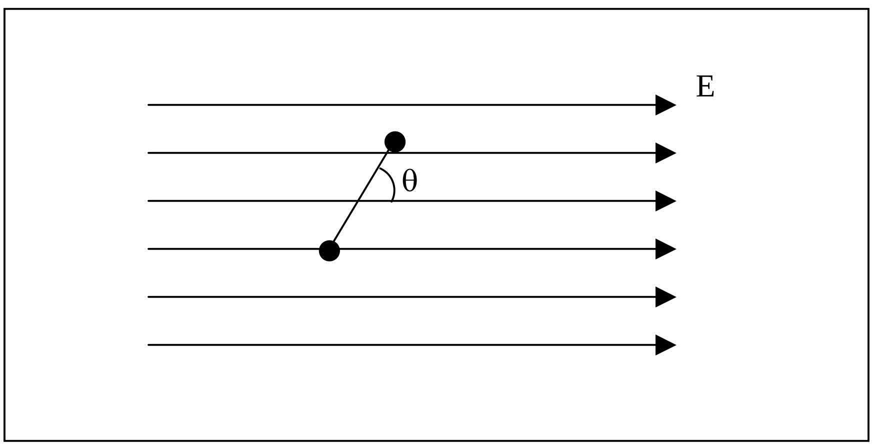
Physics 206b

Homework Assignment V
SOLUTIONS

1. **Recall that the definition of the dipole moment p is $p \equiv Qd$. This is obtained by having a charge of $+Q$ and a charge of $-Q$ separated by a distance d. (Note that in chemistry the dipole moment is called μ.) A molecule of sodium chloride has a dipole moment of $3\times10^{-29}\ Coulomb\cdot meters$. Consider the charges giving rise to the dipole moment of this molecule to be the charge on the proton and electron, respectively.**
   a. **If the axis (the line connecting the two atoms) of such a molecule makes an angle of 1.1 radians with respect to an electric field of $\vec{E} = 1\times10^5 \dfrac{Volts}{meter}\hat{x}$, as shown, what is the total torque on the molecule?**
   b. **What is the angular acceleration of such a molecule in such a field at the stated angle?**



Okay, I'm a real stinker for giving you this one! But I'm not the least bit sorry. I wanted to drive home the fact that this subject is a whole. The fact that we break it into semesters and then further break it into segments is an organizational necessity and is not a fundamental feature of the subject. Therefore, **you do not get to forget**! Last semester's work is still a component of this semester's subject and I feel free to include topics from 206a on homework assignments and exams! In this case, this problem is essentially identical to Problem #3 on Assignment #10 in Physics 206a from last Spring. You might want to review the solution to that problem as well as several others on that assignment for more insight into the solution to this problem.

So, dredging up the depths of our memories from the previous semester (or simply looking it up in the book), we recall that torque is defined as $\vec{\tau} = \vec{d} \times \vec{F}$ where $\vec{F}$ is the force acting at a point and $\vec{d}$ is the displacement vector from a point in the universe which we are free to choose (let's call it the "origin" for this) to the place where the force acts. Now, we have huge freedom in choosing the origin. However, certain choices will make our lives very much easier: Since we are interested in the total torque on this system, we will have to add the torques resulting from two different forces—one on each of the two charged objects in the dipole. By placing the origin on one of the charges, the effect of that charge on the torque will vanish. Another possible choice of origin that might make sense is the midpoint between the charges. (As an aside, a dipole is *defined* as being the two charges of the same size but opposite sign separated by some distance. Of course, if the charges aren't of the same size there is still a torque. It turns out [*insert heavy duty math here*] that in that case we do *not* have the option of where to put the origin. That's way beyond this class, but it's worth mentioning.) But that point is still more work. So let's put the origin on one of the two charges.

The torque is the product of the component of the force perpendicular to the "moment arm," that is the line connecting the two charges in this case, and the moment arm's length. (Alternatively, and totally equivalently, we can say that the torque is the product of the force and the component of the line connecting the point at which the force acts that is perpendicular to the force—this component is called the "moment arm.") Thus, we have $\vec{\tau} = \vec{d} \times \vec{F} = dF\sin(\theta)\hat{\tau}$, where I have included the unit vector $\hat{\tau}$ to remind us that the torque is a vector quantity. For convenience, let's neglect the direction for now and just find the magnitude of the torque—we can do this as long as we don't fall into the trap of believing that the direction is of secondary importance. It is not, but we are not concerned with it at this time. So we can say $\tau = dF\sin(\theta)$. We are now left with the challenge of finding the moment arm and the force.

