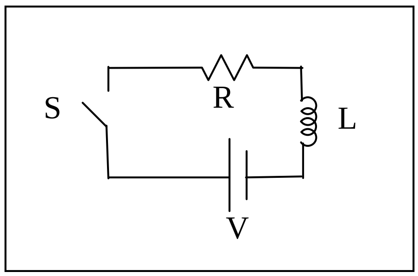
Physics 206b
Homework Assignment XII
SOLUTIONS

1. In the circuit below, the resistance is 37Ω and the inductor is 140 mH. After switch S is closed, how long will it be before the voltage difference across the inductor is ½ V?



This is just like the analogous question (question #1) in assignment 8, except that inductors behave “opposite” to capacitors in some sense. (This is *not* literally true! It’s a good concept to get you started, but don’t take it too far.) Let’s walk through what happens when the switch is thrown.

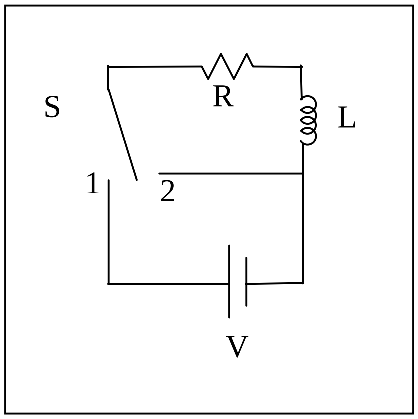
The instant the switch is thrown, the EMF source (e.g., a battery) will start a current flowing in the wire. If there were no inductor there, the current would instantaneously jump from 0 to $I = \frac{V}{R}$, as predicted by Ohm’s law. (Maxwell’s equations tell us that this will actually happen over a time scale determined by the speed of light and the length of the wire, but that timescale is so tiny that we can ignore it for this problem—it might as well be zero so far as we are concerned.) But this would mean that the loops of the inductor would have a very large ΔI occur in a very small Δt. Since the B field inside the inductor is directly related to the current flowing through it, by Faraday’s law, this means that the inductor will develop a *huge* EMF of its own very rapidly—essentially instantaneously. Well, that can’t persist for more than a teensy instant of time since whatever EMF is generated by the inductor will push in the direction *opposite* (because of Lenz’s law) to the EMF that created the current in the first place. The net result of all of this is that the inductor will *fight* the current and prevent it from becoming large too quickly. It *will* slowly grow, however, and will asymptotically approach the value predicted by Ohm's law as time passes.

The EMF produced by the inductor must be opposite to that of the "battery" and must start out having the same size as that of the "battery," by the argument above. It will evolve according to the equation $\mathcal{E}_L = -\mathcal{E}_{max} e^{-\frac{t}{\tau}}$ where $\mathcal{E}_{max}$ is the EMF applied to the inductor and $\tau = \frac{L}{R}$.

It is *very* important to remember that the EMF created by the inductor starts high and becomes low. It is *because* of this evolution that the current through the inductor starts low and becomes high! I can almost guarantee some sort of question on the Final Exam which will test whether you remember this aspect of these systems!

