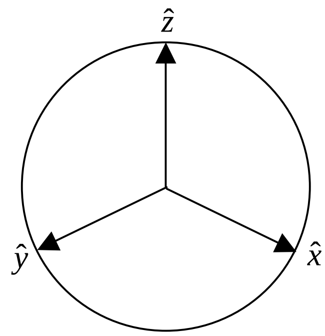
Physics 206b

Homework Assignment X—partial (Problem #3 solution is not yet complete. I will post a final version of this solution set once that is done.)

SOLUTIONS

Some important properties of the cross product for your reference:

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$
$$\vec{a}\times\vec{a} = 0$$
$$\hat{x}\times\hat{y} = \hat{z}$$
$$\hat{y}\times\hat{z} = \hat{x}$$
$$\hat{z}\times\hat{x} = \hat{y}$$

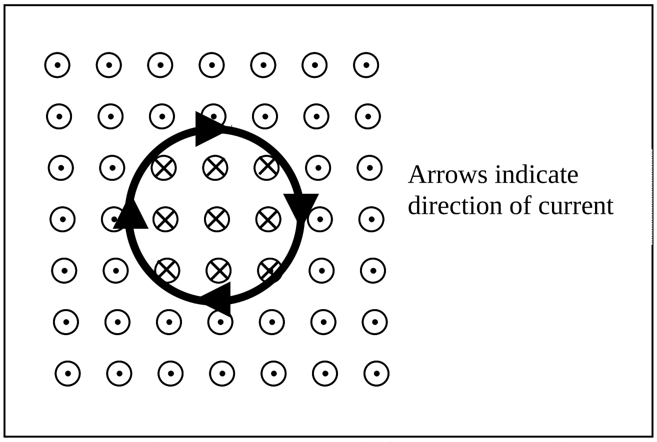


1. **Consider a circular loop of wire in the plane of this sheet of paper (or, at least, the one you're writing your solutions on). If a constant current is flowing in the wire going clockwise, use Ampere's law and the Right Hand Rule to determine the direction of the magnetic field inside the circle and outside of the circle. Make a sketch showing this.**

This one is *really* tough to draw and almost as tough to describe, but it's actually quite easy. Think about it this way: Ampere's law says that the magnetic field will make circles around any given length of wire. That is, given a tiny piece of wire with a current flowing in it, the magnetic field will be constant in size at any point a given distance away from the wire. The field will "look like" a series of circles around the little piece of wire. Since the field is a vector, we also need to specify the direction of it. To find this, imagine grabbing the wire with your right hand and laying your thumb along the wire pointed in the direction of the current. With your fingers so wrapped around the wire, your fingertips will point in the direction of the field.

Now, a circle of wire is just a whole bunch of little, straight segments of wire—or, at least you can visualize it that way. So imagine grabbing a wire loop with your right hand with your thumb pointed so that it is pointed "clockwise." If you hold it so that your fingers are on the inside of the circle, they will point into the page (assuming, as stated in the problem that the loop lies in the plane of the page). If you hold it so that your fingers are outside of the circle, your fingers will point out of the page.

Thus, the field would be:



Note one *very* important thing: As drawn here, the field seems to be of constant strength everywhere, just of different sign inside and outside the wire loop. This is ***not*** the case! The strength of the field varies as one's distance from the wire loop changes, both inside and outside of it. I just can't draw that well. Don't be misled!

2. An electron is accelerated to a velocity $\vec{v}$ by passage through a potential difference of 589 V (consider its initial velocity to be zero). It then enters a region occupied by a constant magnetic field of magnitude 1.7 T with a direction perpendicular to the velocity. What is the radius of the circular path its trajectory will take? Choose directions for the velocity vector of the electron just prior to entering the magnetic field and the magnetic field vector. Sketch the vectors and indicate the path the electron will take after entering the field.

You may have noticed that I'm ramping-up the complexity of your problems. This is intentional. I'm not making them more difficult, necessarily, but I am consciously increasing the number of steps you'll need to put in to solve them. Your problem solving skills are maturing enough, at this point, that you will benefit from the added push. (You're welcome!)

