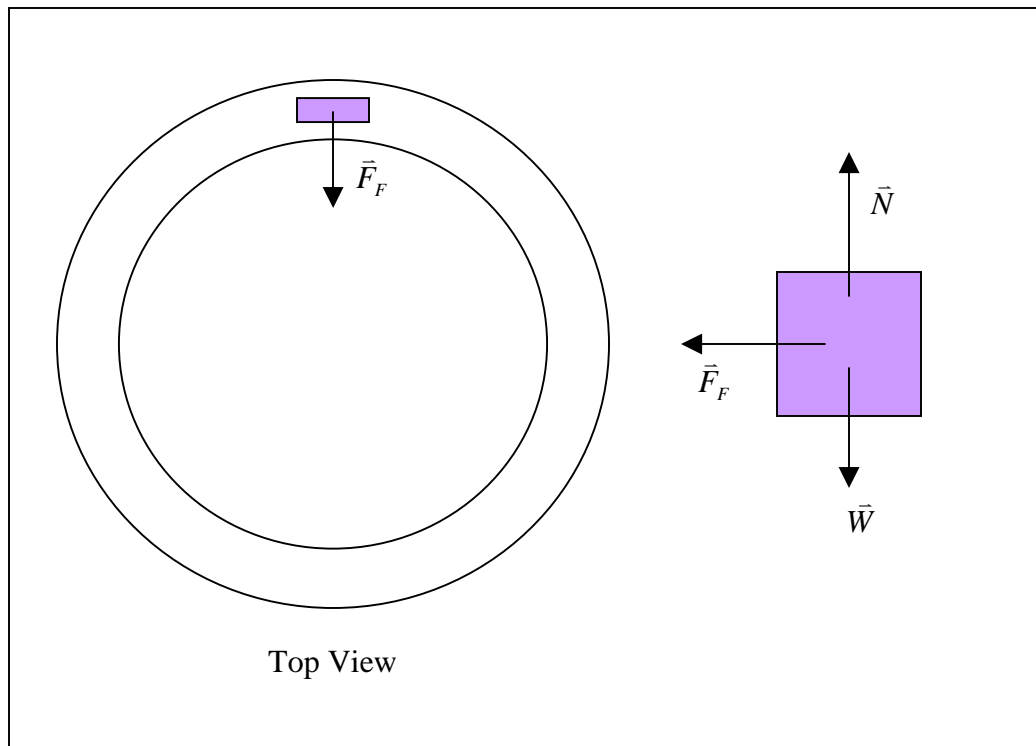


**PHYSICS 206a**  
**HOMEWORK #6**  
**SOLUTIONS**

1. A car travels at a constant speed around a circular track. The track is perfectly horizontal. Draw a free-body diagram showing all the external forces on the car. If the force of static friction between the car's tires and the road is  $\mu_s$ , what is the maximum speed with which the car can travel if it is to remain on a circular path? Express your result as a function of the mass of the car, the radius of the track, and  $\mu_s$ .

Please note an *egregious* typographical error on my part in the above questions: The *force* of static friction isn't  $\mu_s$ .  $\mu_s$  is the *coefficient* of static friction. It doesn't turn into a force until it is multiplied by the normal force. This was a horrible error on my part and one which I would harshly penalize you for making. So I guess this is a "do as I say, not as I do" situation. Please don't forget that the coefficient of friction isn't the same as the force of friction. You have my sincere apologies for this error!

I'm not a good enough draftsman to draw the free-body diagram for this in a single drawing, so I'll do a top-view and a view from the rear of the car. We have:



Notice that the vertical forces cancel, as they should: There is no acceleration vertically. Horizontally, however, there is a net force. This is directed toward the center of the circular track and is dubbed the "centripetal force." Please understand that there is no particular force called "centripetal" any more than

there is a particular force called “up” or “down” or “left.” To say that a force is centripetal simply says what direction it points—toward the center of the circle. In this case, the centripetal force is provided by friction. (If we wanted to do the same problem but with the car on, say, a frozen lake which doesn’t provide much friction, we could get a centripetal acceleration by strapping rockets to the side of the car pointed away from the center of the track.)

As was noted in the answer to the previous problem,  $F_f \leq \mu_s N$ . This can be read “the frictional force has a maximum value of  $\mu_s N$ .” When the car is just about to slip, the friction is at its maximum value and the equality can be used so we can write  $F_f = \mu_s N$ . Since, in this case, the normal force and the weight have the same size, this can also be written  $F_f = \mu_s W = \mu_s mg$ . (As stated above, don’t forget to multiply the coefficient of friction,  $\mu_s$ , by the normal force. If this distinction isn’t clear, come chat with me about it.)