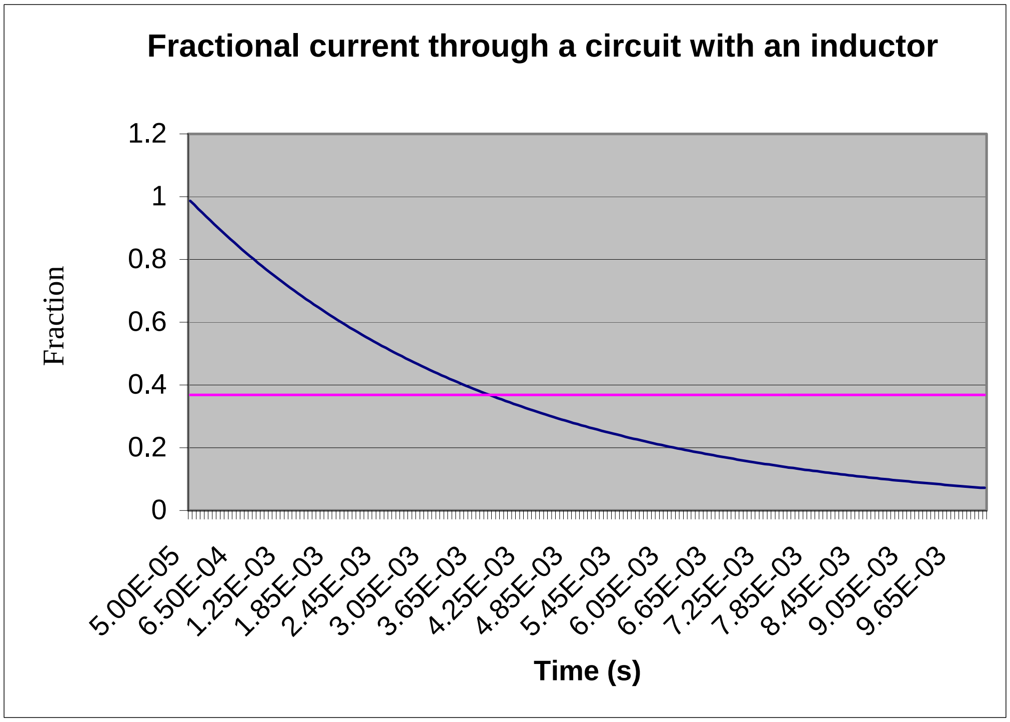
Using the above relation we can determine the time at which the inductors induced EMF is ½ of the EMF applied to it by the "battery." This is $\mathcal{E}_L = \frac{1}{2}\mathcal{E}_{max} = \mathcal{E}_{max} e^{-\frac{t}{\tau}}$. Here I've simply ignored the – sign since we really don't care that the inductor's EMF is opposite to that of the battery. Dividing the common term and taking a natural logarithm of both sides gives $\frac{t}{\tau} = \ln(2)$ so $t = 0.693\frac{L}{R}$. Substituting the numbers gives $\tau = \frac{L}{R} = \frac{140\times10^{-3}H}{37\Omega} = 3.8\times10^{-3}s$. (Normally, I would *strongly* recommend doing this calculation of τ right off the bat—as soon as you see the problem. Having this number in your head will guide you through an understanding of the evolution of the system.) So $t = 0.693\frac{L}{R} = 0.693\times3.8\times10^{-3}s = 2.6\times10^{-3}s.$

2. Now consider the circuit below. Initially, the switch is in position 1. After a long time, the switch is moved to position 2. Make a sketch of the current flowing through the inductor as a function of time after the switch is moved. Use the values for R and L from the previous problem. Be sure to indicate on your sketch the time at which the current will be at the $\frac{1}{e}$ level. If the EMF provided by the battery is 12 V, after 1 millisecond, what will be the current flowing through the resistor (this will be an actual value, not just a fraction)?



After a long time, an inductor is just a wire. The current flows through it as though it were any other piece of wire in the circuit. The current can readily be calculated using Ohm’s law. When the switch is thrown to position 2 the applied EMF goes away. According to Ohm’s law, the current in the inductor ought to vanish as well. However, Faraday’s law says that the very fact that the current *changes* creates an EMF in the inductor which actually keeps the current going for a little while. Of course, since the thing causing the EMF in the inductor is the *change* in the current, the very existence of this EMF impedes the current from changing and the EMF drops away.

Taking this full evolution into account, we can express the current as a function of time as $I = I_{max} e^{-\frac{t}{\tau}}$ where $I_{max} = \frac{V}{R}$ and $\tau = \frac{L}{R}$. We calculated τ in the previous problem and found it to be $\tau = \frac{L}{R} = \frac{140\times10^{-3}H}{37\Omega} = 3.8\times10^{-3}s$ so the current will drop to 1/e of its maximum value in $3.8\times10^{-3}s$. Sketched, this has the appearance shown below. The 1/e level is indicated by the purple line.



So after 1 ms, the current will be given by $I = I_{max} e^{-\frac{t}{\tau}} = I_{max} e^{-\frac{1\times10^{-3}s}{3.8\times10^{-3}s}} = 0.77 I_{max}$. Taking $I_{max} = \frac{V}{R} = \frac{12V}{37\Omega} = 0.32A$, we have $I(1ms) = 0.32A \times 0.77 = 0.25A$.

3. The heating coil in a toaster oven has an inductance of 1 μH. Assume that the toaster is being powered by a D.C. source (of course, they are usually plugged in to A.C. outlets, so this is a simplification) which results in 10 A flowing through the coil. A person making a piece of toast notices it start to burn and quickly unplugs the toaster. A 2 mm spark is observed to be produced by the tip of the plug when this is done. Take the breakdown voltage of air to be $3\frac{kV}{mm}$. Given this information, what was the initial rate of change of the current through the coil? (I.e., what was $\frac{\Delta I}{\Delta t}$ initially?)