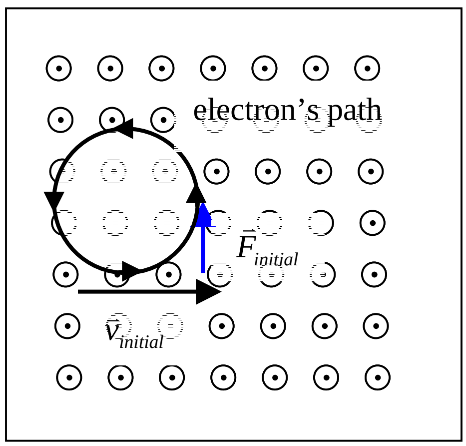
To solve this problem, we'll need to go through a minimum of two steps, but first let's see what the Physics says will happen qualitatively. The electron passes through a potential difference. This means that its potential energy changes. Since this is a conservative system (i.e., no energy leaves the system) a change in potential energy must show up as an equivalent change in kinetic energy. I.e., the electron’s speed changes—in this case, it increases. It then enters a magnetic field. **A magnetic field does no work on a moving charge!** It applies a force that is always perpendicular to the velocity of the charge. This means that it cannot change the energy of the charge. Which, in turn, means that it cannot change the speed of the charge. All it can change is the direction the charged object travels. Since the magnetic field is, in this case, perpendicular to the velocity and of constant magnitude, we have a situation in which the object will experience a constant acceleration which is always perpendicular to its velocity. This is precisely the recipe for circular motion, so the particle will travel in a circle after entering the magnetic field.

Now let’s do it quantitatively, starting with finding the speed of the electron. Invoking conservation of energy, we have $q\Delta V = \frac{1}{2}mv^2$. (Note that this is true only because our initial *K.E.* is zero; otherwise, we would have had to use a *change* in *K.E.*). This gives $v = \sqrt{\frac{2q\Delta V}{m}} = \sqrt{\frac{2\times 1.6\times 10^{-19}\,C\times 589V}{9.11\times 10^{-31}\,kg}} = 1.44\times 10^{7}\,\frac{m}{s}$. However, I implore you not to use the numerical answer at this point! It’s always a good policy to save plugging in numbers until the very end.

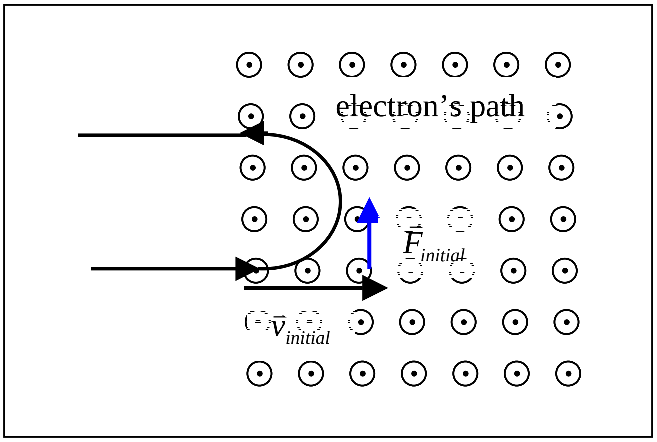
Now, we know the electron will travel in a circle. The relationship between the centripetal force and the speed of an object traveling in a circle is $F_c = \frac{mv^2}{r}$. The force a magnetic field exerts on a moving charge is given by $\vec{F}_m = q\vec{v}\times\vec{B}$. In this case, since we know that the velocity and the magnetic field are perpendicular to each other, we can ignore the vector nature of the cross product (this won’t always be true!) and put the direction of the final vector in by hand later. This allows us to write $F_m = qvB$ (note that by doing this we can also ignore the sign on the charge for now—we’ll need to remember to put it in later!). Since this is the centripetal force, we can write $qvB = \frac{mv^2}{r}$. This can readily be solved for *r* to give $r = \frac{mv}{qB}$. We can now substitute for the speed and we have $r = \frac{mv}{qB} = \frac{m\sqrt{\frac{2q\Delta V}{m}}}{qB} = \sqrt{\frac{2m\Delta V}{qB^2}}$.