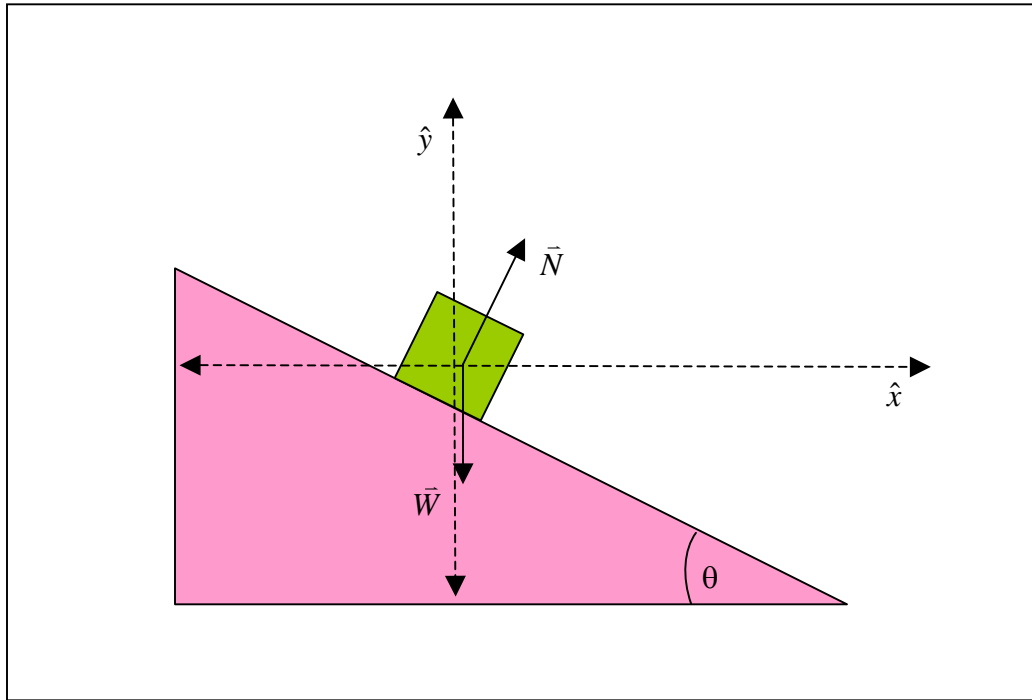
If a centripetal force is to keep an object moving in a circle at a constant speed, then its size must be  $F_c = \frac{mv^2}{r}$ . Since the centripetal force, in this case, is provided by the frictional force, we can write  $F_c = \frac{mv^2}{r} = \mu_s mg$ . When we solve this for  $v$ , we get  $v_{\max} = \sqrt{r\mu_s g}$ . Notice that the result does not depend on the mass of the vehicle.

- 2. Assuming the car in Problem #1 has a mass of 1300 kg (that’s about a VW Golf—which is what I drive), the track has a radius of 63 meters, and  $\mu_s$  is 0.8, what is the maximum speed of the car? (Sorry for the typo in the original problem—of course I meant “Problem #1” not “Problem #8” as originally written.)**

This is just an opportunity to plug numbers into the answer you got in the previous problem since so many of you just don’t consider a problem “done” until there’s a number. Taking  $v = \sqrt{r\mu_s g}$  we get (notice that the mass is irrelevant)

$$v = \sqrt{r\mu_s g} = \sqrt{63 \text{ meters} \times .8 \times 9.8 \frac{\text{meters}}{\text{second}^2}} = 22.2 \frac{\text{meters}}{\text{second}}.$$

3. Consider again the car in the previous problem. If there is no friction between the tires and the road, at what angle must the road be banked to allow the car not to skid? Draw a free-body diagram of this situation.



The free-body diagram appears above. I've put in a coordinate system with the  $x$  and  $y$  axes in the "usual" directions. How did I know to do this and not the weird direction we've used before, with the  $x$  axis parallel to the surface? Well, I know (because it was stated in the problem) that there will be no motion of the car in the direction parallel to the surface of the incline. So there's no good reason to put a coordinate axis in that direction. On the other hand, I know the value of the weight, so putting an axis in that direction is attractive. Finally, as we'll discuss below, I also *know* that the net acceleration will be in the  $\hat{x}$  direction if I use the axes as drawn. In general, a wise choice of coordinate axes is that which allows the largest number of things to be written with only one component. Yes, there's an art to making the right choice and you won't always get it right.

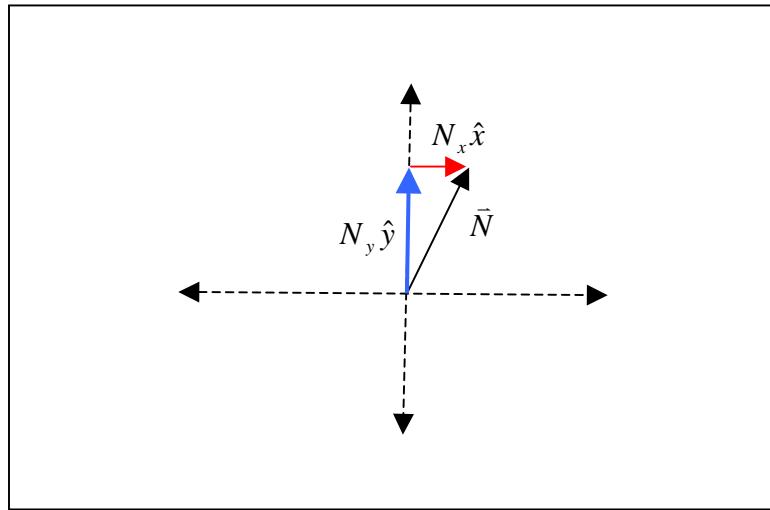
The absolutely **KEY** feature to this problem is that the free body diagram is utterly identical to that for the block on the ramp in problem #7 of assignment #4 and the one for Lab Exercise #3. There is *no* additional force in the problem.

Think, for a moment, about what happens in the "block on a ramp" situation with no friction: I think we'd all agree that the block very nicely slides down the ramp. You did this in lab with the airtrack and found that the acceleration down the ramp was  $g \sin(\theta)$ . So, we have a bit of a conundrum: Ignoring friction, this problem has exactly the same forces as the "block on a ramp" situation. So why

doesn't the car just go sliding down the banked road? Think about that for a moment, I'll wait here.

