PHYSICS 206a
HOMEWORK #3
SOLUTIONS

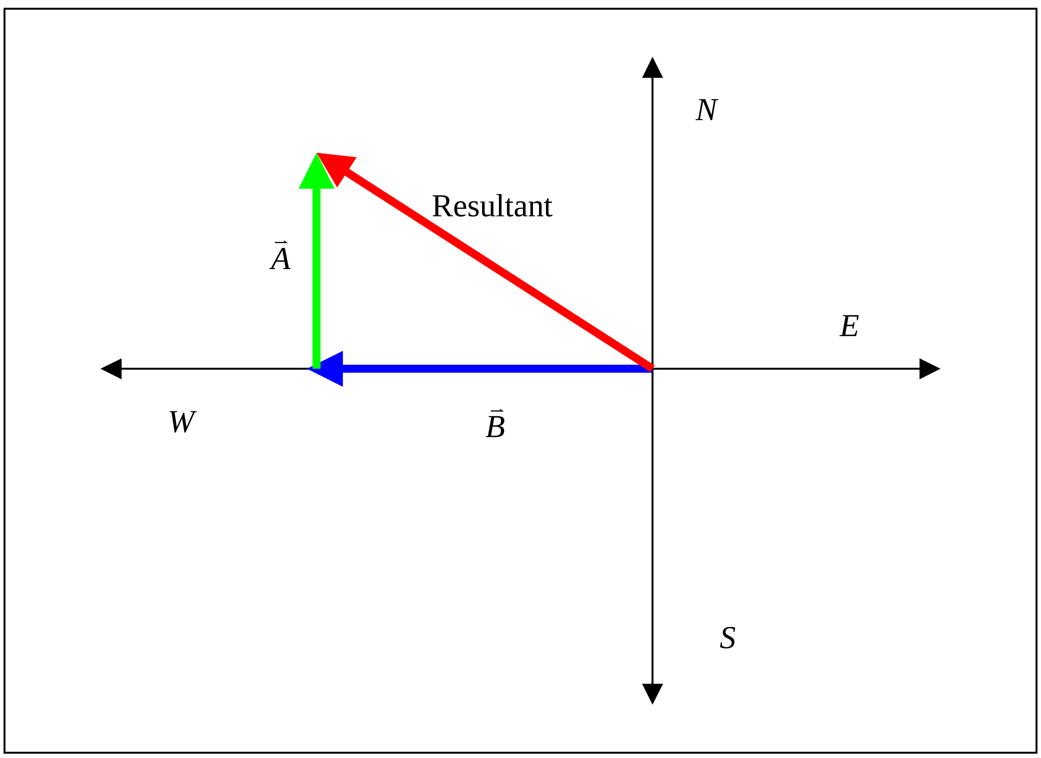
1. **Two runners, A and B depart from the same starting point. Runner A goes directly North with a speed, relative to the ground, of $5\,\frac{meters}{second}$ while runner B goes directly west with a speed, also relative to the ground, of $7\,\frac{meters}{second}$.**

   a. **What is the speed of runner A as seen by runner B?**

   b. **What is the velocity of runner A as seen by runner B?**

   c. **What is the velocity of runner B as seen by runner A?**

For part a., we realize that the *speed* of runner A as seen by runner B is absolutely identical to the *speed* of runner B as seen by runner A. A better way of stating this would be "the speed with which the two are moving apart." To find out what this is, let's use a drawing:



Now, let's use a little common sense: One runner is moving (relative to the ground) at $5\,\frac{\text{meters}}{\text{second}}$ while the other is moving at $7\,\frac{\text{meters}}{\text{second}}$. The *fastest* they can be moving apart must be $12\,\frac{\text{meters}}{\text{second}}$ while the slowest they can be moving apart must be $2\,\frac{\text{meters}}{\text{second}}$. Go no further with this solution until you understand that! I'll wait here....

O.K., from the above, we realize that the answer we're going to get *must* be somewhere between the maximum and minimum as found above. (I *strongly* recommend that you get into the habit of going through this sort of exercise automatically. Never try to solve a problem until you have some sense of where you're going. If your final answer disagrees with your estimate, you made a mistake *somewhere*—if it was in your estimation method, view it as an opportunity to rethink your intuition.) From the picture we've made, we see that the speed with which they're moving apart is just the hypotenuse of a right triangle. This has the length $v=\sqrt{\left(5\,\frac{\text{meters}}{\text{second}}\right)^2+\left(7\,\frac{\text{meters}}{\text{second}}\right)^2}=8.6\,\frac{\text{meters}}{\text{second}}$. This is the speed with which they are moving apart. Note that it *is* consistent with our estimate for the range of possible answers. This is a *scalar* quantity. To get the vector result, we'll have to do a bit more work.