Many people came to me concerned that I had not given the length of the dipole. Of course, since I told you the charges you can figure out the length. However, this is an unnecessary step, as you'll see in a moment. Once again we have a situation in which doing the problem with symbols and saving any numerical substitutions for the very end can save valuable time.

Since the force experienced by a charge immersed in an external electric field is given by $\vec{E}Q = \vec{F}$, we can immediately say (again, neglecting the direction of the force since we're neglecting the direction of the torque for now) $\tau = dF\sin(\theta) = dQE\sin(\theta)$. But note that $dQ = p$ so we have $\tau = dQE\sin(\theta) = pE\sin(\theta)$. We didn't need the length of the molecule at all (in this part of the problem)!

Putting these together, we have

$$\tau = pE\sin(\theta) = 3\times10^{-29}\ Coulomb\cdot meters \times 1\times10^{5}\ \frac{Volts}{meter} \times \sin(1.1)$$
$$= 2.67\times10^{-24}\ N\cdot m.$$

Now, to complete this problem we'll need to find the direction of the torque. Once again, I refer you to the homework solutions from last semester for details. We use the "right hand rule" applied to the displacement vector and the force. This gives us a direction of $-\hat{z}$. I won't require that the direction be specified for credit on this assignment, but in all future problems in which torque is calculated direction must be specified unless I explicitly state otherwise.

Now, to solve for the angular acceleration we use the rotational analog of Newton's second law $\vec{\tau} = I\vec{\alpha}$. For this we'll need the moment of inertia, I. In order to find the moment of inertia, we need to specify the geometry of the molecule and the point about which the molecule is to rotate. I didn't really give you enough information to calculate this unambiguously, so any reasonable choice of shapes and points will do. I'll treat the molecule as a "dumbbell" and use the midpoint as the center of rotation. Technically, we should use the center of mass for the best possible answer. The difference will be small. Another totally reasonable geometry would be to consider the molecule to be a solid rod. If you used that geometry, your answer will be very similar to what I find below.

We'll need the length of the molecule at this point. I gave you the total dipole moment of NaCl: $p \equiv Qd = 3\times10^{-29}\ Coulomb \cdot meters$. Since we are taking the charge to be that of the electron or proton, $Q = 1.6\times10^{-19}\ C$. Thus, we have

$$d = \frac{p}{Q} = \frac{3\times10^{-29}\ Coulomb\cdot meters}{1.6\times10^{-19}\ C} = 1.875\times10^{-10}\ meters.$$

(The distance $1\times10^{-10}\ meters$ is known as an "angstrom" in an old-fashion system of units. It is abbreviated by the symbol Å. This unit is no longer considered standard, but you may encounter it from time to time, especially in older works.)

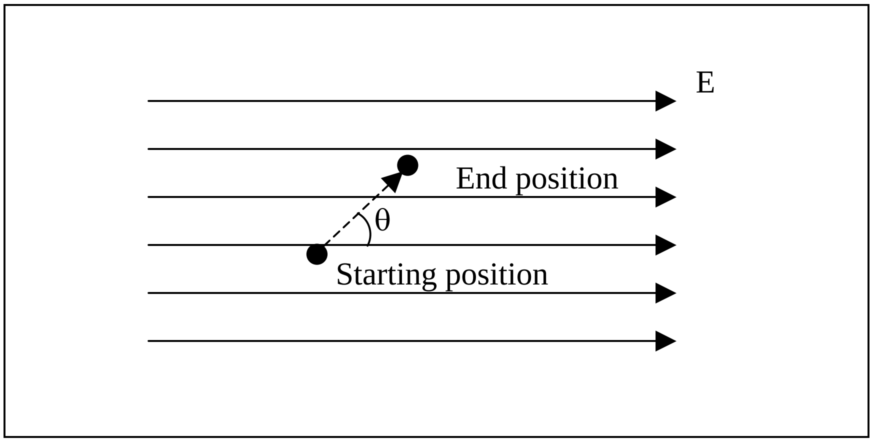
The moment of inertia for a point-mass a distance *r* from a pivot point is $I = mr^2$. We'll have a contribution for each of the masses in this, so our total moment of inertia will be $I = (m_{Na} + m_{Cl}) \times \frac{d^2}{4}$. (The 4 in the denominator comes from the fact that we had to divide the length of the molecule by 2 since we're spinning around the midpoint and then we squared the resulting length.) Now, the mass of sodium is $m_{Na} = 23\,amu \times 1.66\times10^{-27}\ \frac{kg}{amu} = 3.82\times10^{-26}\ kg$ while that for chlorine is $m_{Cl} = 35.5\,amu \times 1.66\times10^{-27}\ \frac{kg}{amu} = 5.89\times10^{-26}\ kg$. Thus, we have