Done? Good, here's the answer: The result from the "block on a ramp" experiment was not that the block would *move* down the ramp, but that it would *accelerate* in that direction. Remember: "To accelerate" does *not* mean "to move!" We can have acceleration without any motion in the direction of the acceleration. In this case, if the car moves with just the right speed, the acceleration will change the direction of the car's velocity without changing its size at all. In some sense, the car can be thought of as slipping down the bank, but every time the car slips a tiny (really, infinitesimal) amount down the bank, it also moves a bit around the track so that it sits higher up the bank. Matching the slipping-down with the catching-up happens only at a single, specific speed. We must find the speed that the car has to go so that it catches itself and the acceleration only changes the car's direction and not its speed. (There's a lot of meat in that paragraph. Don't just read on! Go over what you've just read a few times until it makes sense. The words below will still be there when you're done.)

Think about what would happen if the car were moving too fast: It would tend to creep *up* the bank, ultimately falling off the road. Now think about what would happen if the car were moving too slowly: The tendency to accelerate down the bank would result in actual motion in the direction of the acceleration. Only when the speed is exactly matched to the angle of the bank does the "falling but getting caught" exactly work. (We'll encounter this balance again when we deal with planetary orbits.)



So, now we understand what's going on. The next step is to get the numbers right. As stated above, I've put in a coordinate system with the  $x$  and  $y$  axes pointed in the "usual" directions, rather than parallel and perpendicular to the surface of the bank. This is because we don't expect motion along the bank and there's no particular force (e.g., friction) parallel to it, so there's no good reason to use this as a reference. (One nice thing about the math in problems like this is that the penalty for not getting the ideal coordinate system is simply that the problem will be more difficult. You'll still get the right answer, you'll just have to work

harder for it.) In previous problems, we've decomposed the weight into  $x$  and  $y$  components. Since the weight is now all in the  $y$  direction, we don't need to do this. However, now we have to decompose the Normal force. Using the methods with which we should all now be very familiar, we get  $N_x = N \sin(\theta)$  and  $N_y = N \cos(\theta)$ .

From here, the next step should be second nature to you: Having come up with a coordinate system and having decomposed all the forces so that we know their components in the directions of the axes, we have turned our 2-dimensional problem into two 1-dimensional problems. We treat the  $x$  and  $y$  parts separately.

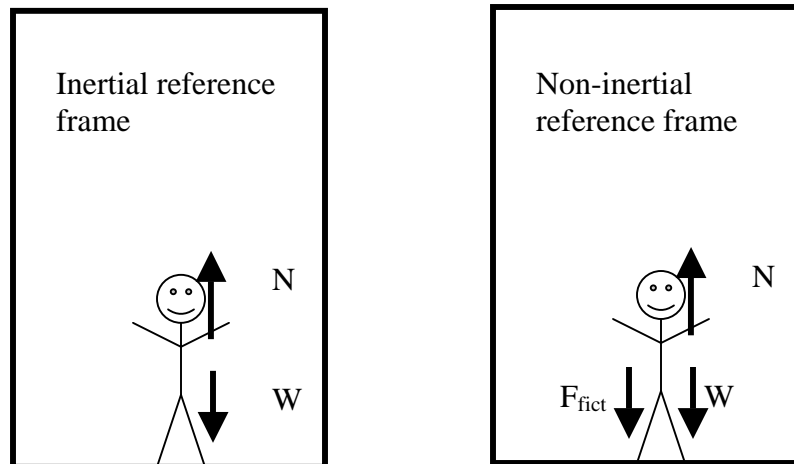
In the  $y$  direction, we have gravity pulling down and the  $y$  component of the Normal force pushing up. These exactly cancel. In the  $x$  direction, there is one and only one force: The  $x$  component of the Normal force. It is this force that provides the centripetal acceleration. So, we can say that  $N_x = N \sin(\theta) = m \frac{v^2}{r}$ . We lack a value for  $N$ , but this can be found from the  $y$  component. Since we know that gravity exactly cancels the  $y$  component of the Normal force, we can also write  $N_y = N \cos(\theta) = mg$ . So a wee bit of algebra gives us  $N = \frac{mg}{\cos(\theta)}$ .

Now, substitute the expression for  $N$  into the expression for  $N_x$  to get  $N_x = \frac{mg \sin(\theta)}{\cos(\theta)} = m \frac{v^2}{r}$ . And, finally, using the relation found in Problem #5 of Assignment #5, we get  $\tan(\theta) = \frac{v^2}{gr}$ .

This result should ring some bells. In Problem #5 of Assignment #5, you found that the coefficient of static friction for a block on an incline is  $\mu_s = \tan(\theta)$ . In Problem #1 of this assignment, you found that  $\mu_s = \frac{v^2}{gr}$  for a car on a non-inclined track. What we've now shown is that the force given by  $mg \tan(\theta)$  makes the block slide down the ramp if there isn't anything to catch it, but if the ramp is part of a curved track, the downward slide gets "caught" and the block doesn't slide down but rather goes in a circle. Don't worry if this insight is not quite clear to you. Being able to work out the problems independently is what's crucial at this stage. Just be aware that the results of problems #1, 3, and, from assignment #5, 5 are related at a deep level. This is not mere coincidence!

4. A man with a mass of 110 kilograms stands in an elevator. The elevator is lifted by a single rope. The elevator has a mass of 1030 kilograms. The elevator accelerates upward at  $0.8 \frac{\text{meters}}{\text{second}^2}$ .

- Draw a free body diagram for the person indicating all relevant forces.
- State the size of each of the forces indicated in (a).
- Draw a free body diagram for the elevator.
- State the size of each of the forces indicated in (c).



