Physics 206a

HOMEWORK #9

SOLUTIONS

(Because of the large amount of algebra in Problem #2, this is only a preliminary solution set. This set contains the solution of Problem #2 for the special case that the two masses are the same—this is not the situation in the problem as stated. I will post a complete solution, which covers the condition for the two masses as given, once I've finished typing up the algebra.)

1. A billiard ball whose mass is 300 grams and whose velocity is $5\,\frac{\text{meters}}{\text{second}}\hat{x}$ strikes another ball. The second ball is at rest, initially. The impact is perfectly elastic. The collision is "head on," i.e., this is a one-dimensional problem. Find the velocity of the *both* balls after the collision if:

   a. The mass of the second ball is 250 grams.

   b. The mass of the second ball is 300 grams.

   c. The mass of the second ball is 350 grams.

Here's where those of you who obsess on numbers will get your comeuppance! If you do the general case, treating the masses as symbols, you only have to do the problem once, only plugging in numbers at the end. If you insist on sticking numbers in there early on, you have to do the same problem three times. Me, I'm lazy. I'd rather do it once.

Once again, we start by stating everything we *know*: We know the masses of the balls. We know the initial speeds. We also know that the collision will be head-on, so this is a 1-d problem. Most importantly, we know that momentum is conserved, because it's *always* conserved. And we know that *K.E.* is conserved because we are told that the interaction is perfectly elastic.

We write the condition for conservation of momentum as $p_i = p_f$, where, again, I've left the vector symbol off because we're working in 1-d. (Feel free to include vector attributes explicitly even in 1-d: It's not wrong, it's just not absolutely essential.) We can rewrite this as $p_i = m_1 v_i = p_f = m_1 v_{1f} + m_2 v_{2f}$.

Our condition for conservation of energy can be written similarly $K.E._i = K.E._f$. Rewriting this with the full expressions for *K.E.*, we have $\frac{1}{2} m_1 v_i^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$. Since a factor of ½ occurs in every term of this, I'll just divide through, for convenience, and rewrite this as $m_1 v_i^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2$.

So, we have two equations and two unknowns. That's enough to start working. Let's rewrite the set of equations cleanly, so we know where we are:

$$m_1 v_i = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1 v_i^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2$$

Now, I'll square the first of these and divide through by m_1. (Sorry for combining steps, but it will just be too painful to include *every* step!) This gives $m_1 v_i^2 = m_1 v_{1f}^2 + \frac{m_2^2}{m_1} v_{2f}^2 + 2m_2 v_{1f} v_{2f}$. (Make sure you understand where the final term, called the "cross term" came from. It's frequently left out in a common error.) Now, the left hand sides of the two expressions (the square of the momentum equation and the *K.E.* equation) are the same, so I can set the right hand sides equal to each other. $m_1 v_{1f}^2 + m_2 v_{2f}^2 = m_1 v_{1f}^2 + \frac{m_2^2}{m_1} v_{2f}^2 + 2m_2 v_{1f} v_{2f}$. Now, I cancel the $m_1 v_{1f}^2$ term that occurs on both sides of the "=" sign and note that all the remaining terms have a factor of $m_2 v_{2f}$ in them, so I divide through by that quantity. This gives $v_{2f} = \frac{m_2}{m_1} v_{2f} + 2v_{1f}$. Now it's just a matter of collecting terms. We get $v_{2f} = \frac{2v_{1f}}{1 - \frac{m_2}{m_1}}$ (note that we'll have to be *very* careful with this in the case that $m_1 = m_2$!).

Going back to our momentum conservation expression, we have $m_1 v_i = m_1 v_{1f} + m_2 v_{2f}$ so we substitute the above expression for v_{2f} into this and get $m_1 v_i = m_1 v_{1f} + m_2 v_{2f} = m_1 v_{1f} + m_2 \frac{2v_{1f}}{1 - \frac{m_2}{m_1}}$. Divide both sides by m_1 to get $v_i = v_{1f} + \frac{m_2}{m_1} \frac{2v_{1f}}{\left(1 - \frac{m_2}{m_1}\right)}$. From here it's just a bit of algebra (give it a shot for yourself) to get $v_i = v_{1f} \left(\frac{m_1 + m_2}{m_1 - m_2} \right)$. Of course, we know the initial speed, so we should solve this for the final speed of the first ball, which is $v_{1f} = v_i \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$. We're not quite there yet. We also need the speed of the second ball. But we have a relation for the second ball in terms of the first: $v_{2f} = \frac{2v_{1f}}{1 - \frac{m_2}{m_1}}$, so we can substitute in to this to get $v_{2f} = \frac{2v_i\left(\frac{m_1 - m_2}{m_1 + m_2}\right)}{1 - \frac{m_2}{m_1}} = \frac{2v_i m_1}{m_1 + m_2}$.

Before we go sticking numbers in, let's interpret these. Note that when the two masses are the same, the first ball will stop and the second ball will emerge with the same speed as the first ball. When the first ball is more massive than the second ball, the first ball emerges with a speed that is positive and the second ball also emerges with a positive speed, so they will travel in the same direction. When the first ball is less massive than the second ball, it will emerge from the collision with a *negative* speed, which means it will be traveling backwards! The second ball will emerge with a positive speed, which means it will move in the direction the first ball originally was moving.

Now we can stick numbers in:

i. Taking $m_1 = 0.3kg$ and $m_2 = 0.25kg$ we have

$$v_{1f} = v_i\left(\frac{m_1 - m_2}{m_1 + m_2}\right) = 5\,\frac{\text{meters}}{\text{second}} \times \left(\frac{.05kg}{.55kg}\right) = 0.45\,\frac{\text{meters}}{\text{second}} \text{ and}$$

$$v_{2f} = \frac{2v_i m_1}{m_1 + m_2} = 5\,\frac{\text{meters}}{\text{second}} \times \frac{2 \times .3kg}{0.55kg} = 5.45\,\frac{\text{meters}}{\text{second}}$$

ii. Taking $m_1 = 0.3kg$ and $m_2 = 0.3kg$ we have

$$v_{1f} = v_i\left(\frac{m_1 - m_2}{m_1 + m_2}\right) = 0 \text{ and}$$

$$v_{2f} = \frac{2v_i m_1}{m_1 + m_2} = 5\,\frac{\text{meters}}{\text{second}} \times \frac{2 \times .3kg}{0.6kg} = 5\,\frac{\text{meters}}{\text{second}}$$