$$I = (m_{Na} + m_{Cl}) \times \frac{d^2}{4} = \left(3.82\times10^{-26}\ kg + 5.89\times10^{-26}\ kg\right) \times \frac{\left(1.875\times10^{-10}\ meters\right)^2}{4}$$

$$= 8.5\times10^{-46}\ kg\cdot m^2$$

Solving for the angular acceleration, we have $\vec{\alpha} = \frac{\vec{\tau}}{I} = -\frac{2.67\times10^{-24}\ N\cdot m}{8.5\times10^{-46}\ kg\cdot m^2}\hat{z} = -3.13\times10^{21}\ \frac{1}{s^2}\hat{z}$. (Once again, you won't be penalized for leaving off the direction this time.) That's a *ferocious* acceleration! This thing is really going to start spinning!

2. **A proton in a constant electric field of** $\vec{E} = 1\times10^{5}\ \frac{Volts}{meter}\hat{x}$ **is moved 3 cm at an angle of 0.9 radians relative to the field, as shown. How much work is done in the motion?**



This is a real piece of cake! Recall that the force on a charged object in an electric field is just $\vec{F} = \vec{E}q$. The work done moving an object is $W = \vec{F}\cdot\vec{d} = Fd\cos(\theta)$, that is, in words, the work is the distance an object moves times the component of the force acting on the object parallel to the direction of the motion. This last part is crucial! The angle, θ, is the angle made by the force vector and the displacement vector. Putting these two together, we have, in this case

$$W = qEd\cos(\theta) = 1.6\times10^{-19}\ C\times1\times10^{5}\ \frac{Volts}{meter}\times.03\ m\times\cos(0.9\,rad)$$

$$= 2.98\times10^{-16}\ Joules$$

But there's another way to do this: Instead of looking at the force, let's look at the potential. The potential difference between two points in a *constant* electric field (careful! this is a special case) is $\Delta V = Ed\cos(\theta)$. Thus, the potential difference between the starting and ending point of the proton is

$$\Delta V = Ed\cos(\theta) = 1\times10^{5}\ \frac{Volts}{meter}\times0.03\ m\times\cos(0.9\,rad)$$

$$= 1.86\times10^{3}\ Volts$$

The change in potential energy of an object with a charge *q* that changes its potential by ΔV is just $\Delta U_E = q\Delta V = qEd\cos(\theta)$. Identical to what we got above.

3. **An electron is released from rest very far away from a proton whose position is fixed. When the electron is 1 mm away from the proton, what is its speed?**

   The concept underlying this question is *very* important: We pick the zero point of electrical potential (and, hence, of electrical potential energy) to be at a point infinitely far away from any point of interest. Thus, the total potential energy of an object is the difference between the object's potential energy at infinity (zero) and the object's potential energy at the point under consideration. For this to be useful, we must actually know what that difference is. Fortunately, we do know that for certain, simple geometries. In general, you'd have to start by figuring this out. But in this case, the thing which is creating the potential is a proton. The potential due to a point charge a distance *r* from that point charge is $V = k\frac{q}{r}$. To find the **potential energy** (again, this is relative to a zero at infinity) of a charged object, we just multiply the potential at the location where the object is by the charge of that object $U_E = VQ = k\frac{q}{r}Q$, where Q is the charge of the other object. (I know this gets confusing. I like to think of it as a source and a target: The source charge *creates* the potential. The target charge *experiences* the potential. In this case, the proton is the source and the electron is the target.) Here, I am using U_E to represent the potential energy of the object.