We are given the velocities of the two runners relative to the ground. We must recognize, however, that the ground is not the only possible choice of frame of reference. It is a relatively common choice, but *any* non-accelerating frame is equally valid. Each runner might well be inclined to select himself as a perfectly dandy frame of reference and their would be nothing wrong with this choice (as long as he wasn't accelerating). We know how to deal with choices of reference frame that are different from the one in which we have knowledge: A coordinate in reference frame #1 is the coordinate in reference frame #2 plus the coordinate of reference frame #2, itself, in reference frame #1.

This leads, ultimately, to the "Galilean velocity transformation": The velocity of an object in frame #1 is the velocity of the object in frame #2 plus the velocity of frame #2 in frame #1. Let's see how this helps us in this case.

We begin by picking a reference frame that is "attached to" runner B. This will be reference frame #1, as described above. Now, we know the velocity of runner A relative to the ground, so the ground forms reference frame #2. All that's left is to find the velocity of reference frame #2 as seen from reference frame #1. Well, since we are given the velocity of runner B relative to the ground, the velocity of the ground relative to runner B is just the negative of this. We'll write this as $\vec{v}_{gB}$ for "velocity of ground relative to B." I'm going to go one more step: I'm going to define the unit vectors $\hat{x}$ and $\hat{y}$ to correspond with the directions "East" and "North," respectively. (For clarity, I might have chosen to call these $\hat{E}$ and $\hat{N}$ instead. But I decided to go with what you've seen before. It really doesn't matter.)

Using this, we have $\vec{v}_{gB}=7\,\frac{\text{meters}}{\text{second}}\hat{x}$. Notice that this is a *positive* quantity because the runner is running in the $-\hat{x}$ direction. Be careful! The velocity of runner A in the ground's reference frame was given in the problem as $\vec{v}_A = 5\,\frac{\text{meters}}{\text{second}}\,\hat{y}$. So, following the prescription laid out above, the velocity of runner A as seen by runner B is $\vec{v}_A = 5\,\frac{meters}{second}\,\hat{y} + 7\,\frac{meters}{second}\,\hat{x}$.

To find the velocity of runner B as seen by runner A, we follow the same procedure. However, from the frame of A, the velocity of the ground is $\vec{v}_{gA} = -5\,\frac{\text{meters}}{\text{second}}\,\hat{y}$. We have $\vec{v}_B = -5\,\frac{meters}{second}\,\hat{y} - 7\,\frac{meters}{second}\,\hat{x}$.

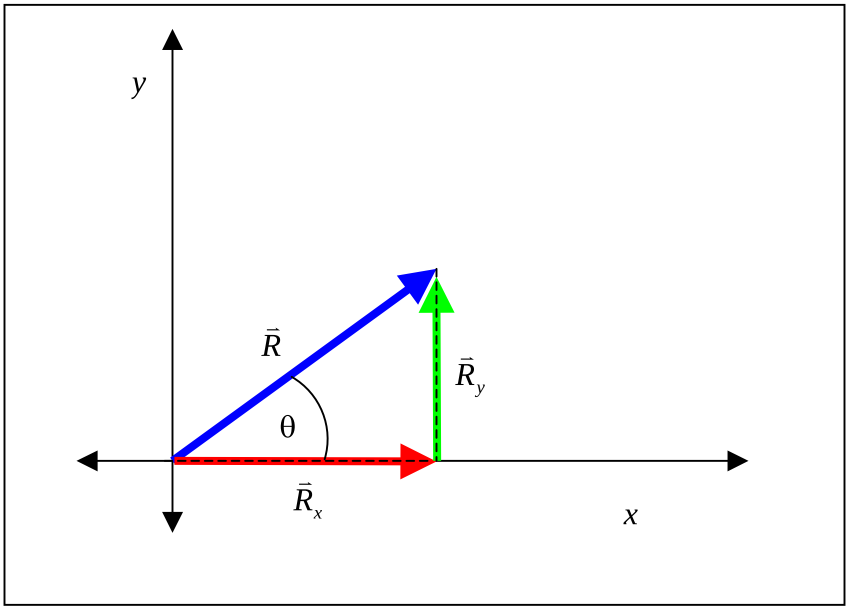
The signs in both $\vec{v}_B$ and $\vec{v}_A$ are *very* important!