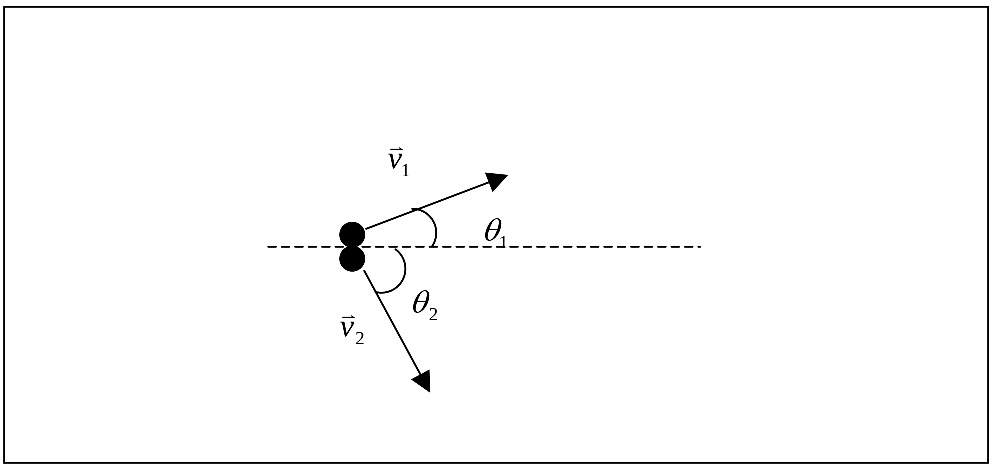
iii. Taking $m_1 = 0.3kg$ and $m_2 = 0.35kg$ we have

$$v_{1f} = v_i\left(\frac{m_1 - m_2}{m_1 + m_2}\right) = -5\,\frac{\text{meters}}{\text{second}} \times \left(\frac{.05kg}{.65kg}\right) = -0.38\,\frac{\text{meters}}{\text{second}} \text{ and}$$

$$v_{2f} = \frac{2v_i m_1}{m_1 + m_2} = 5\,\frac{\text{meters}}{\text{second}} \times \frac{2 \times .3kg}{.65kg} = 4.6\,\frac{\text{meters}}{\text{second}}$$

2. **Consider once again the billiard balls in the previous problem. Now, the second ball's mass is 350 grams. Once again, the impact is perfectly elastic, the second ball is initially at rest, and the first ball's velocity is $5\frac{\text{meters}}{\text{second}}\hat{x}$. But now the impact is not head-on. The first ball's velocity after the collision is directed 27° from the *x* axis. Now, what are the velocities of the two balls?**

As stated above, getting the algebra for this typed up is taking me a while—I'll get it to you as soon as it's done. But, for the time being, let's work out the special case in which the two masses are the same. In this case, a crucial principle holds: The angle between the trajectories of the two balls will be $\frac{\pi}{2}$ radians, i.e., 90°. We have the situation pictured below.



Using this, we see that $\theta_2 = 63°$. (Or –63° depending on how you include signs. I'll call it a positive quantity and just make sure to put the negative in in the right place.) Now we can invoke conservation of momentum, as before. In this case, we have two dimensions to worry about. But we also have two unknowns: v_1 and v_2. Each direction (*x* and *y*) will provide one equation, so we'll have two equations and two unknowns. Perfect! (Note: We also have conservation of energy. The result that the sum of the two angles is 90° was gotten via the conservation of energy expression, so this has already been used "behind the scenes.")

Since the original momentum was purely in the *x* direction, we know that the sum of the momenta in the *y* direction after the collision must be zero. Thus we can write $v_1 \sin(\theta_1) - v_2 \sin(\theta_2) = 0$. (Note that I put in the minus sign by hand here. If I'd used –63° before, I'd have used a plus sign.) This gives us $v_2 = \frac{v_1 \sin(\theta_1)}{\sin(\theta_2)}$.

Now, we also know that the sum of the *x* components of the momenta after the collision is equal to the original momentum, which lay completely in the *x* direction. So we can write $v_1 \cos(\theta_1) + v_2 \cos(\theta_2) = v_i$. (Note that I've just left the mass out of everything. Since there's a factor of *m* in every term, it just divides out. Again: This is *only* possible because the two masses are the same! This is a very different problem if the masses are different.) Substituting in our expression for v_2, we have $v_1\cos(\theta_1)+\frac{v_1\sin(\theta_1)}{\sin(\theta_2)}\cos(\theta_2)=v_i$. You really *could* just plug in numbers here, but this can be made a lot prettier with some algebra. Let's get a common denominator and add fractions: $v_1\left(\frac{\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)}{\sin(\theta_2)}\right)=v_i$. Now, since $\theta_1+\theta_2=\frac{\pi}{2}$ and $\cos(\frac{\pi}{2}-\theta)=\sin(\theta)$ we can say that $\cos(\theta_1)=\sin(\theta_2)$ and $\sin(\theta_1)=\cos(\theta_2)$. So

$$v_1\left(\frac{\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)}{\sin(\theta_2)}\right)=v_1\left(\frac{\sin(\theta_2)\sin(\theta_2)+\cos(\theta_2)\cos(\theta_2)}{\sin(\theta_2)}\right)$$

$$=v_1\left(\frac{\sin^2(\theta_2)+\cos^2(\theta_2)}{\sin(\theta_2)}\right)=v_i$$

We now use the trig relationship (I gave you this in class): $\sin^2(\theta)+\cos^2(\theta)=1$ for any angle. So, we're left with $v_1\left(\frac{\sin^2(\theta_2)+\cos^2(\theta_2)}{\sin(\theta_2)}\right)=\frac{v_1}{\sin(\theta_2)}=v_i$. Now *that's* pretty!

Using this and the relation we found earlier $v_2=\frac{v_1\sin(\theta_1)}{\sin(\theta_2)}=v_i\sin(\theta_1)$ we get $v_1=5\,\frac{\text{meters}}{\text{second}}\times\sin(63°)=4.46\frac{\text{meters}}{\text{second}}$. And $v_2=v_i\sin(\theta_1)=2.27\frac{\text{meters}}{\text{second}}$ at the angles given earlier.

