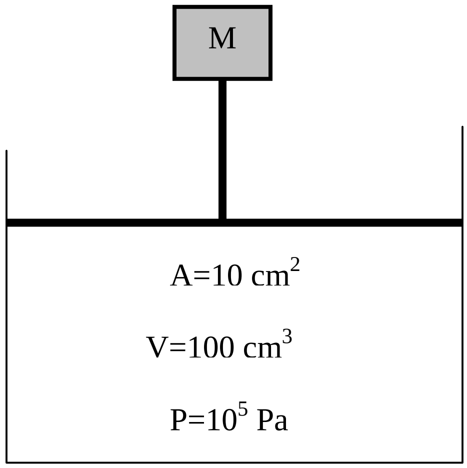
PHYSICS 206b
HOMEWORK #3
SOLUTIONS



1. **Consider the sample of ideal, monatomic gas in a cylinder sealed by a piston shown above. Neglect any pressure of atmosphere outside of the cylinder. If the mass is $m = 10.2\,kg$, how much heat must flow into the gas to raise the mass a distance of 2 cm? What will be the temperature of the gas when this is achieved?**

I must apologize for a glaring error in the statement of this problem: I overspecified the system! That is, I told you two things about the pressure that are not totally consistent with each other. First, I told you that the pressure of the gas is 10^5 Pa. The, I told you that the piston is supporting a mass of 10.2 kg. Either one would have been sufficient. If you calculate the pressure needed to support the mass (this was done in Problem #1 of Assignment #12 last semester), you find that the numbers are consistent to five significant figures, so there was nothing inconsistent in the statement of the problem. But it is an error to overspecify things, in general, even if the overspecification is consistent. I hope this didn't cause any confusion!

This is an "isobaric" process—i.e., the pressure does not change during it. The reason for this is that the pressure is provided by the weight of the mass and the area of the piston, neither of which change. So we can use the ideal gas law and treat P as a constant quantity. This is $PV = Nk_BT$. Now, according to the First Law of Thermodynamics, the total heat put into the system is equal to the work done *by* the system added to the change in internal energy of the system. This can be written $Q = \Delta U + W$. We just need to find the work and the change in internal energy and we're done. Let's do these one at a time.

The total work done is going to be precisely that which is necessary to lift the mass the prescribed distance. In this case, we have $W = mgd$, where d is 2 cm, as stated in the problem. That was easy. We'll see in a moment that we're not done yet, however.

Alternatively, we can calculate the work by saying $W = P\Delta V$. Note that this is totally consistent with the statement above: Since $P = \frac{mg}{A}$ and $\Delta V = Ad$ (see the discussion below), $P\Delta V = mgd$. Either statement is equally valid.

We now need to find the total change in internal energy. How are we going to do that? Well, we weren't told anything about the temperature, but we know that it increases, making the volume expand. If we call the original height of the cylinder *h*, then the original volume is $V_i = Ah$, where *A* is the area of the piston. Since the cylinder's height increases by an amount *d*, the final volume is given by $V_f = A(h+d)$. Thus the change in volume is $\Delta V = V_f - V_i = Ad$, as stated above. We insert this into the ideal gas law. Since the only things that can change are volume and temperature, this gives us $P\Delta V = Nk_B\Delta T$. That is, the change in pressure causes a proportional change in temperature.

We can solve for the change in temperature to get $\Delta T = \frac{P\Delta V}{Nk_B}$. Of course, we know the change in volume. Including this, we have $\Delta T = \frac{P\Delta V}{Nk_B} = \frac{PAd}{Nk_B}$. We *could* stop here. But let's take it one more step, just 'cause it's pretty. Recall that the pressure is given by $P = \frac{mg}{A}$. Substituting for this, we have $\Delta T = \frac{PAd}{Nk_B} = \frac{\frac{mg}{A}Ad}{Nk_B} = \frac{mgd}{Nk_B}$. It is tempting, at this point, to substitute numbers. But then we'd have to figure out *N*, which is a real pain. My strongest advice to you is to refrain from inserting any numbers until the very end of a problem—things frequently work out to our benefit if we keep symbols. This is certainly true here, as we'll see in a moment.