   It is very important to understand the difference between a **potential** and a **potential energy**: A point in space has a potential. An object placed at that point has a potential energy. Gravity provides a good example of a system in which this can be discussed. The (gravitational) potential energy of an object is related to its height above ground (assuming that we've taken the potential energy of the object to be 0 on the ground) *and* its mass. We have $P.E. = mgh$. There is a (gravitational) potential in this system as well, this depends on the height only, $V = gh$. Any object placed at a height *h* will be at the same potential—be it a feather or an anvil. But the potential **energy** of an object at that height will depend on its mass.

Now, energy is conserved in this system. What that means is that at all times the *total* energy of the system will be a constant. There are only two places where energy can hide in this situation: Kinetic energy and electrical potential energy. So, we have $K.E.+U_E=const.$ If we can figure out what that constant is at any point, we know what it is everywhere. Do we know it at some point? Yup: At infinity. Now, infinity is a concept, not a quantity that can be worked with. But saying "very far away" is a way of saying "close enough to being infinitely far that the difference doesn't matter." To a mathematician, this is blasphemy. To a Physicist, it's our daily bread.

So, what is the total energy when the electron is at infinity? Well, the potential energy, by definition, is zero. Since we're told the electron is "released," we know (or are at least safe assuming) that its initial speed is zero, so its kinetic energy is also zero at infinity. Thus, we can say, in this case $K.E.+U_E=0$ everywhere. A smidge of algebra gives us $K.E.=-U_E$ everywhere. So all we need to do is find the potential energy at any point and we'll know the kinetic energy.

Well, at a distance of 1 mm from the proton, the electron's potential energy is $U_E=k\frac{q}{r}Q=9\times10^9\frac{N\cdot m^2}{C^2}\times\frac{(1.6\times10^{-19}\,C)}{1\times10^{-3}\,m}\times(-1.6\times10^{-19}\,C)$. Note the signs on the charges! Keeping the signs in this case is *very* important. Working through the numbers, we have $U_E=k\frac{q}{r}Q=-2.3\times10^{-25}\,J$. This gives $K.E.=\frac{1}{2}mv^2=2.3\times10^{-25}\,J$.

From here, it's just "plug and chug" to get the speed: $v=\sqrt{\frac{2\times2.3\times10^{-25}\,J}{9.11\times10^{-31}\,kg}}=7.11\times10^2\,\frac{m}{s}$.

4. **Electrons in the "gun" of a television set are thermionically emitted from a filament. They are accelerated through a potential difference of 4 kV. Neglecting their initial speed, what speed will the electrons have after the acceleration?**

Piece of cake! Again, energy is conserved. We don't even need to consider the potential energy at infinity because we're told that the electrons travel through a potential *difference*. It's not the potential energy that matters but the *change* in potential energy. Any change in potential energy must show up in an identical (save for a sign) change in kinetic energy. Thus, we have $\Delta U_E = (\Delta V)Q = -\Delta K.E.$ This gives $\Delta U_E = (\Delta V)Q = 4\times10^3\ V \times(-1.6\times10^{-19}\ C) = -6.4\times10^{-16}\ J$ (recall that $1V = 1\frac{J}{C}$). And, as in the previous problem, we just set this equal to the kinetic energy $K.E. = \frac{1}{2}mv^2 = 6.4\times10^{-16}\ J$ which gives $v = \sqrt{\frac{2\times6.4\times10^{-16}\ J}{9.11\times10^{-31}\ kg}} = 3.7\times10^7\ \frac{m}{s}$.

As an aside: It happens so often that we have a certain number of electrons or protons (recall that these have the same charge except for a sign) undergoing an acceleration across a particular number of volts that we've created a special unit of energy. One "electron volt," abbreviated "eV," is the energy change experienced by an electron or proton moving through a potential difference of 1 volt. This is $1\,eV = 1.6\times10^{-19}\ J$. This is a very natural unit for doing Chemistry or many sorts of Physics. In the case of this problem, we'd say "the electrons' energy changed by $4\,keV$."