Once again, this solution was worked out for a case that was not assigned! I'll get the real solution to you soon.

3. One more time for the billiard balls. Now, the balls hit head on. They have the same mass. The initial velocity of the first ball is $5\,\frac{\text{meters}}{\text{second}}\,\hat{x}$ **and the second ball is at rest. But now someone has coated the second ball with a layer of glue so that the balls stick together (no, they don't stick to the table!). What is the velocity of the pair after impact?**

You did this in lab, so it should be very familiar to you. We just need to use conservation of momentum: $\vec{p}_i = \vec{p}_f$. In words: The momentum of the system (which, in this case, consists of the two balls) after the collision is the same as the momentum of the system before the collision. Since the balls stick, kinetic energy is not conserved, so we cannot use conservation of energy. However, again because the balls stick, we have one less variable than we would if they'd bounced: The velocities of the two balls are the same. Indeed, since we are always free to orient the coordinate system any way we'd like, we can put the *x* axis along the velocity vector of the initial momentum. In this particular problem, this has already been done for you, but it could be done in any problem like this.

Writing this out explicitly, we have $\vec{p}_i = m_1\vec{v}_i = 300\,grams \times 5\,\frac{\text{meters}}{\text{second}}\,\hat{x}$ and $\vec{p}_f = (m_1 + m_2)\vec{v}_f = (300\,grams + 300\,grams) \times \vec{v}_f$. Setting these equal to each other, we have $\vec{p}_i = m_1\vec{v}_i = \vec{p}_f = (m_1 + m_2)\vec{v}_f$. This can easily be solved for the final velocity by dividing, so we wind up with $\vec{v}_f = \frac{m_1\vec{v}_i}{(m_1 + m_2)}$. Substituting numbers into this, gives

$$\vec{v}_f = \frac{m_1\vec{v}_i}{(m_1 + m_2)} = \frac{300\,grams \times 5\,\frac{\text{meters}}{\text{second}}\,\hat{x}}{(300\,grams + 300\,grams)} = 2.5\,\frac{\text{meters}}{\text{second}}\,\hat{x}.$$

It is a worthwhile exercise to see how the kinetic energy changed. You should be able to show for yourself that the final kinetic energy is only half of the original kinetic energy, in this case. 50% of the kinetic energy was lost.

Notice that the fraction of energy lost depends only on the ratio of the masses and the original speed. It does not depend on the mechanism that causes the sticking. That's a rather counter-intuitive fact!

4. A cannon shoots a cannonball at $200\,\dfrac{\text{meters}}{\text{second}}$ at an angle 30 degrees above the horizontal (i.e., pointed up 30 degrees). Two seconds after being shot, the cannonball explodes into two, equally sized pieces—call them "A" and "B". Relative to the original cannonball at the instant of the explosion, piece "A" has a velocity of $100\,\dfrac{\text{meters}}{\text{second}}$ exactly horizontal. How long after the original cannonball is shot will piece "A" hit the ground? If the cannon is at $x = 0$, what is the *x* coordinate where piece "A" will hit the ground?

Contrary to what's shown in all the cartoons and movies you've every seen, when something breaks apart (or, more dramatically, explodes), its momentum does not change. However, each piece of the object gains some new, additional momentum. If we were to add up all of the momenta of all of the fragments, they would sum to the original momentum of the thing that broke. I.e., the sum of all of the additional momenta is zero!

Think of it this way: When the cannonball is shot out of the cannon, it begins moving along some trajectory. That trajectory is fixed unless some *external* force acts on the cannonball that wasn't present initially (gravity was there in the first place and acts on all the pieces). If the cannonball explodes, the force of the explosion is an *internal* force—it comes from the cannonball, not some external entity. So all of the pieces that result from the explosion will continue to move along the cannonball's original trajectory. Each piece will retain the original cannonball's velocity and will be subject to the same accelerations due to gravity as the original cannonball. (Now, if we were to include external forces which were different for each fragment, things wouldn't work out so neatly. For example, if we included air resistance, each fragment would experience a different external force and so each fragment would have to be analyzed separately. Each piece would still emerge from the explosion with the original velocity of the cannonball plus some piece. But we'd have to analyze the new trajectories much more carefully.)

This presents us with a straightforward strategy for solving this problem: We will determine the velocity of the full cannonball at the moment of the explosion and then add the new velocity to it to determine the trajectory of the fragment.

Notice that we can answer the first part of the question immediately: Since the fragment's additional velocity is exclusively in the *x* direction (horizontal), the time before it hits will be unaffected by the change in velocity. (Make sure you understand that last sentence!) So let's figure out what that time is. We use our standard method: We begin by breaking the initial vector up into two components that are perpendicular to each other. Since $v_o = 200\,\dfrac{\text{meters}}{\text{second}}$, using SOHCAHTOA, we have $v_{ox} = v_o \times \cos\left(\dfrac{\pi}{6}\right)$ and $v_{oy} = v_o \times \sin\left(\dfrac{\pi}{6}\right)$, where I have expressed the angle, 30° in radians (get used to it). This gives

$$v_{ox} = 200\ \frac{\text{meters}}{\text{second}} \times 0.866 = 173.2\ \frac{\text{meters}}{\text{second}}$$

and

$$v_{oy} = 200\ \frac{\text{meters}}{\text{second}} \times 0.5 = 100\ \frac{\text{meters}}{\text{second}}.$$

Now, let's figure out where this thing is two seconds after being shot. Let's find its position (both *x* and *y*) and its velocity. We use $s = \frac{1}{2}at^2 + v_0 t + s_0$ for each component. (Careful: There's a v_0 in there that's different from v_o—the v_0 is the original speed of each component, not the total original speed.) Let's do the *y* component first. Here we have

$$y = -\tfrac{1}{2} \times 9.8\ \frac{\text{meters}}{\text{second}^2} \times (2\,\text{seconds})^2 + 100\ \frac{\text{meters}}{\text{second}} \times 2\,\text{seconds} = 180.4\,\text{meters}.$$

Note carefully the sign used for the initial speed and the acceleration: The initial speed is up and the acceleration due to gravity is down, so the signs are opposite! The position along the *x* direction is even easier since there is no acceleration in that direction, we just have $x = 173.2\ \frac{\text{meters}}{\text{second}} \times 2\,\text{seconds} = 346.4\,\text{meters}$.