We've all had this experience, or, at least one very similar to it. It's useful and instructive to look at the Physics of it. To throw a spark of any length requires a *large* potential difference between the ends of the spark. In air, that potential difference must be at least $3\frac{kV}{mm}$. (As an aside: The amount of potential difference needed can be greatly reduced if one has the help of the cosmos. I'm not being facetious nor is this unusual. Frequently, cosmic rays will enter a system with a potential difference in it. The cosmic ray [a high-energy particle created by some nuclear "event" in space] strikes an atom somewhere between the two points that will, ultimately, be at the ends of the spark. This ionizes the atom by knocking off one or more of its electrons. There is now a free electron and a free positive ion sitting between a pair of electrodes with a potential difference between them. The two objects accelerate in opposite directions. They smack into neutral atoms as they go, ionizing those atoms. The result is an "ion trail" which can carry a current requiring a *much* lower potential difference than would have been needed without the help from the cosmic ray. This process is actually essential for starting things like neon signs and some fluorescent tubes.) Clearly, no spark of significant length can be created with the 120V available from an electrical outlet in a typical home! But we've all seen such sparks, so what gives?

Well, the sparks are created not by the 120V potential difference between the poles of an outlet but rather by the potential difference between the poles of the plug that result from induced EMF in the circuit of the device being unplugged or turned off. (Many light switches, for example, throw a millimeter-long spark *every* time they are switched off. This is a major cause of long-term failure of the switches.) According to Faraday's law, when we try to turn something off very quickly, if it has any inductance in it it will create an EMF due to the rapid change in current. This induced EMF is proportional to the rate of change of the current, so it can be huge!

In the problem you are given, the length of the spark is 2 mm. This implies that the EMF induced is *at least* $\mathcal{E} = 2mm \times 3\frac{kV}{mm} = 6\times10^{3}V$. This is related to the changing current by $\mathcal{E} = -L\frac{\Delta I}{\Delta t}$. Since you are told that the inductance of the toaster is $L = 1\times10^{-6}H$, so, ignoring the sign since this just tells us that the current is "turning off," which we already know, $\frac{\Delta I}{\Delta t} = \frac{6\times10^{3}V}{1\times10^{-6}H} = 6\times10^{9}\frac{amperes}{second}$.

The number above is all that was asked for in the problem. But it's fun to take it one more step. We know that the current will drop according to an exponential function. But a quick-and-dirty approximation of the time it takes to go from its maximum to zero can be gotten just by assuming the dependence is linear, not exponential. It's just an estimate, but it gets one on the right track. We know that the current starts at 10 amperes. We also know that $\frac{\Delta I}{\Delta t} = 6\times10^{9}\frac{amperes}{second}$. Thus we can approximate $\Delta t \approx \frac{10\,amperes}{6\times10^{9}\frac{amperes}{second}} = 1.7\times10^{-9}s$. So the current "went away" in a couple of nanoseconds. This is also the duration of the spark that was thrown when the toaster was unplugged. That's a really short interval of time, but a lot happens in it!

4. An inductor consisting of a solenoid with 20,000 windings per meter that is 3 cm long and 1 cm in diameter is placed in series with a resistor whose resistance is 9 mΩ and a capacitor with a capacitance of 187 mF. The circuit is driven at 60 Hz. What is the impedance of this system?

Just plug-and-chug on this one. The impedance of a circuit is given by $Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$ where *R* is the resistance and X_L and X_C are the inductive and capacitive reactances, respectively. These last two are given by $X_L = \omega L$ and $X_C = \frac{1}{\omega C}$. We need do only one preliminary calculation before finding the final answer: We must find the inductance of the solenoid. This is given by $L=\mu_0 n^2 \pi r^2 l = 4\pi\times10^{-7}\frac{T\cdot m}{A}\times\left(2\times10^4\frac{1}{m}\right)^2\times\pi\times(.005m)^2\times.03m = 1.18\times10^{-3}H$. We must also realize that the angular frequency used in the formulas for the reactances differs from the circular frequency in which electrical signals are usually expressed. (Yes, this *is* a huge source of confusion in the "real world"! Both types of frequency are measured in Hz. In this class, if you're uncertain, *ask*! In other contexts, it is important not to assume one or the other without considering the options. I have seen very serious gaffs made by high-level professionals due to this ambiguity. *Usually* ω is used for angular frequencies and either *f* or ν [that's the Greek letter "nu"] are used for circular frequencies.) Fortunately, the difference is "only" a factor of 2π. (My boss once asked me if I'd mind if he reduced my salary by a mere factor of 2π. The lesson stuck!) Here, let's assume that the frequency given is the circular frequency—without some sort of guidance, the other choice would have been equally justified. I'm assuming it's the circular frequency because U.S. household current is driven with a circular frequency of 60 Hz.