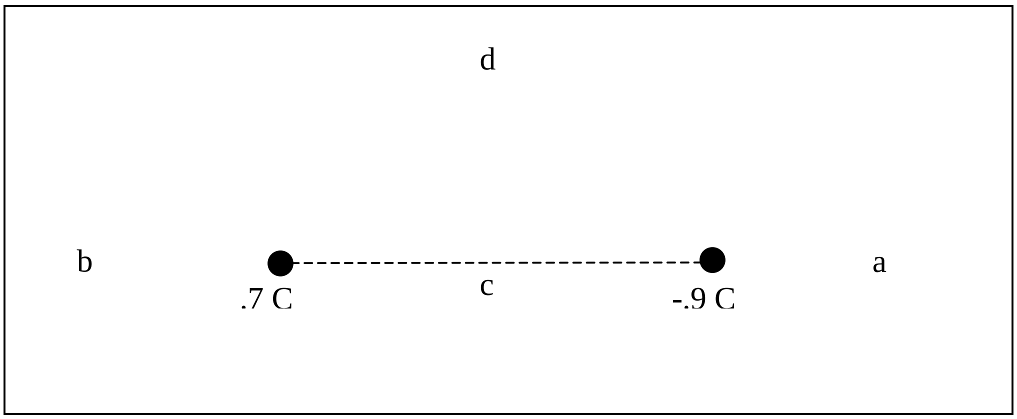
5. **Two charges lie on the *x* axis, as shown. The one on the right is -0.9 C and the one on the left is 0.7 C. They are separated by 37 cm. What is the electric potential at the following points:**

   a. **10 cm to the right of the charge on the right?**

   b. **10 cm to the left of the charge on the left?**

   c. **The point on the *x* axis midway between the two charges?**

   d. **The point 12 cm in the $\hat{y}$ direction directly "above" the midpoint between the charges?**



This is *much* easier than its brother in the previous assignment! Potential is a scalar. While it has a sign, that sign comes exclusively from the sign of the charge creating the potential—the distance is just a distance, not a displacement, so it's "unsigned" (i.e., always positive). When we have multiple charges creating the potential at a point, we just add each of their contributions, one at a time. The potential for each charge will be $V_n = k\frac{q_n}{r_n}$ where I've used the subscript *n* to keep track of the different charges—it's just a label and has no mathematical meaning other than to "name" the charges.

In this case, we have two charges. I'll do each of the four points we're considering in order. Let's call the –0.9 C charge q_1 and the 0.7 C charge q_2.:

a) Here, $r_1 = 10\,cm$ and $r_2 = 47\,cm$ so

$$V = k\left\{\frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} = 9\times10^9\,\frac{N\cdot m^2}{C^2}\left\{\frac{-0.9C}{0.1m} + \frac{0.7C}{0.47m}\right\} = -6.76\times10^{10}\ V$$

b) Here, $r_1 = 47\,cm$ and $r_2 = 10\,cm$ so

$$V = k\left\{\frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} = 9\times10^9\,\frac{N\cdot m^2}{C^2}\left\{\frac{-.9C}{.47m} + \frac{.7C}{.1m}\right\} = 4.58\times10^{10}\,V$$

c) Here, $r_1 = r_2 = 18.5\,cm$ so

$$V = k\left\{\frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} = 9\times10^9\,\frac{N\cdot m^2}{C^2}\left\{\frac{-.9C}{.185m} + \frac{.7C}{.185m}\right\} = -9.73\times10^9\,V$$

d) It’s clear that $r_1 = r_2$ again in this case, but we must do a little work to find what that distance is. Just apply the theorem of Pythagoras to get $r_1 = r_2 = \sqrt{(.12m)^2 + (.185m)^2} = .22m$

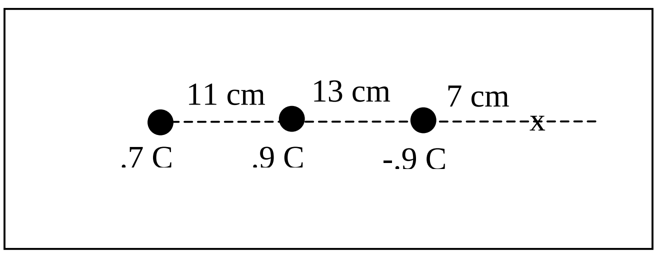
Thus $V = k\left\{\frac{q_1}{r_1} + \frac{q_2}{r_2}\right\} = 9\times10^9\,\frac{N\cdot m^2}{C^2}\left\{\frac{-.9C}{.22m} + \frac{.7C}{.22m}\right\} = -8.18\times10^9\,V$