Now, the *x* component of the velocity is unchanged by gravity. The *y* component of the velocity is simply

$$v_y = v_{oy} + at = 100\ \frac{\text{meters}}{\text{second}} - 9.8\frac{\text{meters}}{\text{second}^2} \times 2\,\text{seconds} = 80.4\frac{\text{meters}}{\text{second}}.$$

(Confusion alert: That $100\ \frac{\text{meters}}{\text{second}}$ comes from taking the *y* component of the original velocity. Don't confuse it with the $100\ \frac{\text{meters}}{\text{second}}$ gained by the fragment in the *x* direction when the cannonball explodes!) We now know where the cannonball will be when it explodes and what its velocity is at that instant. Both of these will be the same for fragment "A" as for the entire cannonball in the instant prior to the explosion. In the instant after the explosion, the position of fragment "A" will be unchanged, but its velocity will be different. The *x* component of its velocity will be $100\ \frac{\text{meters}}{\text{second}}$ greater than it was. Thus $v_x = 173.2\ \frac{\text{meters}}{\text{second}} + 100\ \frac{\text{meters}}{\text{second}} = 273.2\ \frac{\text{meters}}{\text{second}}$.

Now, we'll need to know how long it takes the ball to hit the ground. This can be found from $s = \frac{1}{2}at^2 + v_0 t + s_0$. It is important to realize that, since the *y* component of the velocity is unaffected by the explosion *in this case*, we could have figured this out right at the beginning. However, one can imagine, quite easily, a very similar problem in which some fraction of the change in velocity is in the *y* direction. So let's do this the hard way. The position and velocity of the fragment immediately after the explosion define a brand new problem. The best way to solve the full problem is to treat it as two problems: One to figure out everything about the cannonball between being shot out of the cannon and exploding and the other two figure out things subsequent to the explosion. With this in mind, we figure out how long before the fragment hits the ground: $s = \frac{1}{2}at^2 + v_0 t + s_0 = 0$. Where did the 0 come from? That's the *y* position of the fragment when it hits the ground! We use the quadratic formula to figure out the time when this happens. (Note: I've noticed that a disturbing fraction of you either don't know *when* to use the quadratic formula or don't know *how* to use it. Come see me if in doubt about this.) This gives $t = \frac{-v_0 \pm \sqrt{v_0^2 - 2\times a \times s_0}}{a}$. Inserting numbers for these terms, we have

$$t = \frac{-80.4\frac{\text{meters}}{\text{second}} \pm \sqrt{6464.16\frac{\text{meters}^2}{\text{second}^2} + 2\times 9.8\frac{\text{meters}}{\text{second}^2}\times 180.4\,\text{meters}}}{-9.8\frac{\text{meters}}{\text{second}^2}}.$$

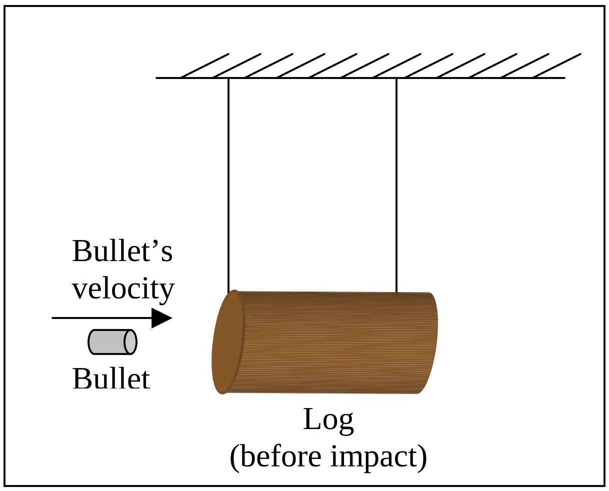
$$= \frac{80.4 \mp 100}{9.8}\text{seconds}$$

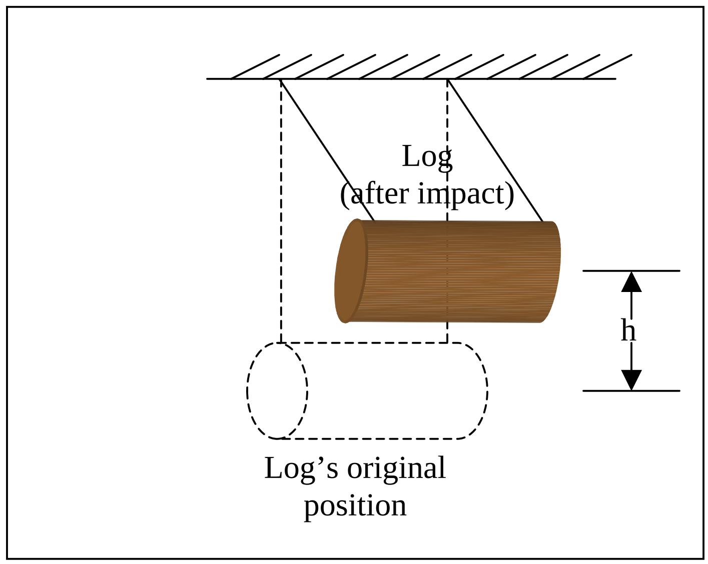
This gives us two "roots"—two possible answers for *t*. We'll need to discriminate between them. First, let's take the "-" sign's root. This gives $t = -2\,\text{seconds}$. Of *course* it does! This is the time that the cannonball was originally shot! We solved for the times at which the height of the cannonball is zero. One of those times is when it is originally shot. This had better come as no surprise! The other root should be what we want. This is $t = 18.41\,\text{seconds}$.

Now, we use the time we just found to locate the *x* position of the impact. This is again found using $s = \frac{1}{2}at^2 + v_0 t + s_0$ with the values found after the explosion. Here we have (using the fact that the *x* acceleration is zero)

$x = \frac{1}{2}at^2 + v_0 t + x_0 = 273.2\frac{\text{meters}}{\text{second}}\times 18.41\text{seconds} + 346.4\,\text{meters}$. This gives $x = 5376\,\text{meters}$.