Now we substitute the numbers and get $r = \sqrt{\frac{2m\Delta V}{qB^2}} = \sqrt{\frac{2\times 9.11\times 10^{-31}\,kg\times 589V}{1.6\times 10^{-19}\,C\times (1.7T)^2}} = 4.82\times 10^{-5}\,m$.

Since we are free to choose the direction of the electron initially, let’s pick it’s velocity to be in the $\hat{x}$ direction. We are also free to pick the direction of the *B* field, so long as it is perpendicular to the electron’s velocity. So, if we draw our *x* axis to the right, as seen on this page, let’s pick the *B* field to be out of the page. This is the $\hat{z}$ direction. Sketching this, we have:



Now, here is where I have to confess to a little fib which some of you (I hope!) may have picked up on: The symmetry of the problem is such that the electron can’t stay in the magnetic field, assuming that the region starts abruptly. In this case, the electron can only make a semicircle and then it will leave. If you didn’t notice this, that’s fine, I didn’t want you to. But if you did, I apologize for any confusion and congratulate you on having the picture right! So I really should draw:



(The semicircle here is just half of the circle discussed earlier.)

Anyway, the direction of the force can be found by the right hand rule. Remember to include the – sign for the electron, however! Another way to find it is to us the vector relations directly. Remember that we're only going to find the *initial* force, which will be based on the *initial* velocity. Once the force has acted on the electron, the direction of the velocity will change. Since the force is always perpendicular to the velocity, once the velocity changes, the force also changes, and so on.

Using the force law, $\vec{F}_m = q\vec{v}\times\vec{B}$, we can find the direction (which is all we care about now—we already found the size) by rewriting this without sizes for any of the vectors. I'll indicate the vector in the direction of $\vec{F}_m$ but without its size in the usual way, as $\hat{F}$. Thus we have $\hat{F} = -\hat{x}\times\hat{z} = \hat{y}$ (note the – sign for the electron). This is (no surprise) the same as we found using the right hand rule. Of course, you may have picked different directions than I did for the velocity and *B* field, so your answer may be different.

3. **In a mass spectrometer, a molecular ion is accelerated through a potential difference of 400 V. Assume the ion is singly-ionized glucose ($C_6H_{12}O_6$)—i.e., one of its electrons has been removed. The ion then travels in the $\hat{x}$ direction through a constant magnetic field of 300 G oriented in the $\hat{y}$ direction. The region with the magnetic field is between the plates of a parallel-plate capacitor with a separation of 1 cm oriented in the $\hat{z}$ direction. What potential difference needs to be placed across the plates of the capacitor if the ion is to emerge from this "Wien filter" undeflected?**

I'm still working on the solution to Problem #3 and will post a final version of this solution set once this is done.

4. **Consider again the electron in the previous problem. Now assume that there is an electric field parallel to the magnetic field and pointed in the same direction as the magnetic field. What electric field strength is necessary so that on its *first* "cycle" in the magnetic field the electron travels as far vertically as it does horizontally? (Take "vertical" to mean "parallel to the two fields" and "horizontally" to mean "in the plane perpendicular to the two fields.") This is a tough problem and you'll have to reach way back in your bag of tricks to do it!**

Alrighty, now for the tough one. Let's continue to assume that the electron is able to make a full circle in the magnetic field. (This could be accomplished by having the magnetic field instantaneously turn on rather than having the charge enter the field directly—but I don't want to confuse things more than necessary.) The *total* force on the electron is given by the Lorentz force law: $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$ where $\vec{E}$ is the electric field. Did you see what happened there? I took the magnetic force that we discussed in the previous problem and simply added it to the force due to an electric field that we discussed in previous assignments. One place where many of you made life difficult for yourselves, I'm sure, is by assuming something incorrect about the electric field. You made an assumption that the field was caused by a point charge. This was not stated in the problem and, indeed, is not consistent with the problem as stated! $\vec{E}$ is simply $\vec{E}$. That's all it is. Just remember that the electric field at a point in space is *defined* to be the force that a 1 coulomb positive charge would experience if it were placed at that location in space.