- There are two ways to do this problem: We can either work in the inertial reference frame of the Earth or we can work in the non-inertial reference frame of the elevator. The two approaches will give exactly the same answer provided we do them right. Let's do it both ways and see.

In the inertial frame, the elevator accelerates. Since the person is in the elevator, the elevator must exert a force on the person such that the person accelerates at the same rate as the elevator. (If it didn't, the person would fall through the floor of the elevator or fly through its ceiling!) There are only two forces acting on the person: Gravity pulling down and the normal force, provided by the floor of the elevator pushing up.

In the non-inertial reference frame, the elevator does not accelerate! Instead, the person experiences a fictitious force in addition to the force of gravity downward and the normal force upward.

- The weight of the person, which is the force of gravity pulling on the person, is simply  $W = mg$ . Inserting the mass of the person and the strength of gravity, we get  $W = 110 \text{ kg} \times 9.8 \frac{\text{meters}}{\text{second}^2} = 1078 \text{ Newtons}$ . This is true in both the inertial and the non-inertial reference frames.

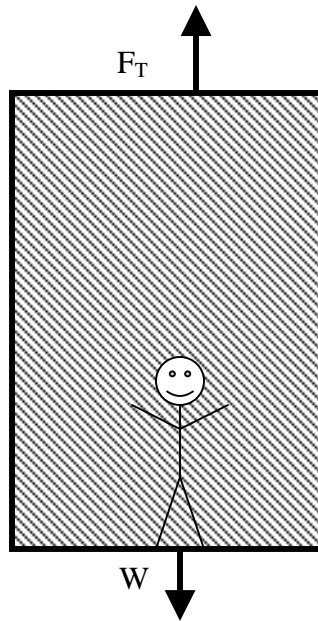
Finding the normal force is a potential trap. Recall that the normal force is the only upward force acting on the person. In the inertial reference frame, the person is accelerating upward, so there must be a *net* force pushing up on the person. Your first instinct was probably to equate the normal force with the weight of the person. This would be correct in the non-inertial reference frame (see below). But in the inertial reference frame, this is incorrect. Using Newton's second law, we have  $F_{net} = ma = N - W$  (since this is a 1-d problem, we can leave off the vector symbols, but we do still need to retain the signs correctly to indicate direction). Inserting the mass of the man and his acceleration, we have  $N - W = ma = 110 \text{ kilograms} \times 0.8 \frac{\text{meters}}{\text{second}^2} = 88 \text{ Newtons}$ . This is the *net* force acting on the man. Using this and the weight we found earlier, we get  $N = 1078 \text{ Newtons} + 88 \text{ Newtons} = 1166 \text{ Newtons}$ .

Now, in the non-inertial reference frame, the man doesn't accelerate. So, in this case, the normal force is equal in size to the sum of the downward forces. The net force in the non-inertial reference frame is zero. (This is why our minds tend to choose these reference frames when we are in them. Usually, using a non-inertial reference frame to solve a problem is a very bad idea. But, psychologically, we use these all the time. This is the source of many misconceptions about Physical laws. So we should learn to solve problems in non-inertial reference frames so that we can understand the origins of the fictitious forces that cause our misconceptions—that will allow us to attack those misconceptions.) But we must solve for the fictitious force. Recall that the fictitious force an object experiences in a frame that is accelerating at  $\vec{a}_f$  is  $\vec{F}_{fict} = -m\vec{a}_f$ . Note two things about this relation: First, the minus sign tells us that the fictitious force will be in a direction exactly opposite to the acceleration of the frame. In this case, the frame is accelerating "up" so the fictitious force will be "down." Second, the mass that experiences the force is the mass of the object in question—in this case, the man. The mass of the frame (the elevator, in this example) doesn't matter at all.

Using this, we can write  $F_{net} = ma = N - W - F_{fict} = 0$ . Since  $\vec{F}_{fict} = -m\vec{a}_f = -110 \text{ kilograms} \times 0.8 \frac{\text{meters}}{\text{second}^2} \hat{y} = -88 \text{ Newtons } \hat{y}$  (the same direction as  $W$ ), and we know the man's weight, we can immediately solve for the normal force—this gives the same final result as we found previously.

Note one last detail: When I wrote  $F_{net} = ma = N - W - F_{fict} = 0$  I put in the signs "explicitly." But then I calculated  $F_{fict}$  and put in the sign "implicitly." This is a bit confusing. Be careful when you're doing this sort of calculation: The minus sign in front of  $F_{fict}$  in  $F_{net} = ma = N - W - F_{fict} = 0$  already accounts for the fact that the fictitious force is downward. Don't double-count!

c.



Now, the elevator *is* a candidate for a non-inertial reference frame, but it does not exist *within* a candidate for a non-inertial reference frame. So we must treat the elevator as an object within an inertial reference frame—which makes life easier. As with the free-body diagram of the man, there are only two forces acting on the elevator. These are the weight of the elevator (including its contents!) downward and the tension on whatever cable is pulling the elevator up.