2. **A vector has a length of 7.2 and is oriented 51° relative to the *x* axis of a Cartesian coordinate system. (Positive angles correspond to counterclockwise rotation.) Express this vector in terms of the unit vectors parallel to the *x* and *y* axes.**

This activity is called "vector decomposition" because we decompose the vector into its two components. Please remember both why we *can* do this and why we *want to* do this: We can do this because the "real" vector can be thought of as the resultant of an appropriate set of component vectors. In general, there are an infinite number of combinations of component vectors that could add up to give any particular vector. Once we've picked a set of coordinate axes, however, life gets very much easier if we choose a set of component vectors that are parallel to the coordinate axes. By doing this, we ensure that the component vectors are perpendicular to each other. This, in turn, ensures that they are *independent* of each other. This last gives the true power of the decomposition: Vectors which are perpendicular to each other are independent of each other! Keep repeating that to yourself until it's burned into your memory.

Once we've decomposed a vector into its perpendicular components, we can treat each component as its own problem. If our full problem exists in three dimensions, for example, by decomposing it into perpendicular components we have recast it as three separate problems. One hard problem has been broken into three easy (or, at least, easier) problems. This can make something that was intractable into something quite straightforward. After we're done with the three solutions, we just add them back together.

So, in this case, we have the following situation:



In this case, the *magnitude* (the "size" or "length" of the vector) of $\vec{R}$ is 7.2 (note that, in this case, the vector has no units; this will not be the case typically—be careful). The angle is $\theta = 51°$. We find the magnitudes of the components by using SOHCAHTOA. The x component is found from $\cos(\theta) = \frac{R_x}{R}$ (recall that when we write the vector without its little arrow we mean the magnitude of that vector), which gives $R_x = R\cos(\theta) = 7.2 \times \cos(51°) = 4.53$. Likewise, the y component is found from $\sin(\theta) = \frac{R_y}{R}$, which gives $R_y = R\sin(\theta) = 7.2 \times \sin(51°) = 5.6$.

We're not done yet. We've now found the magnitudes of the components. In order to express the original vector, we must multiply these components by "unit vectors" in the directions of the two axes. Recall that unit vectors are vectors which carry only direction. They must be multiplied by a scalar to be used. Thus, we have $\vec{R} = R_x\hat{x} + R_y\hat{y} = 4.53\hat{x} + 5.6\hat{y}$. At this point, we are *done*! Resist the urge to go another step. This is it. We're finished. No more. STOP!

3. **Three vectors are added together. The first has a length of 2.9 and is oriented at 22°. The second vector has a length of 5.1 and is oriented at –37°. The third vector has a length of 3.5 and is oriented at 113°. Find the resultant algebraically. *After* finding the resultant algebraically, express it in terms of a size and direction. Note: All angles are relative to the *x* axis.**

I'll do this one with a minimum of explanation until the end. We perform the procedure we used in the previous problem, except we do it three times, to find the components of the three vectors given. Let's call these three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. Expressed in component form, these are:

$$
\begin{aligned}
\vec{A} &= 2.9\cos(22°)\hat{x} + 2.9\sin(22°)\hat{y} = 2.69\hat{x} + 1.09\hat{y} \\
\vec{B} &= 5.1\cos(-37°)\hat{x} + 5.1\sin(-37°)\hat{y} = 4.07\hat{x} - 3.07\hat{y} \\
\vec{C} &= 3.5\cos(113°)\hat{x} + 3.5\sin(113°)\hat{y} = -1.37\hat{x} + 3.22\hat{y}
\end{aligned}
$$

One quick statement is in order here: Please do *not* just memorize the relations for the components in terms of sines and cosines as shown above! Always draw a picture before applying SOHCAHTOA. Once you get good at it, you'll be able to make the picture in your head. But angles won't always be specified in the same way, so be careful and make sure you do things systematically, not from rote.