6. **Three charges are in the configuration shown below.**

   a. **What is the potential energy strored in the configuration?**

   b. **How much work would be required to move a charge of 0.3 C to the position marked “x” from a large distance away?**

   c. **Assume the 0.3 C charge has a mass of 5 grams. With what speed would it have to be thrown to get it to the position marked “x” if thrown from a large distance away?**



The potential energy stored in the configuration is just the sum of the potential energies needed to add each of the charges to it. Just imagine building this from scratch—we move each charge into position from infinity, where its potential energy is zero. Moving the first charge into place takes no work. Moving the second charge into place requires that we fight the force between it and the first charge. Moving the third charge into place actually gets some work back because both of the first two are pulling on it so we don’t need to add energy to the system to get it into place. In fact, we need to extract energy from the system if we want the third charge to arrive with negligible speed. So the work done in that case is negative and includes contributions from each of the other two charges.

To do this systematically, let’s introduce a little notation: I’ll call U_{En} the total energy resulting from adding charge *n* to the system. I’ll call U_{Enm} the energy resulting from the interaction between charge *n* and charge *m*. I hope this becomes clear from the context below. Mathematically, we have

$$U_{E1} = 0$$

$$U_{E2} = U_{E21}$$

$$U_{E3} = U_{E31} + U_{E32}$$

Thus, the *total* energy will be given by

$$U_E = U_{E1} + U_{E2} + U_{E3} = U_{E21} + U_{E31} + U_{E32}$$

(Be careful! Resist the urge to double-count.) From here, it’s just plug and chug, again, substituting the correct values of *r* and *q* into $U_{Enm} = k\frac{q_n q_m}{r_{nm}}$ (where, again, I’m using the subscripts to indicate charge *n* or charge *m* and the distance between them). This gives

$$U_E = U_{E1} + U_{E2} + U_{E3} = U_{E21} + U_{E31} + U_{E32}$$

$$= k\left\{\frac{q_2 q_1}{r_{12}} + \frac{q_3 q_1}{r_{13}} + \frac{q_3 q_2}{r_{32}}\right\}$$

$$= 9\times10^9 \frac{N\cdot m^2}{C^2}\left\{\frac{.7C\times.9C}{.11m} + \frac{.7C\times(-.9C)}{.24m} + \frac{.9C\times(-.9C)}{.13m}\right\}$$

$$= -2.8\times10^{10}\ J$$

Next, we are asked to add a fourth charge to the setup. Since we approached the first three systematically, to extend that system to the fourth charge is a piece of cake. We have

$$U_{E1} = 0$$
$$U_{E2} = U_{E21}$$
$$U_{E3} = U_{E31} + U_{E32}$$
$$U_{E4} = U_{E41} + U_{E42} + U_{E43}$$

Since we've already calculated the first three of these and added them together, all we need to do is find the fourth one in the progression. Using the same technique as previously, we have

$$U_{E4} = k\left\{\frac{q_4 q_1}{r_{41}} + \frac{q_4 q_2}{r_{42}} + \frac{q_4 q_3}{r_{43}}\right\}$$

$$= 9\times10^9 \,\frac{N\cdot m^2}{C^2}\left\{\frac{.3C\times.7C}{.31m} + \frac{.3C\times.9C}{.20m} + \frac{.3C\times(-.9C)}{.07m}\right\}$$

$$= -1.65\times10^{10}\ J$$

We could continue this technique forever. Each additional charge interacts with all the charges that are already there. The order in which we choose to assemble the system doesn't matter (try this for yourself!), so, once we're done, we can talk about the system having some total energy stored in it just by dint of it existing in the first place.

Part (c) just applies what you already know about conservation of energy: In a system without dissipative forces, all the energy is conserved although it can be in different forms. In particular, in this case, our energy can be either kinetic energy or electrical potential energy (we'll just call it "potential energy" for short since there's no other kind in this system). Mathematically, we write this as $U_E + K.E. = constant$. Since the constant is just that, constant, if we can figure out what it is at any one point in the system we will know it for all points.

Now, here I have to confess to having made a blunder in writing this question: I slipped a sign! (Gee, that's never happened before...) When I originally calculated this, I got a positive potential. I do apologize for any confusion caused by this error.