5. **A "ballistic pendulum" is a low-tech way of determining the speed of a bullet. This is pictured below. A bullet traveling exactly horizontally strikes a log hanging from a pair of long strings and embeds itself in the log. The mass of the bullet is 20 grams. The mass of the log is 10 kg. The strings are 2 meters long. If the speed of the bullet is $500\,\frac{\text{meters}}{\text{second}}$, how high will the log rise? (Hint: The kinetic energy of the bullet is not conserved. Find the momentum of the log after impact and then use conservation of energy for the *log* to find its height.) This is a terrific example in which K.E. is not conserved but momentum is.**





Again, this is a problem that has two "phases"—two distinct regimes: The time before the impact of the bullet and the time after it strikes the log. The kinetic energy of the bullet is not conserved between these two phases. This is an inelastic collision. However, the momentum of the bullet *is* conserved. Also, the kinetic energy of the log *after* the collision, in the time in which it is swinging up to its new height, *is* conserved. Be sure you recognize the distinction between these two regimes! (Note that the momentum of the log+bullet system is not conserved in this second phase. This is because of the strings—the Earth would need to be considered part of the system if we wanted to use conservation of momentum after the log begins to move.)

The condition of conservation of momentum is $\vec{p}_i = \vec{p}_f$—the initial momentum of the entire system before the impact is equal to the momentum of the entire system after the impact. I'll treat the momentum of the bullet as a scalar since this is a one-dimensional problem. The total momentum before the impact is just that of the bullet: $p_i = 0.02\,\text{kg} \times 500\,\frac{\text{meters}}{\text{second}} = 10\,\frac{\text{kg}\cdot\text{meters}}{\text{second}}$.

The momentum after the impact is $p_f = (m_{log} + m_{bullet}) \times v_{log}$. Normally, I'd just use the mass of the log in this: The bullet changes the mass by only .02%. But, for completeness, I'll leave it in. I'd recommend neglecting it, however. We can use this with the momentum conservation condition to solve for the speed of the log. This is $v_{log} = \frac{p_i}{m_{log} + m_{bullet}} = \frac{10\,\frac{\text{kg}\cdot\text{meters}}{\text{second}}}{10.02\,\text{kg}} = 0.998\,\frac{\text{meters}}{\text{second}}$. (Normally, I try to avoid giving you questions with answers like this—having it just come out to "1" really tends to obscure crucial information. It makes the problem seem too pat.)

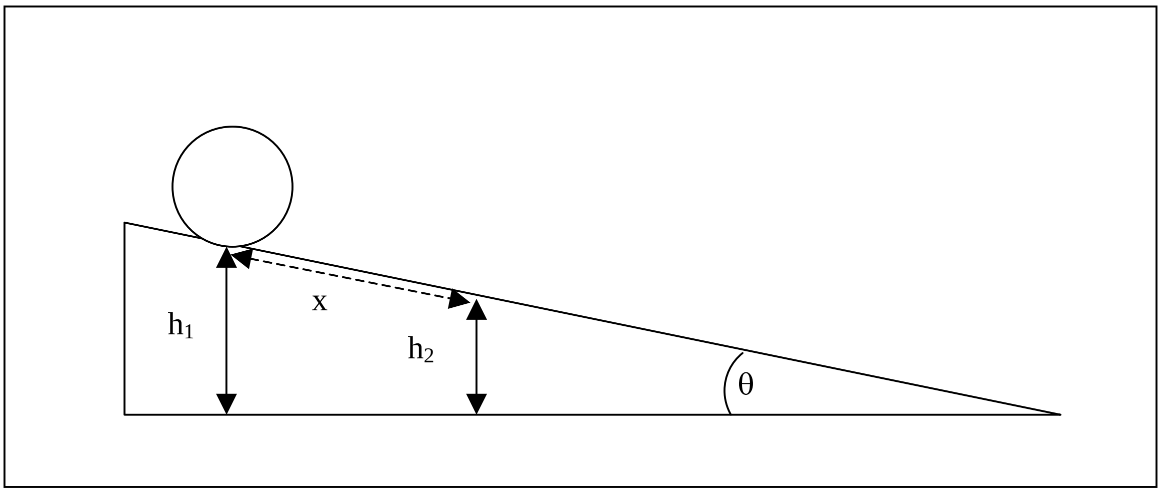
But this doesn't answer the question we were asked. Imagine a real world situation: If you're actually trying to *use* this device (and they really are used), you'd be left trying to measure the speed of the log immediately after the impact. Since the log is swinging on its ropes, it will start "decelerating" (yuck!) right away, so your measurement wouldn't be very good. On the other hand, you could quite easily measure the final height of the log as it swung. So, let's figure out what that is. For this, we use conservation of energy.

Let's take the zero of potential energy to be when the log is at the bottom of its swing. Thus the PE of the log is zero the moment after the bullet strikes it. It does have a kinetic energy, however. The condition of conservation of energy can be written $KE + PE = constant$. *PLEASE* write it this way! I've seen a huge amount of confusion due to people memorizing special cases and then misapplying them! Our challenge in a conservation of energy problem is to determine the value of the constant. The easiest way to do this is to find some point at which we know the values of the PE and the KE. Ideally, one of these will be zero. This is, indeed, the case when the log has just been hit by the bullet: At that instant, $PE = 0$ and $KE = \frac{1}{2}\left(m_{log} + m_{bullet}\right)v_{log}^2$. This gives us the value of the constant (the total energy in the system) $constant = \frac{1}{2} \times 10.02\,\text{kg} \times \left(0.998\,\frac{\text{meters}}{\text{second}}\right)^2 = 4.99\,\text{Joules}$.

Now, at the end of the swing, the kinetic energy will be zero (the log stops). Thus all of the energy will be in the potential energy and we have $PE = 4.99\,\text{Joules}$. Using the expression for gravitational potential energy, we have $mgh = 4.99\,\text{Joules}$. Solving this for *h*, we have $h = \frac{4.99\,\text{Joules}}{(m_{log} + m_{bullet}) \times 9.8\,\frac{\text{meters}}{\text{second}^2}} = .0508\,\text{meter}$.

6. A thin metal hoop and a solid disk both roll down an incline, starting at the same point. The masses of the two objects are the same—call it *M*. The incline makes an angle θ relative to the horizontal. The objects roll without sliding. Find a mathematical expression for the speed of each of the objects as a function of their position, *x*, along the incline. It is highly recommended that you use energy conservation to do this.