We know that the change in internal energy of a system is related to its change in temperature via the specific heat. For a collection of *N* atoms in a gas, the *total* change in energy is given by $\Delta U = \frac{3}{2}Nk_B\Delta T$. This was derived last semester. The derivation can be found in the pages of your text discussing the Kinetic Theory of Gases. (Note that the proportionality factor for between the *total* energy of a system and the temperature of the system is called the "heat capacity." We didn't discuss this in class. It's quite a simple concept, though: The "specific heat," which we discussed, is the proportionality constant between the energy of a specific *mass* of the substance and its temperature—that is, usually, per kg or per gram. The heat capacity just scales this by the *amount* of the substance that we actually have. So the heat capacity of *N* atoms of an ideal gas is $\frac{3}{2}Nk_B$.) So, using the expression for the change in temperature we found above, we can write that the total change in internal energy is $\Delta U = \frac{3}{2}Nk_B\Delta T = \frac{3}{2}Nk_B \times \frac{mgd}{Nk_B} = \frac{3}{2}mgd$.

Notice that all the annoying stuff—*N* and k_B, just go away! *That* is why you should wait until the end to substitute numbers.

To find the total heat which must flow into the cylinder to achieve our desired goal, all we do is add the two results we've found. That is, $Q = \Delta U + W = \frac{3}{2}mgd + mgd = \frac{5}{2}mgd$. Notice the inefficiency of this: We've had to use 150% *more* energy heating the gas up than we actually got out of the system in terms of work! That's not very efficient. But that's a fact of life: Most of the time, we waste more energy than we use and not because of poor engineering, just because of fundamental laws of Physics. (Poor engineering just exacerbates this.) This situation would be even worse if we'd used some gas with more degrees of freedom than a monatomic gas, as you'll see in the next problem.

Inserting numbers into this, we get $Q = \Delta U + W = \frac{5}{2}mgd = 5\,J$.

Now, finally, to find the final temperature we will need the original temperature. I didn't give you this, so we'll have to do it symbolically, without using numbers. (As I've told you repeatedly: If I don't specify a value for something that you think you'll need, just work the answer out symbolically. Often, you'll discover that you didn't need the number at all. If it turns out that you *do* need a specific value at the end, you are welcome to substitute a reasonable number for the quantity—just make sure to state this explicitly.)

Starting with the ideal gas law, we have $PV = Nk_BT$. Now, the change in temperature is given by $\Delta T = \frac{mgd}{Nk_B}$, as found previously. Since we don't know the number of atoms in the gas, N, we'll have to get this from the ideal gas law. This gives $N = \frac{PV_{original}}{k_BT_{original}}$. Thus, $\Delta T = \frac{mgd}{Nk_B} = \frac{mgdT_{original}}{PV_{original}}$. This gives us $T_{final} = T_{original} + \Delta T = T_{original} + \frac{mgdT_{original}}{PV_{original}} = T_{original}\left(1 + \frac{mgd}{PV_{original}}\right)$. We can make this even prettier if we remember that the pressure is just the weight divided by the area. So we can write

$$T_{final} = T_{original}\left(1 + \frac{mgd}{\frac{mg}{A}V_{original}}\right) = T_{original}\left(1 + \frac{Ad}{V_{original}}\right) = T_{original}\left(1 + \frac{Ad}{Ah}\right) = T_{original}\left(1 + \frac{d}{h}\right)$$

where, as before, *h* is the original height of the cylinder.

Even though we don't know what $T_{original}$ is, we can find the multiplicative factor. This is $\left(1+\frac{mgd}{PV_{original}}\right)=\left(1+\frac{d}{h}\right)=1.2$. So we need to increase the temperature by 20% (measured on an absolute temperature scale!) to lift the mass the prescribed distance. If we started with the gas at 300 K, we would have to increase the temperature to 360 K.