- d. The weight is the force of gravity pulling on the elevator plus the force of gravity pulling on the person. This is  $W = (m_{person} + m_{elevator})g$ . Inserting the masses and the strength of gravity, we get

$$W = (110 \text{ kg} + 1030 \text{ kg}) \times 9.8 \frac{\text{meters}}{\text{second}^2} = 11,172 \text{ Newtons} .$$

The tension on the cable must provide the force needed not only to support the elevator against gravity but also to provide the needed upward acceleration. Using Newton's second law, we have  $F = ma = F_T - W$ . Inserting the mass of the elevator containing the man and their acceleration, we have

$$F_T - W = ma = (110 \text{ kg} + 1030 \text{ kg}) \times .8 \frac{\text{meters}}{\text{second}^2} = 912 \text{ Newtons} . \text{ This is the } \textit{net}$$

force acting on the elevator. Using this and the weight we found earlier, we get  $F_T = 11172 \text{ Newtons} + 912 \text{ Newtons} = 12,084 \text{ Newtons} .$



5. **A man is standing in a train accelerating at  $0.1 \frac{\text{meter}}{\text{second}^2}$ . He drops a nickel ( $m=5$  grams) from a height of 1.5 meters. In his frame of reference, what force appears to act on the nickel? How far off the vertical path is the nickel deflected by the time it strikes the ground?**

Sometimes we introduce a non-inertial frame of reference simply because a human observer is involved and it is in our psychological makeup to define whatever space we are in as a good frame of reference. We usually know that the frame that we're in is accelerating, but our brains are essentially egocentric—they're hardwired to think of forces as acting on us rather than to think of us as flying about, according to Newton's first law, and having our environment change around us. Other times, we'll introduce a non-inertial frame because purely pragmatic considerations make it more convenient to do so (this is rare).

In either of these cases, what we must be aware of is that objects within the frames will behave as though forces are acting on them that simply don't exist. Someone with "x-ray vision" standing in an inertial frame (for example, standing on the ground outside of the train) looking through the walls of the train in this problem would see the nickel fall down while retaining whatever forward velocity the train had at the instant the nickel was released. Relative to the Earth, the path of the nickel would have the same shape, for the same reason, as the bomb that we studied in Problem #8 of Assignment #4. He would see that the *train* is accelerating relative to the nickel just as it does relative to the Earth. This is a key feature of non-inertial frames of reference and a distinction between them and inertial frames: *All* observers, in whatever frame, recognize that a frame is non-inertial and agree on the size and direction of its acceleration. As we have seen, there is *no* observation that someone can make to distinguish whether one or another inertial frame is "really" moving.

So the reality really is that the train is accelerating while the nickel is moving with a constant velocity (in the  $\hat{x}$  direction—everyone will agree that it is accelerating in the  $\hat{y}$  direction due to gravity). But it is natural for the man to regard the train as a reference frame. Relative to the train, it is the nickel that accelerates despite the fact that relative to an inertial frame it is the other way around.

Since we are free to pick any *inertial* frame we want, we have a couple of obvious choices for analyzing this problem: We can consider this problem from a frame fixed to the Earth or we can consider it from a frame moving, relative to the Earth, with the velocity of the train at the instant the nickel is released. Please recognize that either of these frames is inertial and so is equally valid. Let's work in the frame that has the same velocity as the train relative to the Earth at the instant the nickel is released.

As soon as the man lets go of the nickel, it begins to fall. In our inertial frame, the nickel drops straight down. (If we'd picked the Earth-fixed frame, the nickel would have the same parabolic path as the bomb in the prior assignment.) Once

the man is not holding on to the nickel, there is no force keeping the nickel in the man's reference frame. So the *man's* reference frame (the train) accelerates relative to the nickel. The nickel gets "left behind" in some sense. The train accelerates away from the nickel at  $0.1 \frac{\text{meter}}{\text{second}^2}$  in the  $\hat{x}$  direction. In the frame of the train, the nickel seems to accelerate in the opposite direction, at  $0.1 \frac{\text{meter}}{\text{second}^2}$  in the  $-\hat{x}$  direction. To this, we should add the acceleration due to gravity, which both the man and an observer in any inertial frame would agree is acting on the nickel. Thus, the total apparent acceleration (in the non-inertial frame) is  $\bar{a}_{\text{non-inertial}} = a_x \hat{x} + g \hat{y} = -0.1 \frac{\text{meter}}{\text{second}^2} \hat{x} - 9.8 \frac{\text{meter}}{\text{second}^2} \hat{y}$ .

Of course, the man sees the nickel accelerate and concludes that something has shoved it. What did the shoving? Nothing—it didn't get shoved at all. But when we observe something to accelerate we infer that a force has acted on it. If the acceleration is just due to the frame being non-inertial, this inferred force is called a "fictitious" force. This fictitious force acts along with gravity. The force which *appears* to be acting on the nickel in the non-inertial frame is just

$$\begin{aligned} \bar{F} &= m a_x \hat{x} + m g \hat{y} = .005 \text{ kg} \left( -0.1 \frac{\text{meter}}{\text{second}^2} \hat{x} - 9.8 \frac{\text{meter}}{\text{second}^2} \hat{y} \right) \\ &= -5 \times 10^{-4} \text{ N} \hat{x} - 4.9 \times 10^{-2} \text{ N} \hat{y} \end{aligned}$$

To determine how far off the vertical path the man views the nickel as traveling, we use the same technique as we did for the bomb problem: We use the height and the acceleration of gravity to determine the time it will take for the nickel to hit the ground. We'll then use this time to calculate the distance the nickel will travel horizontally. The crucial difference between the two cases is that we must include the fact that the nickel is accelerating horizontally when calculating the horizontal distance traveled—this is very much like the rocket problem in Assignment #5. Since vectors which are perpendicular to each other are independent of each other, this will have no effect on the time it takes the nickel to fall, however.