Taking the very good advice of the very smart man who wrote the problem, we use conservation of energy for this system. The only difference between the objects is their moment of inertia, so we can do this once and then just substitute for the moment of inertia when the time comes.



Unlike previous conservation of energy problems, we have two distinct kinetic energies to consider: One of these is due to the overall motion of the object. The other is due to internal motion of the object, motion which would require energy even if the object weren't moving through space. Fortunately, we just have one type of potential energy to deal with here! Our conservation of energy equation is $K.E. + K.E._{rot} + P.E. = constant$ As always, the constant is the total energy of the system.

Let's assume the object starts rolling when it is at h_1. Thus, our constant will just be the potential energy of the object at that point. This is $P.E._{initial} = mg\mathrm{h}_1$. Now, after the object has rolled a distance x, it will be at a different height. We can easily calculate the change in the object's height using trigonometry. This is $\mathrm{h}_1 - \mathrm{h}_2 = \mathrm{x}\sin(\theta)$. Since the new potential energy will be $P.E._{new} = mg\mathrm{h}_2$, our conservation of energy equation gives us $K.E. + K.E._{rot} + mg\mathrm{h}_2 = mg\mathrm{h}_1$. A smidge of algebra gives $K.E. + K.E._{rot} = mg\mathrm{h}_1 - mg\mathrm{h}_2 = mg\,\mathrm{x}\sin(\theta)$.

Now, for our kinetic energies all we need to do is plug in formulas: $K.E. = \frac{1}{2}mv^2$ and $K.E._{rot} = \frac{1}{2}I\omega^2$. Thus $K.E. + K.E._{rot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mg\,\mathrm{x}\sin(\theta)$.

We now have an equation that involves both the linear speed and the angular speed. However, since the objects are rolling, the two speeds are related. A point that is a distance r from the rotational axis of a spinning object moves at a speed of $v = \omega r$ (be very careful when applying this! be sure you know which distance it is that you are talking about and what the direction of the velocity of that point is). Since these objects are rolling and not sliding, whatever speed a point on the edge has will be the speed at which the object moves. So we can write $\omega = \frac{v}{r}$. Substituting this into the equation above, we have $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2} = mg\,\mathrm{x}\sin(\theta)$. Which can be cleaned up a bit to give $\left(\frac{1}{2}m + \frac{1}{2}\frac{I}{r^2}\right)v^2 = mg\,\mathrm{x}\sin(\theta)$.

Finally, dividing and taking a square root, we have $v = \sqrt{\frac{mg\,\mathrm{x}\sin(\theta)}{\left(\frac{1}{2}m + \frac{1}{2}\frac{I}{r^2}\right)}}$.

We can now substitute for the moments of inertia of the two objects. For the disk, the moment of inertia is $I_{disk} = \frac{1}{2}mr^2$. This gives $v_{disk} = \sqrt{\frac{mg\,\mathrm{x}\sin(\theta)}{\left(\frac{1}{2}m + \frac{1}{2}\frac{\frac{1}{2}mr^2}{r^2}\right)}} = \sqrt{\frac{g\,\mathrm{x}\sin(\theta)}{\left(\frac{1}{2}+\frac{1}{4}\right)}} = \sqrt{\frac{g\,\mathrm{x}\sin(\theta)}{\frac{3}{4}}}$.

For the ring, we have $I_{ring} = mr^2$, so $v_{ring} = \sqrt{\frac{mg\,\mathrm{x}\sin(\theta)}{\left(\frac{1}{2}m + \frac{1}{2}\frac{mr^2}{r^2}\right)}} = \sqrt{\frac{g\,\mathrm{x}\sin(\theta)}{\left(\frac{1}{2}+\frac{1}{2}\right)}} = \sqrt{g\,\mathrm{x}\sin(\theta)}$.

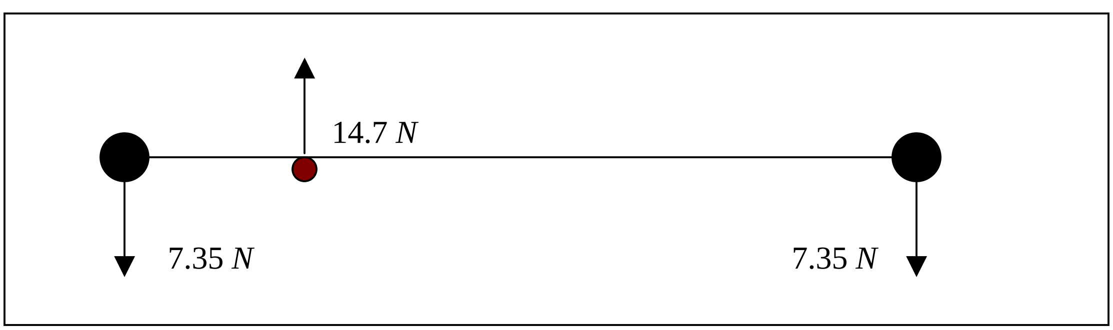
There are several things to notice about these. First, notice that they neither depend on the mass nor on the radii of the objects. Galileo primarily used rolling objects for his experiments in order to minimize the effects of friction. This proves that his method was acceptable. Notice also that the speed of the disk has a number which is less than one in the denominator. Thus the disk is expected to be moving faster at a given distance down the ramp than the ring. This is consistent with what we observed in class.

7. A rod is exactly one meter long. There is a 0.75 kg mass on each end of it. A man attempts to support it by placing his finger under the rod 20 cm from one end. The rod can be considered to have zero mass. What is the total torque on the rod? (You may express the direction of the torque as "up", "down", "right", "left", "into the page", or "out of the page" making reference to the above picture.) As an ungraded variation: Repeat the calculation using masses that are different.

When we calculate torque, we are free to pick the point about which the torque is calculated (the "origin"). This can be any point in the universe—it doesn't need to be the "axis" of the object (after all, we often don't know what the axis is until we've solved the rest of the problem), in fact, it doesn't need to be on the object at all! However, an unwise choice of this point will result in a tremendous amount of unnecessary work. A wise choice will minimize the amount of effort needed to solve the problem.