2. **Again consider the system above. If the monatomic ideal gas were replaced by a diatomic ideal gas, how much heat would need to be added to the system to achieve the same result? Will the final temperature be different in this case?**

Recall that previously we obtained $Q=\Delta U+W=\frac{3}{2}mgd+mgd=\frac{5}{2}mgd$. The $\frac{3}{2}$ in the first term of this came from the heat capacity of a monatomic ideal gas. If we look even more closely, it came from a thing called "the equipartition theorem" which states that each "degree of freedom" of a system gets $\frac{1}{2}k_B$ in the heat capacity. Now, a complete discussion of the equipartition theorem is way beyond this course, but we looked at how it applies to a diatomic system in class. Recall that the "3" for the monatomic system came from the fact that each molecule (atom) can travel in three possible directions (*x*, *y*, and *z*). A diatomic molecule can do this as well, but a diatomic molecule can also rotate in two possible directions and can also vibrate. The result is that a diatomic molecule has 7 degrees of freedom (six of these are relatively easy to see, the seventh is very subtle and I've known *many* professionals who miss it—kinda scary!) so it has a heat capacity of $\frac{7}{2}k_B$. That is, a sample containing N molecules requires an energy of $\frac{7}{2}Nk_B\Delta T$ to change its temperature by an amount ΔT.

This accounts for the change in internal energy of our sample. The work done is the same. So the first law of thermodynamics for the diatomic molecules in a cylinder/piston system is $Q=\Delta U+W=\frac{7}{2}mgd+mgd=\frac{9}{2}mgd$. Inserting numbers, this gives $Q=\Delta U+W=\frac{9}{2}mgd=9\,J$.

3. **One hundred moles of an ideal, monatomic gas fills a cylinder at a pressure of $1\times10^5\,Pa$ at 300 K. The sample then undergoes the following set of processes:**
    a. **The sample's pressure is kept constant (as in the problem above) but heat is added to it to increase its volume by 500 ml.**

b. **Next, the piston is clamped in place so that the volume is fixed while the gas is cooled down. Its temperature is reduced until its pressure is reduced by** $2.5\times10^4\ Pa$ **.**

c. **The mass sitting on the piston is reduced to maintain the new pressure and the gas is compressed by 500 ml (back to its original volume). This may require the addition or removal of heat.**

d. **Finally, the volume of the gas is fixed and then heat is added or removed from the gas to increase its temperature back to** $1\times10^5\ Pa$ **. Note that this returns the system back to its original configuration.**

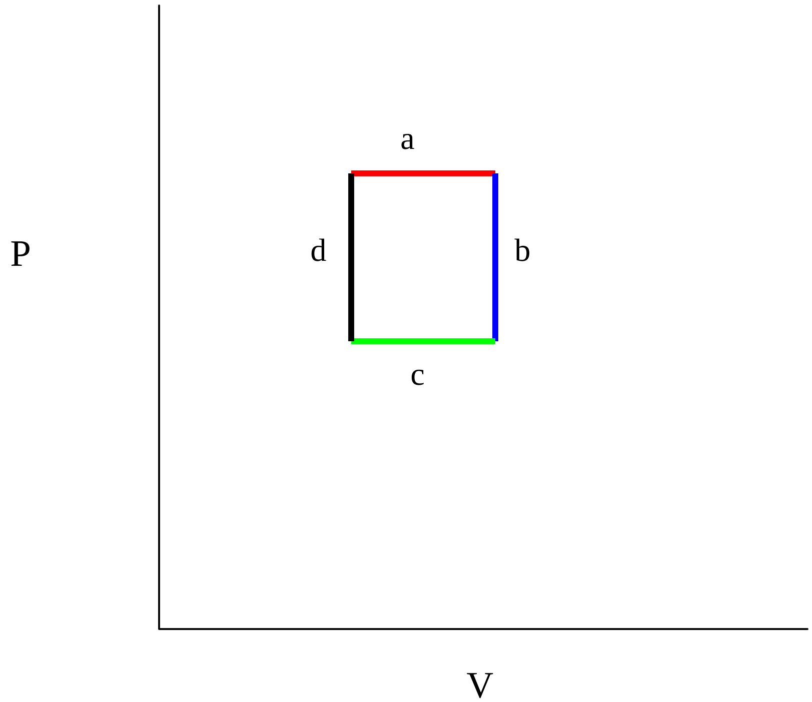
i. **How much net work is done *by* or *on* the piston in this cycle?**

ii. **For each of the four steps, determine how much heat must be added to or removed from the system.**

iii. **Assuming that any heat removed from the system is lost, what is the efficiency of the process? This is the ratio of net work done by the system to total energy added to the system.**

This seems *way* more complicated than it actually is. As with any complicated-looking problem, the best way to attack it is in bite-sized pieces. Let's do it step by step.