The time for the nickel to hit the ground can be found from the now-familiar relation  $h = \frac{1}{2} g t^2$  where I've just skipped a couple of steps and substituted  $g$  for the acceleration and the initial height of the nickel for the distance traveled. This

gives us  $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1.5 \text{ meters}}{9.8 \frac{\text{meters}}{\text{second}^2}}} = 0.55 \text{ s}$ . Now, we once again use our

distance-traveled relation to calculate the  $x$  distance traveled under the acceleration of the fictitious force. This gives

$$x = \frac{1}{2} a_x t^2 = -\frac{1}{2} \times 0.1 \frac{\text{meter}}{\text{second}^2} \hat{x} \times (0.55 \text{ s})^2 = -0.015 \text{ meters} \hat{x}.$$

6. A car is traveling at a constant speed of  $23.4 \frac{\text{meters}}{\text{second}}$  around a perfect circular, perfectly horizontal track. The radius of the track is 63 meters. A cup of coffee with a total mass of 300 grams sits on the dashboard of the car. The dash is perfectly flat and horizontal. What is the magnitude of the force the cup appears to experience in this non-inertial reference frame? What coefficient of friction between the dash and the cup is necessary to keep the cup from sliding off the dash?

Calculationally, this problem is a piece of cake! Conceptually, it's a bit harder. Let's walk through it. In order for *any* object to travel in anything but a straight line, it must be accelerated. This is true even if the speed remains constant. We've already dealt with the problem of figuring out the frictional force needed to keep the car on the track. Thus, the car receives a centripetal acceleration via its interaction with the track. The result of that is to turn the car, itself, into a non-inertial reference frame for anything inside of it. By Newton's first law, the cup of coffee is going to move in a straight line at a constant speed unless a net, external force acts on it. That force must be provided by *something*. The coffee cup doesn't care what the car does—it's going to move in a straight line at constant speed. Within the frame of reference of the car, this makes it appear that the coffee cup is experiencing a force, even though it's not. In order to keep the cup from accelerating in the non-inertial frame of reference, a force must be applied to it by something: The dash, a cupholder, the driver's hand—*something*.

Another way of looking at this, which is more obvious when one is looking at the problem on a sheet of paper but less obvious when one is sitting in the car, is that the cup must receive the same centripetal acceleration as the car if it is to move in the same circle and at the same speed as the car. The car gets its centripetal acceleration from friction with the ground. The cup must get its centripetal acceleration from friction with the dash.

The centripetal force the cup requires is  $F_c = \frac{mv^2}{r}$  directed *toward* the center of the track. In the non-inertial frame of the car, this is equivalent to saying that the cup experiences a fictitious force of  $F_{\text{fictitious}} = \frac{mv^2}{r}$  directed *away* from the center of the track. Plugging in numbers, this is

$$F_{\text{fictitious}} = \frac{mv^2}{r} = \frac{0.3 \text{ kg} \times \left(23.4 \frac{\text{meters}}{\text{second}}\right)^2}{63 \text{ meters}} = 2.6 \text{ Newtons}$$

directed away from the center of the track.

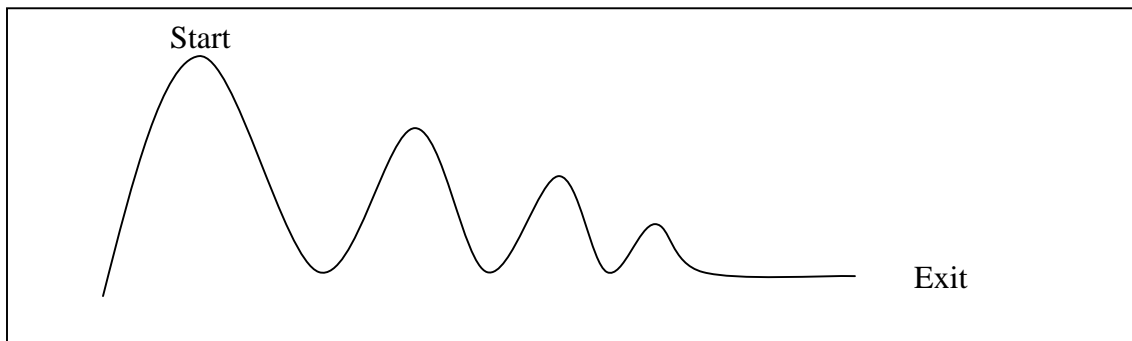
Since this force is provided by friction, we have  $F_c = \frac{mv^2}{r} = F_f = \mu_s N$

(again, we use the equality because we want the maximum force—don't get complaisant with this!). Now, since the dash is perfectly horizontal and there is no acceleration in the  $\hat{y}$  direction (draw a free-body diagram of this for reference), the normal force has the same magnitude as the weight of the cup (another warning about complacency! the normal force does not have anything to do with gravity, in general, they just wind up related in many problems). Thus, we can

write  $\frac{mv^2}{r} = \mu_s mg$ . A bit of algebra then gives

$$\mu_s = \frac{v^2}{rg} = \frac{\left(23.4 \frac{\text{meters}}{\text{second}}\right)^2}{63 \text{meters} \times 9.8 \frac{\text{meters}}{\text{second}^2}} = 0.89.$$

This is quite a large coefficient of friction! Better hold on to that cup.



7. A roller coaster is built with four “humps,” as shown. The hump from which the coaster is launched is 40 meters tall. The second hump is 30 meters tall. The third hump is 20 meters tall. The final hump is 10 meters tall. Assuming the coaster is launched with negligible initial speed, what is the speed at the top of each of the humps? What is the speed at the exit (at a height of zero)? Assume no friction.

The key to this problem is to recognize that the total energy of the system is conserved but that it changes form as the system evolves. That is, we can write at any time  $K.E. + P.E. = \text{constant}$ , or, in English: The sum of the potential energy and the kinetic energy is always equal to the same thing. Solving the problem now simply boils down to figuring out what the constant is and how it is split between kinetic and potential energies.