Let's run with it, though. First, let's deal with the situation that you do have. Since the potential energy of this charge is negative, work was done *on it* by the other charges. If we released it from rest from a long distance away, it would accelerate up to some speed and continue to get closer. We could figure out where it would stop with a bit of work—somewhere close to the positive charges. This would be basically the same problem as #3 above.

Now, let's rework this. Assume that I had gotten a positive value for the energy. Let's just flip the sign on what we *did* get and take $U_E = 1.65 \times 10^{10}\ J$. (That's what we would have gotten if the new object's charge were –0.3 C.) This means that we had to do this much work on the system to push the charge into position.

Now, if we threw the charge, starting at infinity, all of its energy would be kinetic. When it arrived at the final point, we'd want it to come to a stop, so all of its energy would be potential energy. At the beginning we'd have $K.E._{begin} = constant$ and $U_{E\,end} = constant$ and the constant is the same thing. So we have $K.E._{begin} = U_{E\,end}$. Well, we've just calculated $U_{E\,end}$, so we just set this equal to $K.E. = \frac{1}{2}mv^2$ and solve for the speed. This gives $v = \sqrt{\frac{2 \times 1.65 \times 10^{10}\ J}{5 \times 10^{-3}\ kg}} = 2.57 \times 10^3\ \frac{m}{s}$.

As a final step, let's imagine the Physics of this: We're standing very far away from the original set of three charges. They're just sitting there. We're holding another charge (as mentioned above, its sign would have to be negative to give the energy we used here). We toss it at the speed shown above toward the charges that are already sitting there. As it gets closer, the other charges push on it, repelling it and slowing it down. It slows down more and more as it gets closer. As it gets closer, the repulsive force also gets bigger, increasing the rate at which it slows down. But we've picked the speed *just right* so that when it hits the point marked X it is at a dead stop. Of course, it doesn't end there. There's still a net repulsive force on the object and it begins to fly back toward us. When it gets back to us, the repulsive force is zero, but it's moving back at its original speed when it smacks us right in the face. Maybe throwing things at repulsive forces isn't such a good idea, after all!

7. **Calculate the total capacitance of a pair of parallel plates separated by air. Consider the plates to be circular with a radius of 1 cm. Take the separation between them to be 1/2 mm.**

The only judgment call here is whether it's alright to use the dielectric constant for vacuum in place of air. The two vary by only a fraction of a percent, so, yes, go ahead and pretend it's vacuum—it makes life a lot easier!

After that, this is really just plug and chug. We use $C = \frac{A}{4\pi kd} = \varepsilon_0 \frac{A}{d}$ where A is the area of the plates, d is the distance between them, and ε_0 is the "permittivity of free space." This has a number, which you can memorize: $\varepsilon_0 = 8.8542 \times 10^{-12} \ \frac{C^2}{N \cdot m^2}$. However, it's easier to remember that $k = 9 \times 10^9 \ \frac{N \cdot m^2}{C^2} = \frac{1}{4\pi\varepsilon_0}$. So you can use $\varepsilon_0 = \frac{1}{4\pi k} = \frac{1}{4\pi\left(9 \times 10^9 \ \frac{N \cdot m^2}{C^2}\right)}$.

This might come in handy. It certainly will help you to remember the units and order of magnitude of ε_0.

We'll need to do a quick calculation to find the area of the plates: Since they are circular with a radius of 1 cm, the area is $A = \pi r^2 = \pi \times (.01m)^2 = 3.1 \times 10^{-4} \ m^2$ Using this, we get $C = \varepsilon_0 \frac{A}{d} = 8.9 \times 10^{-12} \ \frac{C^2}{N \cdot m^2} \times \frac{3.1 \times 10^{-4} \ m^2}{5 \times 10^{-4} \ m} = 5.5 \times 10^{-12} \ F$.

Note that I've introduced a new unit: The "farad" abbreviated "*F*". This is the unit of capacitance in the SI system. The value that we got is in the picofarad range. This seems tiny, but it's a not-unusual value. A farad is *huge*! Picofarads are often called "puffs" in part because their abbreviation is "pF" and in part because they are rather small.

Note that the formula we used is good for parallel-plates only! A different geometry for the conductors would lead to a different formula for the capacitance. Be careful!

Problems #8 and #9 have been moved to the next assignment.