First, note that the system has gone through a complete cycle. That is, it is in the exact same state at the end of the process as it was when it began. As an aid, it makes sense to plot the pressure and volume of the system in each of these steps on a "P-V" diagram. In this case, this is particularly easy since steps "a" and "c" are "isobaric" processes (pressure held constant) and will, therefore, appear as horizontal lines on our P-V diagram and steps "b" and "d" are "isochoric" processes (volume held constant) and will, therefore, appear as vertical lines on our diagram. This gives us:



It is true, in general, that the total work done by a system that goes through a complete cycle is just the area enclosed by the cycle on a P-V diagram. If we had Calculus, we could work with much more complicated systems. But, with our limited toolbox, we are stuck with relatively simple systems (although they do not have to be "iso" type systems—there's no rule that says that *anything* has to be held constant). In the case of the system depicted by this cycle, it is clear that the area enclosed by the cycle is just $A = Q = \Delta P \times \Delta V$ where ΔP is the pressure change incurred in steps b and d (except for the sign) and ΔV is the volume change incurred in steps a and c (again, save for the sign). Inserting numbers, we get (remembering to express the volume change in cubic meters) $Q = \Delta P \times \Delta V = 2.5\times10^4 \times 5\times10^{-4}\ m^3 = 12.5\,J$.

Now, let's look at the heat flow in each step separately:

a: This is identical to the system in problem #1. We can just use the expression for the heat capacity of a monatomic ideal gas (I also derived this for you in class): $Q = \frac{5}{2}P\Delta V = \frac{5}{2}\times1\times10^5\,Pa\times5\times10^{-4}\,m^3 = 125J$.

We'll need the final temperature in a little while, so let's calculate that while we're at it. This can easily be done using the ideal gas law adjusted to allow for the fact that we know a change in volume but don't really feel like solving for the volume itself: $P\Delta V = Nk_B\Delta T$. Solving for the change in temperature, we have $\Delta T = \frac{P\Delta V}{Nk_B} = \frac{50J}{6.02\times10^{25}\times1.38\times10^{-23}\ \frac{J}{K}} = 0.06K$.

(Notice that I cheated a smidge and used the fact that $\frac{5}{2}P\Delta V = 125J$ to get $P\Delta V = \frac{2}{5}Q = 50J$.) So the final temperature is given by $T_f = 300.06K$.

(Does this seem like a small change in temperature to you? Well, think about it: We put 125 joules into the gas, total. Fifty joules of this went to do mechanical work. So we've only added 75 joules to the internal energy. But if we were to calculate the initial internal energy [do it!], we would find that it is about $3.74\times10^5 J$. So adding 75 joules is only increasing the internal energy by about 2/100ths of a percent—a tiny change!)

b: Now we've got an isochoric process. All heat that flows into or out from the system in an isochoric process increases or decreases the internal energy (which shows up as temperature change) in an isochoric process since no work is done. Using the ideal gas law, we can write $V\Delta P = Nk_B\Delta T$. Now, it is tempting to start throwing some numbers around to solve for the unknown quantities in this. You certainly *can* do that. But a very useful technique with which you should become familiar is to take a ratio—find the fractional changes in the quantities you're looking for. This is a real effort-saver. Let's divide both sides of the above expression by the "normal" ideal gas formula $VP = Nk_BT$. This gives (after canceling all the stuff that's the same in the numerator and the denominator) $\frac{\Delta P}{P} = \frac{\Delta T}{T}$. So $\Delta T = T\frac{\Delta P}{P}$. To turn this into an energy, we just multiply by $\frac{3}{2}Nk_B$ (since it's a monatomic gas). This gives $Q = \frac{3}{2}Nk_B\Delta T = \frac{3}{2}Nk_BT\frac{\Delta P}{P}$. Now we can just stick in the numbers:

$$Q = \frac{3}{2}Nk_BT\frac{\Delta P}{P}$$

$$= \frac{3}{2}\times100moles\times6.02\times10^{23}\frac{molecules}{mole}\times1.38\times10^{-23}\frac{J}{K}\times300.06K\times\frac{-2.5\times10^4 Pa}{1\times10^5 Pa}$$

$$= -9.35\times10^4 J$$

(note the sign—we are reducing the pressure so the heat flows *out* of the system).