Let's begin with the  $K.E.$  at the top of the first hump. At the very beginning, the coaster isn't moving, so its  $K.E.$  is zero. Thus, the constant is just whatever the  $P.E.$  is. Now, we are free to pick the zero-point of  $P.E.$  to be anywhere we'd like, but, having made that choice, we must stick with it. Two choices that would make sense would be to pick the top of the first hump as the zero of  $P.E.$  or the very bottom of the coaster—at the exit. If we pick the top of the first hump, then  $P.E.$

will be negative through the rest of the problem. This isn't a big deal, as long as we keep track of the minus sign. But it's one more thing to worry about and I have enough to worry about as it is, so let's pick the bottom of the coaster.

Now, remember what potential energy is: Potential energy is energy representing work which has been done on the system but which is still available to the system. (If it becomes negative, then we can think of it as a debt which must be paid back to get back to zero. No problem: There's no interest on that kind of loan.) So if we think of the ground as  $P.E.=0$ , then some work had to be done to get the coaster up to the top of the first hump.

"But, Jack," you say, "we don't know *how* the coaster got to the top of the hump! How can we calculate the work done to get it there?" Good question! Fortunately, the answer is very simple: It doesn't matter! I *strongly* recommend that you try working another problem (maybe I'll stick it on another homework assignment): **"Calculate the work done sliding a block of mass  $m$  up a frictionless ramp which has an angle  $\theta$  with respect to the horizontal. Slide it a distance such that its height above the beginning point is  $h$ ."** If you do this, you will find that the work is *identical* to the work you would have to do to lift the block straight up to the same height. Remember: It's only the component of the force in the direction parallel to the distance traveled that matters when calculating work! So the fact that a path was taken that includes motion sideways is totally irrelevant. This can be summarized in a very important and powerful general rule (**MEMORIZE THIS!**): **If there are no dissipative forces present, the work needed to change the potential energy of an object depends *only* on the beginning point and the end point. The path taken in going from the beginning to the end does not matter.** So, we don't care how the coaster got to the top of the first ramp. Let's just calculate the work done to get it there from the ground taking the simplest route: Straight up.

The force of gravity, as we've seen before, is  $F = mg$  and it is directed straight down. In order to lift something up, therefore, we must exert a force of  $F = mg$  straight *up* on the object. Somebody, sometime, exerted this force (at least as a component of a force in some other direction). They exerted it lifting the coaster up a distance  $h$  to the top of the first hump. So the work done lifting the coaster is  $W = \vec{F} \cdot \vec{d} = mgh$ . Now, this work does not show up as kinetic energy—the coaster isn't moving just yet. But we *can* get this back—indeed, the problem is based on this assumption. So the potential energy at the top of the first hump is equal to the work done getting the coaster there and we can say  $P.E. = mgh$ .

Please, oh **please**, do not fall into the trap of memorizing this as a formula and then regurgitating it at an inappropriate time. (Inappropriate regurgitation can be messy!) This is the potential energy of a mass lifted up a height near the surface of the earth. It is *not* a universal formula for potential energy! Be careful that you understand why it is what it is and then use it only when appropriate.

So, we know the  $K.E.$  at the top (zero) and the  $P.E.$  as well ( $mgh$ ), so we know the constant:  $K.E. + P.E. = constant \Rightarrow constant = mgh$ . Now, another

trap: The height expressed in this constant is the height of the first hump. The thing which makes a constant a constant is the fact that it doesn't change (duh!). So don't go changing  $h$  to be whatever the height is at some particular instant. For clarity, let's write this as  $constant = mgh_1$ , indicating that it's the height of the first hump that matters.

Using the same argument as we used above, the potential energy at *any* height will be  $P.E. = mgh$ , but this will be different from the constant set at the beginning. Also, at any point, the kinetic energy will be  $K.E. = \frac{1}{2}mv^2$ . So, everywhere in this problem we can write  $K.E. + P.E. = constant \Rightarrow \frac{1}{2}mv^2 + mgh = mgh_1$ . Now we're ready to get some answers.

For convenience, let's cancel the  $m$  that appears in every term. Since we're asked for the speed at the top of each hump, we can write  $\frac{1}{2}v^2 = g(h_1 - h)$ . Notice that it's the *difference* in  $P.E.$  that matters (see the "Work-Energy theorem"). That's why we can pick the zero anywhere we want. It's always going to be the difference (or change) in  $P.E.$  that matters, never it's absolute value.

Solving for  $v$ , we get  $v = \sqrt{2g(h_1 - h)}$ . And from here it's just "plug and chug": At the second peak ( $h=30$  meters) the difference in height is 10 meters, so

$$v = \sqrt{2g(h_1 - h)} = \sqrt{2 \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 10 \text{ meters}} = 14 \frac{\text{meters}}{\text{second}}.$$

At the third peak the difference in height is 20 meters, so

$$v = \sqrt{2g(h_1 - h)} = \sqrt{2 \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 20 \text{ meters}} = 19.8 \frac{\text{meters}}{\text{second}}.$$

At the fourth peak the difference in height is 30 meters, so

$$v = \sqrt{2g(h_1 - h)} = \sqrt{2 \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 30 \text{ meters}} = 24.2 \frac{\text{meters}}{\text{second}}.$$

Finally, at the exit, all of the original  $P.E.$  is turned into  $K.E.$  so we have

$$v = \sqrt{2gh_1} = \sqrt{2 \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 40 \text{ meters}} = 28 \frac{\text{meters}}{\text{second}}.$$

Notice that the equation we have here is just another way of writing  $v^2 = 2ad$ , which we worked with earlier this semester, before we had the concept of energy. This is no surprise: Remember that potential energy is energy representing work done on the system that is still available to the system. When we get it *all* back, it's as though we did the work directly—accelerating the object up to a speed  $v$  with a force  $ma$  exerted over a distance  $d$ . The advantage of the new system is that we can do the energy calculation using only scalars.