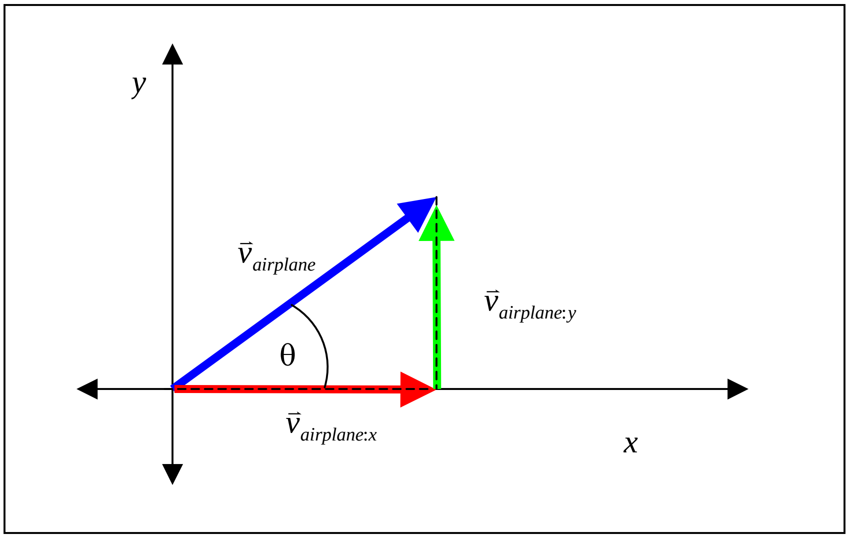
Now, we have our components of the three vectors. To find the resultant of the addition of these three, all we need to do is add the *x* components together and add the *y* components together. This gives

$$
\begin{aligned}
\vec{D} &= (2.69 + 4.07 - 1.37)\hat{x} + (1.09 - 3.07 + 3.22)\hat{y} \\
&= 5.39\hat{x} + 1.24\hat{y}
\end{aligned}
$$

Now, I won't bother making another drawing since the one I used in Problem #2 will do just dandy for us here. Take a look at that picture. Now, note that one can't tell by simply looking at the picture whether the original vector was given as a size and direction and then decomposed into components or if the components were given and used to create a size and direction. The geometry is the same. We can find the size using the theorem of Pythagoras: $D = \sqrt{5.39^2 + 1.24^2} = \sqrt{30.59} = 5.53$. To find the direction, we use the "tangent" function inverted. (Since we now know the size, we certainly also have the option of using the inverse of either the sine or cosine functions. Give these a shot on your own and confirm that you get the same answer.) Since $\tan = \dfrac{opposite}{adjacent}$, we have $\tan(\theta) = \dfrac{D_y}{D_x}$. We want the *inverse* of this, so we write $\theta = \tan^{-1}\left(\dfrac{D_y}{D_x}\right) = \tan^{-1}\left(\dfrac{1.24}{5.39}\right) = 12.96°$. This is relative to the *x* axis.

4. **An airplane is traveling at a constant velocity of $203\,\dfrac{\text{meters}}{\text{second}}$ relative to the ground, oriented 17 degrees north of east. A skydiver jumps out of the airplane and (after opening his parachute) falls at a constant velocity of $3.7\,\dfrac{\text{meters}}{\text{second}}$ straight down. What is the velocity of the airplane as seen by the skydiver? (You can treat "north" as being the $\hat{y}$ direction and "east" as being the $\hat{x}$ direction in both this and the next question, if you'd like. This would make the up/down direction $\pm\hat{z}$.)**

I threw this one in since we've gotten a little bit stuck in two dimensions. I wanted to make sure that you remembered that there is a third dimension out there. A "top-view" drawing of the velocity vector of the airplane would look just like the drawing I did in Problem #2.



Let's break this into its components, as we've done in the previous two problems. This is (I'll skip the gory details—come see me if you get a different result and can't figure out why), $\vec{v}_{airplane} = 194.1\frac{\text{meters}}{\text{second}}\hat{x} + 59.4\frac{\text{meters}}{\text{second}}\hat{y}$.

Now, here's where things will go awry for many of you: As drawn, we are looking straight *down* at the airplane's velocity vector. One mistake that I just *know* a huge fraction of you will make is that you will want to put the skydiver's velocity vector in the $\hat{y}$ direction! This is wrong. If we are looking straight down on the airplane, the skydiver's velocity will be into the page—along the direction we are looking. I can't draw that, so I'll suggest holding a pencil right at the base of the airplane's velocity vector so that it is pointed straight into the page. Pretend that the pencil is the skydiver's velocity vector. In the frame of reference of the Earth (note that the airplane's velocity vector was specified in that frame), this will be given by $\vec{v}_{skydiver} = -3.7\frac{\text{meters}}{\text{second}}\hat{z}$. (It's not the worst mistake in the world if you didn't use a negative sign on this. As long as you're consistent, it doesn't matter at this stage. Later in the course, it will be very important to make "down" the $-\hat{z}$ direction.)