In this case, there are three points that make senses: The location of either of the two masses or the location of the man's finger. I'll do it two ways—using the location of the finger first and then the location of one of the masses. I'll leave it to you to use the location of the other mass as an exercise in problem solving.

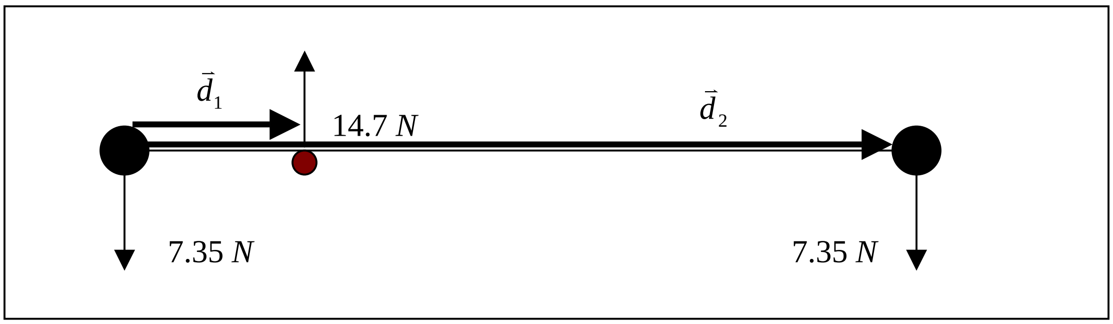
In order to solve for the torques, we first need to find the external forces and their locations. There are three external forces acting on the rod: The force of gravity pulling down on each of the masses and the force of the man's finger pushing up on the rod. The force of gravity is just the weight of each of the masses $W = mg = .75\,\text{kg} \times 9.8 \frac{\text{meters}}{\text{second}^2} = 7.35\,\text{Newtons}$.



If the rod is not accelerating, the force exerted by the man's finger is equal in size to the *sum* of the two weights. This is a common source of error. Be careful not to overlook it. The rod is not accelerating. Newton's second law didn't go away just because we started talking about torque. Therefore, the net external force must be zero. Since the total downward force is that due to gravity acting on the two masses, the force of the man's finger must be the same size, only acting upward. Our free body diagram appears above.

This diagram will be the same for any choice of the point about which the torques are to be calculated. What will be different is the distances—which, we **must** remember, are considered as vectors. Let's use the mass on the left as our origin first.

Picking our mass on the left as the origin, we have two distances, $\vec{d}_1$ and $\vec{d}_2$. Each of these is multiplied by the force at that location. Let's find the size of these two products first and then worry about the directions (remember that the product is a *vector* and so must include both a magnitude [size] and a direction to be complete). This gives $\tau_1 = d_1 \times 14.7\,\text{N} = 0.2\,\text{meter} \times 14.7\,\text{N} = 2.94\,\text{Newton} \cdot \text{meters}$ and $\tau_2 = d_2 \times 7.35\,\text{N} = 1\,\text{meter} \times 7.35\,\text{N} = 7.35\,\text{Newton} \cdot \text{meters}$. What about the force on the mass on the left? Well, since it's at the origin, the distance to it will be zero. So, although there is a force acting there, the contribution of this force to the torque will be zero, so we can ignore it. (Please recall: This does *not* mean that the rod will pivot about that mass! We're just calculating the net torque.)



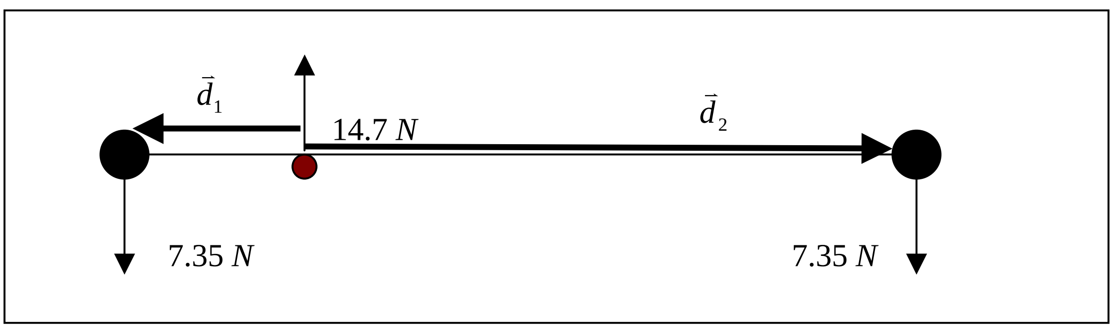
Now for the task of determining the directions of these products. The direction is determined using the "right hand rule." The way this works is, for each of the products, draw the vectors in the product tail-to-tail (this is very important! also, it's confusing because it's different from what you draw when *adding* two vectors). Then, picture the *d* vector rotating into the force vector. The way I like to do it is to draw the two vectors in the product the same length for the purpose of determining the direction of the product: Just as we ignored the direction while we were calculating the size, so we now ignore the size when figuring out the direction. We'll put the two pieces (size and direction) together later. This gives

Now, take your right hand and orient it so that your fingers are curled along the path of the curved arrows in the picture above—your fingertips should point in the directions of the arrowheads. Stick your thumb straight out. Your thumb will be pointing in the direction of the resultant vector. Let's call "out of the page" $+\hat{z}$ and "into the page" $-\hat{z}$. Notice that $\vec{\tau}_1$ is positive and $\vec{\tau}_2$ is negative. (I'm not very concerned with whether you get the final answer with precisely the right sign. It is crucial, however, that you get the *relative* signs of the two torques correct.)

We take these two values and add them, now keeping the signs, to find the final torque:

$$\begin{aligned}\vec{\tau} &= \vec{\tau}_1 + \vec{\tau}_2 = 2.94\,\text{Newton}\cdot\text{meter}\,\hat{\text{z}} - 7.35\,\text{Newton}\cdot\text{meter}\,\hat{\text{z}} \\ &= -4.41\,\text{Newton}\cdot\text{meter}\,\hat{\text{z}}\end{aligned}$$

This is either out of the page or into the page. Since we decided that negative values were into the page and this is negative, we conclude the total torque is $\vec{\tau} = 4.41\,\text{Newton}\cdot\text{meter}$ into the page.