**8. Refer once again to the roller coaster above. What is the gravitational potential energy on each of the humps and at the exit?**

This is just a shot at repetition (“practice makes permanent”). At each height, the gravitational potential energy (potential energy from work done against gravity) is just  $mgh$ . Since we don’t know the mass of the car, all we can do is solve this symbolically. So at the first peak it’s

$$P.E. = mgh_1 = m \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 40 \text{ meters} .$$

$$\text{At the second peak it is } P.E. = mgh_2 = m \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 30 \text{ meters} .$$

$$\text{At the third peak it is } P.E. = mgh_3 = m \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 20 \text{ meters} .$$

$$\text{At the fourth peak it is } P.E. = mgh_4 = m \times 9.8 \frac{\text{meters}}{\text{second}^2} \times 10 \text{ meters} .$$

And, finally, at the exit it is zero. All this assumes that we took the zero of  $P.E.$  to be at the ground. If we’d picked some other height, we’d have to add or subtract accordingly.

**9. A man drags a heavy box across the floor at a constant speed using a rope. The rope makes an angle  $\theta$  with the floor. The man exerts a force  $F_T$  on the rope. If he walks a distance  $d$ , how much work does he do?**

The work done by a force  $\vec{F}$  moving an object a distance  $\vec{d}$  is defined to be  $W \equiv \vec{F} \cdot \vec{d} = Fd \cos(\theta)$ . Here,  $\theta$  is the angle between the force and the direction the object travels. Thus, the work done by the man is simply  $W = F_T d \cos(\theta)$ . Sorry if that was too easy!

**10. For the previous problem, assume the box has a mass of 15 kg,  $\theta=17^\circ$ ,  $F_T=75$  N, and  $d=6$  meters. Now how much work does he do?**

$$W = F_T d \cos(\theta) = 75 \text{ N} \times 6 \text{ m} \times \cos(17^\circ) = 430 \text{ J} .$$

**11. In the previous problem, is there an angle and an  $F_T$  such that the man could do *no* work moving the box across the floor? Explain.**

Yes: If the man lifts straight up on the rope, the angle between the rope and the direction of travel is  $90^\circ$ . The cosine of  $90^\circ$  is zero, so no work is done. He would have to exert enough force to exactly cancel gravity so that the Normal force would be zero, otherwise there would be friction and he’d have exert some forward force to cancel it (hence the rope would not be at  $90^\circ$  anymore).

This seems like a trick, or at least trivial, problem. In fact, it's intended to drive home the notion that work, in the sense used in Physics, has very little to do with the amount of effort a human being puts forth to accomplish a task. Beware of falling into the trap of letting your intuition rule your solutions!

**12. Instead of dragging the box, the man in problem #9 gives the box a mighty shove such that it slides across the floor at speed  $v$ . (He doesn't touch the box after giving it the shove.) If the coefficient of sliding friction between the box and the floor is  $\mu_k$ , how far does it slide? Use energy considerations to calculate this.**

This problem is intended to illustrate the concept of work and its relation to energy: The work done by a force (it does *not* need to be a net force) is the product of that force and the distance an object travels in the direction parallel to the force. After the man shoves the box, I see three forces acting on it: Gravity pulling down, the Normal force countering gravity, pushing straight up, and friction (draw a free-body diagram!). Friction *always* acts to counter motion, so the friction force is exactly opposite in direction to the motion of the box.

Now, when work is done on something it changes the energy of that thing. In some cases, the work simply changes the distribution of energy between kinetic and potential: The work represents energy previously put into potential energy or it represents an increase in potential energy. In this case, however, the potential energy does not change. The only work being done is done by friction. And friction is a dissipative force—a force that removes energy from the system. So friction removes energy from the box. What energy does the box have? It has the kinetic energy given to it by the man! By how much does this change? It goes from  $K.E. = \frac{1}{2}mv^2$  to 0. So  $W = -\frac{1}{2}mv^2$ . (The minus sign comes in because we're looking at the *change* in  $K.E$ . It goes from something to nothing, so the change is that it gets smaller and hence is a negative value.)

That tells us the total amount of work done, now we have to relate that to the distance traveled. But the relation between work and distance traveled is a very natural one, since the distance traveled is built in to the very definition of work.  $W = \vec{F} \cdot \vec{d} = Fd \cos(\theta)$ . But in this case, the force is in the exact opposite direction to the distance traveled so  $\theta = \pi \text{ rad} = 180^\circ$  so  $\cos(\theta) = -1$ . (You really don't need to go through all the trig on this one. Just remember that when the force is opposite in direction to the distance traveled the work is negative.) All we need now is the force. But the force is that of sliding friction and we know that this is  $F = \mu_k N$ , where  $N$  is the Normal force. Since there is no acceleration up or down, the Normal force must exactly cancel gravity. So we can write  $N = mg$  (don't just automatically write " $N = mg$ "! we have a *reason* for concluding this in this case; it won't always be true and I *will* attempt to get you to fall into the trap of assuming that it is true when it isn't on the next exam) which allows us to write  $W = -\mu_k mgd$ . But, from before, we also know that  $W = -\frac{1}{2}mv^2$ , so we can

write  $-\mu_k mgd = -\frac{1}{2}mv^2$ . A teeny bit of algebra gives  $d = \frac{v^2}{2\mu_k g}$ .