Putting these numbers together, we have $X_L=\omega L=2\pi fL=2\pi\times60Hz\times1.18\times10^{-3}H=0.446\Omega$

and

$$X_C=\frac{1}{\omega C}=\frac{1}{2\pi\times60Hz\times0.187F}=\frac{1}{70.46}\Omega=0.014\Omega.$$

Thus

$$Z=\sqrt{R^2+(X_L-X_C)^2}=\sqrt{(9\times10^{-3}\Omega)^2+(0.446\Omega-0.014\Omega)^2}=0.432\Omega.$$

If you treated the 60Hz as the angular frequency instead of the circular frequency, you would have gotten $X_L=\omega L=60Hz\times1.18\times10^{-3}H=0.071\Omega$ and $X_C=\frac{1}{\omega C}=\frac{1}{60Hz\times0.187F}=\frac{1}{11.22}\Omega=0.089\Omega$. Thus $Z=\sqrt{R^2+(X_L-X_C)^2}=\sqrt{(9\times10^{-3}\Omega)^2+(0.071\Omega-0.089\Omega)^2}=0.02\Omega$.

5. In the previous assignment you performed the following calculation: “A sinusoidal electric potential with a peak strength of 120V oscillating at 60 Hz (i.e., U.S. household current) is applied to solenoid with 1100 windings per meter. The total resistance of the wire is 7Ω. The solenoid has an air-core (i.e., no chunk of metal running down its middle). Take the diameter of the solenoid to be 2 cm and its length to be 11 cm. What is the maximum energy stored in its magnetic field under the conditions described?” Now, calculate the contribution the inductive reactance has to the calculated energy. At what frequency would current have to be supplied in order for the inductive reactance to be as important to the stored energy as the resistance?

In the previous problem so referenced, we found the energy to be $U = \frac{1}{2}LI^2 = \frac{1}{2}\times 5.25\times 10^{-5}H\times(17.14A)^2 = 7.72\times 10^{-3}J$. Unfortunately, we found this using an incorrect assumption: We assumed that the current was related to the potential via the “old” form of Ohm’s law, $V = IR$. Because of the inductance of the solenoid, an applied potential that is time-varying will not cause the same current as in the DC case. We should have used the version of Ohm’s law that takes this into account. This is $V = IZ$ where Z is the impedance, which is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. This includes the inductive and capacitive reactance, X_L and X_C, respectively. Now, in this case, there is no capacitor, so we do not include the capacitive reactance. The inductive reactance is given by $X_L = \omega L$, thus $Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$.

Now, we will use this quantity along with $V = IZ$ to find the current flowing through the inductor. We can then find the energy stored in its field. If we subtract the value we found for this *without* the inductive reactance considered from the quantity we so find, that difference will be whatever was caused by the inductive reactance. We have $Z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{49\Omega^2 + 3600\frac{1}{s^2}\times 4\times\pi^2\times 2.76\times 10^{-9}H^2}$. Note that I used $\omega = 2\pi f$ and took $f = 60Hz$ in this. The *circular* frequency of the potential provided by U.S. wall outlets is 60 cycles per second while this equation calls for the angular frequency. As I’ve mentioned before, this is a very common source of confusion and error and one that is uncommonly difficult to avoid. Working out the numbers, we get $Z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{49\Omega^2 + 3.92\times 10^{-4}\Omega^2} = 7.00003$. Clearly, the impact of the reactance is miniscule, but let’s push through and figure out what it is.