Now, here's the nastiness: The electric field and the magnetic field are parallel to each other. But the *forces* exerted by those fields are perpendicular to each other. This is because the force exerted by the electric field is in the direction of the electric field while the force exerted by the magnetic field is always perpendicular to the magnetic field. (You might wonder why we didn't just come up with a different definition of the magnetic field that would allow it to be parallel to the force that it exerts. Unfortunately, the magnetic *force* depends on the velocity of the object as well as the magnetic field. We can't come up with a definition of the field that would make the force parallel to the field and still take this dependence on velocity into account. So we have to live with this little confusion.)

But this problem is a saving grace, in this case, if we remember a profoundly important fact from last semester: **Vectors which act perpendicularly to each other act independently of each other!** Since the *E* field and the *B* field are *parallel* to each other, by definition their respective forces will be *perpendicular* to each other. Thus it is sufficient for us to consider each of these forces separately and then add their effects.

We've already figured out the effect of the magnetic field: It will result in the electron traveling in a circle. Now, recall that the electric field points in the direction of the force that would be experienced by a *positive* charge were it placed at that location in the field. Since an electron has a negative charge, the force on the electron will be in a direction opposite to that of the field. The electric field will result in the electron accelerating in the $-\hat{z}$ direction, therefore.

If we have an object traveling in a circle in the *x*-*y* plane while accelerating in the $-\hat{z}$ direction, the result will be a spiral, like a screw. In fact, since the electron is *accelerating* in the $-\hat{z}$ direction, the "pitch" of the screw will continuously increase: It will travel farther in "vertically" with each revolution. But we're not concerned with that here. What we care about in this case is only what it does on its first cycle.

In once cycle, the electron will travel horizontally a distance $d = 2\pi r$, where the radius is what we found in the previous problem: $r = 4.82\times10^{-5}\,m$. This gives $d = 2\pi r = 2\pi\times4.82\times10^{-5}\,m = 3.03\times10^{-4}\,m$. We use this as the distance it travels vertically, per the statement of the problem. We also know how long it takes for this motion to happen: Since we know the *horizontal* speed of the electron (the $\hat{x}$ and $\hat{y}$ pieces, which are unaffected by electric field and the sum of which remains unchanged through the whole process), we can just divide the distance found above by this speed to find the time it took to travel that distance. This gives $t = \frac{d}{v} = \frac{3.03\times10^{-4}\,m}{1.44\times10^{7}\,\frac{m}{s}} = 2.1\times10^{-11}\,s$. (This time is the reciprocal of an important quantity known as the "cyclotron frequency.")

Now, it's crucial to notice that the electron is traveling this distance under a constant acceleration. Last semester, we exhaustively studied the behavior of objects undergoing constant accelerations. Among the most important equations we used in that study was $d = \frac{1}{2}at^2$. We now know the distance our electron has traveled (in the direction of interest) and the time it took to travel that distance. Using these facts, we can easily find the acceleration: $a = \frac{2d}{t^2}$. We also know the relationship between the acceleration of the object and the force acting on it: $\vec{F} = m\vec{a}$. Since the force in the direction of interest here is just that due to the electric field, $F = qE$ (ignoring directions for now), we can easily write $qE = ma = \frac{2md}{t^2}$. This can be solved readily for *E* to give $E = \frac{2md}{qt^2}$.

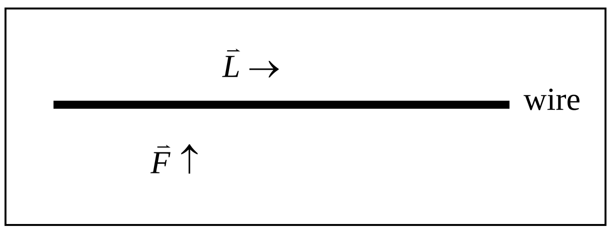
Now, it's just a matter of substituting numbers. We have $E = \frac{2md}{qt^2} = \frac{2 \times 9.11 \times 10^{-31}\,kg \times 3.03 \times 10^{-4}\,m}{1.6 \times 10^{-19}\,C \times \left(2.1 \times 10^{-11}\,s\right)^2} = 7.8 \times 10^6 \frac{Volts}{meter}$.

