar quantity. Next, for each segment along the closed loop, find the component of the magnetic field parallel to the segment. Multiply the field component you've just found by the length of the segments and add all the products together for the full loop. The resulting number will be equal to the current sum you did previously.

It is very important to note that this can not be done as described without calculus except for a very few cases of very high symmetry. Notably, these include: A very long straight wire, a solenoid, and a toroid. We did the first two of these in class.