Now, let's redo this whole problem with the location of the man's finger as the origin. Now our diagram is shown above. As you can see, $\vec{d}_1$ has changed sign, but it's the same size as before. On the other hand, $\vec{d}_2$ is a different size but is in the same direction as before. We get

$$\tau_1 = d_1 \times 7.35\,\text{N} = 0.2\,\text{meter} \times 7.35\,\text{N} = 1.47\,\text{Newton}\cdot\text{meters}$$

(again, ignoring the direction for now). And

$$\tau_2 = d_2 \times 7.35\,\text{N} = 0.8\,\text{meter} \times 7.35\,\text{N} = 5.88\,\text{Newton}\cdot\text{meters}.$$

Following the same procedure as above to find the direction, we see that $\vec{\tau}_1$ is positive and $\vec{\tau}_2$ is negative. Adding these together appropriately, we get

$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = 1.47\,\text{Newton}\cdot\text{meter}\,\hat{z} - 5.88\text{Newton}\cdot\text{meter}\,\hat{z} = -4.41\,\text{Newton}\cdot\text{meter}\,\hat{z}$. This is the same number as before—as it had better be! We could do this calculation an infinite number of ways, all of which would be correct and equally valid and all of them had better give the same number. After all, the net torque tells us how large the angular acceleration will be. This is a real, physical quantity. It mustn't depend on which inertial system we observe it from.

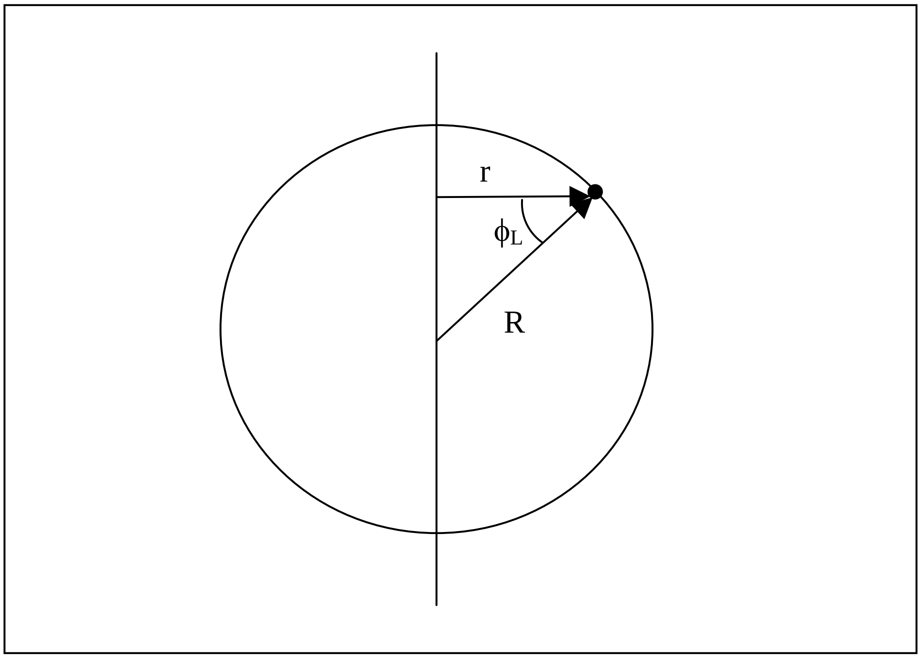
I *strongly* encourage you to try redoing this calculation with at least one different origin as a test of your mastery of the method!

8. Refer again to the rod in problem #7. The man is exerting enough force to keep the rod from falling down (if it didn't rotate). Pick the mass on the left as the "special point" about which the torques are calculated. Calculate the total torque on the rod using this point.

I used a little psychology in this problem: I just *knew* that virtually all of you would pick the finger as the origin for Problem #7 even though I didn't specify that, so I felt safe giving you this problem as a followup. If you *didn't* use the finger as the origin, congratulations! That showed real imagination. Anyway, the answer to this is given in my solution to #7, so I won't repeat it.

9. What is the angular speed of the Earth? Using this, what is the speed (not angular) of Edwardsville?

The angular speed of a rotating object is the angle over which it rotates divided by the time that it takes to rotate that angle. Written as an equation, this is $\omega = \frac{\Delta\theta}{t}$. That part's easy. Now, how do we find what it is for a particular system? When solving a problem, start by writing down the things that you *know*—be careful that they're things that you actually know to be true, not things that you merely think are true (yes, that's a hard call sometimes!). What I *know* about the rotation of the earth is that it makes one complete rotation every day. So, if I pick the angle, $\Delta\theta$, to be 2π (working in radians), I will know the time it takes to rotate that angle. Using this, we have $\omega = \frac{\Delta\theta}{t} = \frac{2\pi}{86400\,\text{seconds}} = 7.272\times10^{-5}\,\frac{1}{s}$. Note that I used the number of seconds in a day. Also, note the weird unit at the end. This is read "per second" or "Hertz". I *could* have expressed this in radians per second as well, either way is equally accurate. Radians per second $\left(\frac{\text{radians}}{\text{second}}\right)$ is probably clearer, however.



Now, since the distance traveled by an object moving along an arc of a circle (or along a circle) is $l = \theta r$ (being careful to express the angle in radians!), the speed of that object is given by $v = \omega r$. So the speed of Edwardsville is the angular speed of the earth times the radius of the circle around which Edwardsville is traveling. Here's where you'll make a mistake if you're not careful: Edwardsville is *not* revolving around a circle whose radius is the same as that of the earth! This can be seen in the picture above. The dot represents Edwardsville. While we are a distance R from the center of the earth, we are only a distance r from the rotational axis of the earth. The angle ϕ_L is the latitude of Edwardsville. So the distance from the earth's rotational axis is $\mathrm{r} = \mathrm{R}\cos(\phi_L)$. Edwardsville's latitude is about 39° and the Earth's radius is about $\mathrm{R} = 6.37 \times 10^6$ meters, so $\mathrm{r} = 4.95 \times 10^6$ meters. This gives $v = \omega r = 7.272 \times 10^{-5} \dfrac{1}{\text{second}} \times 4.95 \times 10^6 \text{ meters} = 360 \dfrac{\text{meters}}{\text{second}}$ (this is about 800 miles per hour).