5. **A length of wire has a linear mass density of $\mu = 7.8 \frac{grams}{meter}$. You wish to suspend a length of such a wire in midair (i.e., against the pull of gravity) by flowing a current through it while applying an external magnetic field to it. What direction does the magnetic field need to point to make this happen? Given this, what direction does the current need to travel? Make a sketch! What size current is needed if the external magnetic field is 41 milliteslas?**

This is very straightforward to solve, once you have the geometry in mind. The force on a current carrying wire in a magnetic field is given by $\vec{F} = I\vec{L} \times \vec{B}$. This is basically just an extension of the Lorentz force law ($\vec{F} = q\vec{v} \times \vec{B}$) that sums the forces on all of the moving charges in a wire. Getting the directions is the hardest part. In this case, we know that the force must act "up" in order to counteract gravity Let's call that the $\hat{y}$ direction. Unfortunately, we can't just divide the equation to get an answer for the rest. One very strict rule is that you cannot divide by a vector! You can divide by the *magnitude* of a vector, but the directional feature must not be included. So we have to use reasoning, rather than just algebra, to find the direction of the other two vectors in the problem.

Because of the nature of the force equation above, we know that the force must be perpendicular to *both* the magnetic field and the wire itself. We do *not* need to have the wire perpendicular to the field to make this happen, but the force will be maximized if it is since $\left|\vec{L}\times\vec{B}\right| = LB\sin(\theta)$. (The absolute value bars, $\left|\;\right|$, mean "the size of" the vector enclosed.) So our lives will be easier if we pick things this way. If we choose differently, all that will change is that we'll have to tote around that $\sin(\theta)$ term. It will *still* be true that the final vector will be perpendicular to *both* of the vectors in the cross product.

Using this, let's pick the length vector to lie in the $\hat{x}$ direction. If we pick the $\hat{y}$ to be "up" on this sheet of paper, this will put the $\hat{x}$ direction to the right. So we will have $\vec{L} = L\hat{x}$ and $\vec{F} = F\hat{y}$:



Since $\vec{B}$ must be perpendicular to $\vec{L}$, its direction can be any combination of $\hat{x}$ and $\hat{z}$. If we're sensible and pick the direction to maximize the force, then the only possible direction remaining is the $\hat{z}$ direction (although it might be $-\hat{z}$, we won't know until we do the math). Just out of a sense of perversity, I'm going to do this the hard way for you. For absolutely no good reason, I'll pick the $\vec{B}$ field to be pointed at an angle of 13° relative to the $\hat{x}$ direction. This gives $\vec{B} = B(\cos(13°\,\hat{x} + \sin(13°)\hat{z}))$. Let's see what this gives us: $\vec{F} = I\vec{L}\times\vec{B} = ILB[\hat{x}\times(\cos(13°)\hat{x} + \sin(13°)\hat{z})] = -ILB\sin(13°)\hat{y}$. As expected, our $\sin(\theta)$ factor is tagging along. Also, we've wound up with a negative sign in front of it all. Since the final force *must* be in the $+\hat{y}$ direction, this just means that our choice of direction for the current was backwards, as I stated it might be.