Now, here's yet another instance in which doing things symbolically is loads more fun and useful than doing them numerically. I'll do it that way (are you surprised?). Solving for the current (note that this is the amplitude of the current—the current itself is a time-varying quantity), we get $I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \omega^2 L^2}}$. This gives $U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{L V^2}{R^2 + \omega^2 L^2}$. Now, substituting numbers we get $U = 7.714 \times 10^{-3} J$. So the difference in the energy in this situation from that calculated previously is $\Delta U = 5.8 \times 10^{-6} J$. So the contribution isn't much at all—at least at 60 Hz. Now, let's see if we can drive this sucker harder!

Notice that the energy depends on $\frac{1}{R^2 + \omega^2 L^2}$. The contribution to the energy of the resistance and the reactance will be the same when the resistance and the reactance are the same. This happens when $R = \omega L$. Solving for the frequency, we have $\omega = \frac{R}{L} = \frac{7\Omega}{5.25 \times 10^{-5} H} = 1.33 \times 10^5 Hz.$ Once again, this is the angular frequency, so we need to divide by 2π to get the circular frequency. This gives $f = 21.22 kHz$.

6. A “narrow pass” filter is to be constructed. It is desired that the filter pass signals at 88.1 MHz (that’s the local radio station KDHX). An inductor with an inductance of 19   H is used in this filter. A variable capacitor is included in the circuit to tune the filter. The separation between the plates is 0.7 mm and the area can be adjusted via a knob. What will be the area of the capacitor plates when the radio station is “in tune”? If the capacitor plates are square, what is the side-length of the plates?

The filter will have a resonance when the inductive and capacitive reactances are equal. Thus we want $\frac{1}{\omega C} = \omega L$. This can easily be solved for the capacitance to give $C = \frac{1}{\omega^2 L}$. Unlike the previous problem, this time we *will* treat the frequency as being the circular frequency. This is standard for radio frequencies. (Don’t be concerned if you didn’t realize this—there’s really no way you could have known without asking.) Thus $\omega = 2\pi \times 8.81\times 10^7\,Hz = 5.535\times 10^8\,Hz$. This gives $C = \frac{1}{\omega^2 L} = \frac{1}{\left(5.535\times 10^8\,Hz\right)^2 \times 1.9\times 10^{-5}\,H} = 1.72\times 10^{-13}\,F$.

Using this capacitance with our formula for the capacitance of a parallel-plate capacitor, $C = \varepsilon_0 \frac{A}{d}$, we can solve for the area of the plates. This is $A = \frac{dC}{\varepsilon_0} = \frac{7\times 10^{-4}\,m \times 1.72\times 10^{-13}\,F}{8.85\times 10^{-12}\,\frac{C^2}{N\cdot m^2}} = 1.36\times 10^{-5}\,m^2$. The side length of this would be $l = \sqrt{1.36\times 10^{-5}\,m^2} = 3.7mm$.

7. In lab, you recorded the frequency at which the current passed by an LRC circuit was greatest. Ideally, this would be the “resonant frequency.” Using the values you recorded for the resistance, inductance, and capacitance of the circuit (if you didn’t record these, I will provide them to you in class), calculate the theoretical impedance of the circuit at the resonance you measured. Also calculate what the theoretical resonant frequency is for that system. What is the impedance of the system at the theoretical resonant frequency?

As I announced in class, the values of the inductance, capacitance, and resistance of the circuit are, respectively, $L = 8\times 10^{-3}\,H, C = 220nF,$ and $R = 1k\Omega$. (Some of you may have used $R = 220\Omega$.)

The instruction to calculate the theoretical impedance at the resonant frequency was a bit of a trick question: The resonance occurs when $X_L = X_C$. The result of this is that, at resonance, $Z = R$. So the impedance is the same as the resistance at resonance.

The frequency you measured was the resonant frequency. This occurs when the angular frequency $\omega = \frac{1}{\sqrt{LC}}$. Note that the resistance doesn't enter into this—the role of the resistance is that it prevents the current from becoming infinite at the resonant frequency. Using the inductance and capacitance given, we should see an angular frequency of $\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{8\times10^{-3}H \times 2.2\times10^{-7}F}} = 23836Hz$. This gives a circular frequency ($f = \frac{\omega}{2\pi}$) of $f = 3794Hz$.

8. What is the wavelength, in vacuum, of the 88.1 MHz signal mentioned above?

Plug and chug: In a vacuum, all electromagnetic waves travel at a speed of c. Thus, we use $\lambda f = c$ to get $\lambda = \frac{3\times10^8 \frac{m}{s}}{88.1\times10^6 Hz} = 3.4m$.