Once again, let's just figure out what the new temperature is. Even if we don't need it later, it's interesting to know. $\Delta T = T\frac{\Delta P}{P} = 300.06K\times\frac{-2.5\times10^4 Pa}{1\times10^5 Pa} = -75K$. So our final temperature is about 225 *K*.

c: The new pressure is now $P = 7.5\times10^4\,Pa$. We do the same thing as in step a, only with the new pressure and in the opposite direction. $Q = \frac{5}{2}P\Delta V = \frac{5}{2}\times 7.5\times10^4\,Pa\times\left(-5\times10^{-4}\,m^3\right) = -93.75\,J$. Once again, heat is removed from the system.

Once again, let's find the temperature in case it's useful in the future. Using the exact same method as in part a, we have $\Delta T = \frac{P\Delta V}{Nk_B} = \frac{-\frac{2}{5}\times 93.75J}{6.02\times10^{25}\times1.38\times10^{-23}\,\frac{J}{K}} = -0.045K$. Again, a really tiny number. We could almost certainly ignore this temperature change and the one that we found in part a without noticing the loss of accuracy. But, inserting this just for completeness, we have $T_f = 225.015K$. Let's just call it 225 *K*.

d: Finally, we warm things up again. We can follow the same procedure as in step b. We have

$$Q = \frac{3}{2}Nk_BT\frac{\Delta P}{P}$$

$$= \frac{3}{2}\times 100moles\times 6.02\times10^{23}\,\frac{molecules}{mole}\times1.38\times10^{-23}\,\frac{J}{K}\times225K\times\frac{2.5\times10^4\,Pa}{7.5\times10^4\,Pa}$$

$$= 9.3461\times10^4\,J$$

The total work done by this system was (as found above) 12.5 joules. To get this, we had to put in $9.35855\times10^4\,J$. We also extracted $9.3594\times10^4\,J$. *If* we built our system so that the extracted energy could be reused, then we'd be in fine shape. But the presumption stated in the problem is that all the extracted energy is lost. (During office hours I mis-advised a couple of people on this. My apologies: My instructions in the problem were to throw the extracted energy away, not account for it in the total efficiency. Of course, you will not be penalized for doing this.) So the total efficiency is the ratio of the work done to the energy put in. This is

$$e = \frac{12.5J}{9.36\times10^4} = 1.3\times10^{-4}.$$

Of course, this is a horribly inefficient system! In reality, we would find a way to reuse much of the extracted energy, but never all of it.

4. **Heat is added to a system consisting of a cylinder filled with a monatomic ideal gas and a piston that is *not* constrained to have a constant pressure. The pressure is (somehow) continuously adjusted so that the volume increases as heat is added to the system without the temperature changing. The temperature is maintained at 500 K. The system begins with a pressure of $3\times10^{5}\,Pa$ and a volume of 2 liters. At the end of this isothermal process, its volume is 3.5 liters. How much work was done in this process? How much heat had to be added to the system to achieve this work?**

Wow, a very simple one compared to that last one! Really, all that you have to realize with this system is that it is isothermal. In this case, we derived an equation in class that gives the total work done. This is $W = Nk_B T \ln\left(\frac{V_f}{V_i}\right)$. Of course, I didn't tell you the number of molecules, but that shouldn't slow you down at this point. You know to use the ideal gas law to get $N = \frac{PV_i}{k_B T}$. So we have $W = PV_i \ln\left(\frac{V_f}{V_i}\right)$

We can directly insert numbers into this and get

$$W = PV_i \ln\left(\frac{V_f}{V_i}\right) = 3\times10^{5}\,Pa\times2\times10^{-3}\,m^{3}\times\ln\left(\frac{3.5l}{2l}\right).$$
$$= 600\times\ln(1.75) = 335.8J$$

(Notice that I got lazy and expressed the volume in liters inside the parentheses. Since I knew that the units on the numerator and the denominator would be the same and would cancel, I knew that I could get away with this. This sort of thing is a time-saver, but make sure you know what you're doing before trying it.)

Of course, since this is an isothermal process, all energy added to the system goes to work. The heat added and the work done are the same. $Q = 335.8J$.

Problems #5 and #6 have been moved to the next assignment.