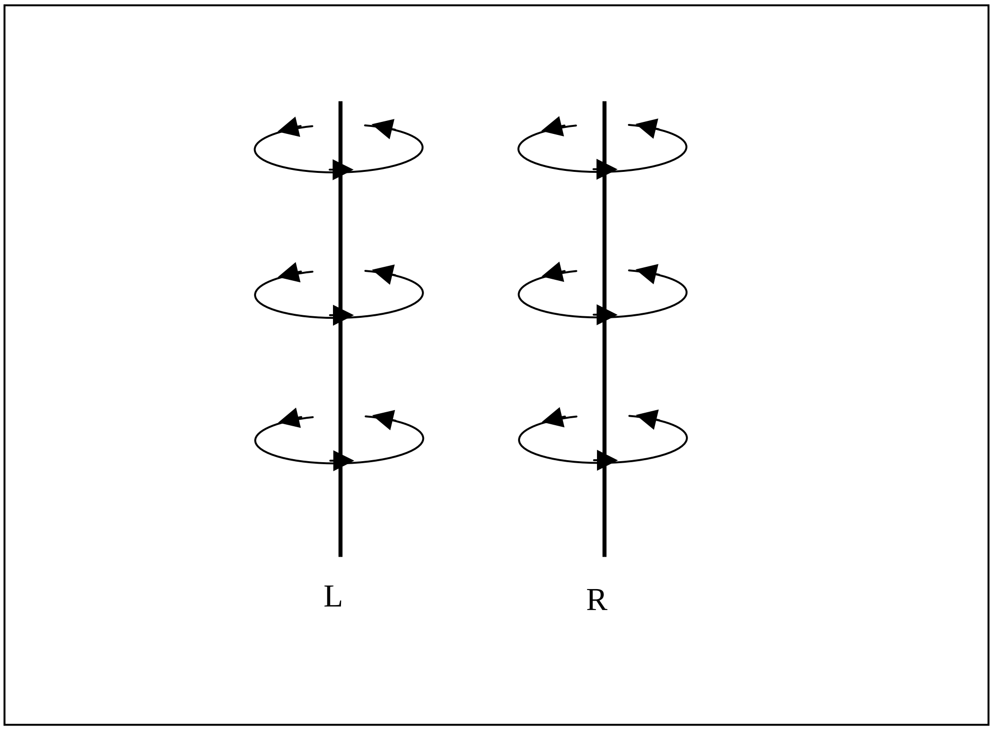
A judicious use of the right hand rule would have told us this without doing any math. A variation on the right hand rule exclusively for the force on a current-carrying wire is this: Using your right hand (duh), point your thumb in the direction of the $\vec{L}$ vector (the direction the current is flowing along the wire). Point the fingers in the direction of the $\vec{B}$ field. The palm of your hand will point in the direction of the force. I like this one because of the evocative nature of the palm *pushing* on the wire!

We're almost done. (We would have been done a while ago if I hadn't decided to do this the hard way!) Now we need the size of the vector. What is F? Well, remember that we're pushing against gravity. The force of gravity on the wire is $F = mg$. But what is the mass? Well, the mass can be found from the linear density and the length $m = \mu L$. This gives us $F = mg = \mu Lg$. Thus, we can write $ILB\sin(13°) = \mu Lg$. The length drops out (think about why this is so) and, solving for the current, we have $I = \frac{\mu g}{B\sin(13°)}$. (Of course, you will probably not have that $\sin(\theta)$ term in there.) Sticking in some numbers, we have $I = \frac{\mu g}{B\sin(13°)} = \frac{7.8\frac{grams}{meter} \times 9.8\frac{meters}{second^2}}{4.1\times10^{-2}T \times .225} = 8.29A$. Of course, if you'd been sensible and chosen $\theta = 90°$, this would be $I = \frac{7.8\frac{grams}{meter} \times 9.8\frac{meters}{second^2}}{4.1\times10^{-2}T} = 1.86A$. This would be in the $-\hat{x}$ direction for a $\vec{B}$ field out of the page ($+\hat{z}$ direction) or in the $\hat{x}$ direction for a $\vec{B}$ field into the page ($-\hat{z}$ direction).

6. **Two 1.4 m long wires oriented vertically and parallel to each other are separated by a distance of 7 cm. A current of 3 amperes flows through each of them. In both wires, the current flows from bottom to top. What is the size and direction of the force experienced by each wire due to this? If the direction of both currents is reversed, does this change? What about if the current in one of the two wires is reversed? (I.e., if one of them is flowing top-to-bottom while the other is flowing bottom-to-top.) Hint: Use Ampere's law.**

This is a *very* easy problem *if* you avoid falling into a very common trap: Many of you are still confusing the field that something (a charge, a current, a magnet, etc.) *experiences* with the field that it *creates*. We have two wires. Each one carries a current. That means two, distinct, totally separate things: First, it means that each one will experience a force if it is in a location with a magnetic field pointed in any direction other than parallel to the wire. Second, it means that each one will create a magnetic field. These two facts have *nothing* to do with each other!

Realizing this, we recognize that, to find the force on one wire, we must find the field created by the *other* wire. The wires and fields are indicated in this picture for currents both flowing from bottom to top:



Notice that the $\vec{B}$ fields (found via the right hand rule) for each of the wires are the same. However, since the wire on the right is to the right of the wire on the left and the wire on the left is to the left of the wire on the right (duh), the wires each encounters a field *opposite* to that encountered by the other one. The wire on the left encounters a field pointed out of the page ($+\hat{z}$ direction) while the wire on the right encounters a field pointed into the page ($-\hat{z}$ direction). Since the currents are each in the same direction, the total force encountered by each wire will be in the opposite direction. (You will also note that this is demanded by Newton's third law.)

To deal with this mathematically, we'll need the size of the $\vec{B}$ fields. Fortunately, the long, straight wire is one of the problems which can be solved using Ampere's law. Ampere's law gives $B = \frac{\mu_0 I}{2\pi r}$. This is the magnitude of the field due to a long, straight wire carrying a current *I* a distance *r* from the wire. The direction is found, as stated above, by the right hand rule. Since in this problem we are only interested in the field a distance of $7cm = 0.07m$ from the wire, we will use a value of $B = \frac{\mu_0 I}{2\pi r} = \frac{4\pi \times 10^{-7} \frac{T \cdot m}{A} \times 3A}{2\pi \times .07m} = 8.57 \times 10^{-6} T$.

We find the total force on each wire by using the relation $\vec{F} = I\vec{L} \times \vec{B}$ (I usually prefer to consider the current as a vector and the wire's length as a scalar, but most books write it this way, so I'll stick with the standard; it makes no difference in the end). For the wire on the right, we have (as previously, using asterisks to denote scalar multiplication for clarity):

$$\vec{F}_R = I\vec{L} \times \vec{B} = 3A * 1.4m\,\hat{y} * 8.57 \times 10^{-6} T(-\hat{z}) = 3.6 * 10^{-5}(\hat{y} \times (-\hat{z}))N = -3.6 * 10^{-5} N\,\hat{x} .$$

As noted above, the only difference for the wire on the left will be the sign on the direction of the magnetic field it encounters. Thus, the only difference in the force will be a sign. We will have:

$$\vec{F}_L = 3.6 * 10^{-5} N\,\hat{x}$$

So, the wire on the right will experience a force to the left and the wire on the left will experience a force to the right. The result is that the wires will *attract* each other.

Clearly, if the direction of *both* currents is reversed nothing changes, overall: The sign on $\vec{B}$ changes in both cases, but so does the sign on $\vec{L}$. The result is that nothing changes.

Now, if the current on only *one* of the wires is reversed, *both* forces will change sign. In one case the sign change is because the $\vec{B}$ field changes sign and in the other case the sign will change because $\vec{L}$ changes sign. The result will be that the wires will *repel* each other in this case.

In all three cases discussed, the magnitude of the force is the same.

7. Write down, in words, Faraday's Law.

Qualitatively first: Faraday's law says that a time-varying magnetic flux passing through a closed loop of wire or, indeed, any closed loop in space, whether there's anything there or not, will result in an EMF if there's a conductor there or an E-field if not.

Stated mathematically: $-\mathcal{E} = \frac{\Delta\Phi_B}{\Delta t}$.

Stated operationally: If you have a closed loop made out of a conductor (or a closed region of space—but we haven't worked with that yet; we will! but let's ignore the possibility for now) and, for whatever reason, the magnetic flux through that loop changes with time, find the flux as a function of time. Find how it varies with time. The EMF will be just the change in flux divided by the time over which that change occurs. This will cause a current to flow in a conductor, if possible.

(Combining these last two: Once the current is caused to flow by the changing magnetic flux, Ampere's law says that a magnetic field will be created. This field may very well interact with the very field that caused the current to flow in the first place. This is "inductance." That's for the next assignment.)