Once you've gotten to this point, it's easy. We just do the same thing we did in the previous assignment: A vector in frame of reference *B* is that vector expressed in frame of reference *A* plus the vector connecting the two frames. (This is called "Galilean relativity.") We want the velocity of the airplane as seen by the skydiver—that is, the velocity of the airplane in the skydiver's frame of reference. We know the velocity of the airplane in the frame of the earth. The earth's velocity in the frame of reference of the skydiver is $\vec{v}_{earth} = 3.7\frac{\text{meters}}{\text{second}}\hat{z}$. (Note the change in sign: From the skydiver's perspective, the earth is coming up toward him—eek! Thus the sign is positive for "up." Again, it doesn't matter if you did this backwards as long as you're consistent. In this case, "consistent" means having opposites signs for the earth's and the skydiver's velocities, respectively.) Thus the velocity of the airplane as seen by the skydiver is $\vec{v}_{plane/skydiver} = 194.1\frac{\text{meters}}{\text{second}}\hat{x} + 59.4\frac{\text{meters}}{\text{second}}\hat{y} + 3.7\frac{\text{meters}}{\text{second}}\hat{z}$. It's still going to the Northeast, but it's also going up.

5. **A car travels at $26\frac{\text{meters}}{\text{second}}$ oriented 30 degrees north of east. It travels for 3 kilometers. It then turns so that it is oriented to the south. It travels at $30\frac{\text{meters}}{\text{second}}$ for 27 minutes. What is its velocity at the end of that time (assuming it keeps on going)? (You can treat "north" as being the $\hat{y}$ direction and "east" as being the $\hat{x}$ direction, if you'd like.)**

Okay, I admit it: This was something of a trick question. As a policy, I detest trick question. But there's a very, very common error made by students new to vector concepts that needs to be addressed as quickly as possible. One of the quickest ways to bring someone's attention to a problem is with a bit of pain! So here it is.

The problem begins by stating the velocity of the car and the distance that it travels with that velocity. You are then told that its velocity changes. Further, you are told how long it travels at the new velocity. What you *must* learn is that the distance traveled affects the displacement (or "position") vector, *not* the velocity vector. Likewise, the time traveled affects the distance traveled (and, hence, the position vector) and *not* the velocity vector. The velocity at the end of the trip is $30\frac{\text{meters}}{\text{second}}$ in the $-\hat{y}$ direction.

Many people have trouble distinguishing between a quantity and the *change* in that quantity. Stated like that, it seems simple. But the fact remains that many (most?) of you will have trouble with this. When a quantity changes, that change can take a variety of forms. If the quantity is a vector, the change can be in the size of the vector, its direction, or both! The change in a vector has nothing to do with the vector itself. Of course, a vector describing some attribute of a Physical system, e.g., the position or velocity, depends on the history of changes that it experienced. But the connection between the position of something and the velocities that it had in the past is not a direct one. More importantly, the history of the velocities that an object had has absolutely nothing to do with the velocity that it has at a given moment. Be careful!

I promise not to ask any more trick questions this semester. (On the other hand, I will often ask you tricky questions—that one letter makes a big difference!)

6. **Two cars are traveling in the $\hat{y}$ direction. At some instant in time (let's call it t=0), one is 123 meters behind the other. The one in front is traveling at $23\,\frac{\text{meters}}{\text{second}}$ while the one in the rear is traveling at $26\,\frac{\text{meters}}{\text{second}}$. At what time (relative to t=0) will the one in the rear catch the one that was initially in front?**

Since this is a one-dimensional problem, we can ignore the vector features. That makes life a smidge easier. There are two ways to do this problem, one is very easy but requires a fairly sophisticated way of looking at things. The other way is very straightforward but a bit more tedious. I'll do it the tedious way first and then the elegant way.

One of the key skills that is necessary for doing Physics is to be able to translate a description of a situation into mathematical relationships. One useful tool in this process is to recognize important words of phrases. In this case, the most important word is "catch." Mathematically, when one car catches the other, we will have the relation $x_1 = x_2$, where x_1 is the location of one of the cars and x_2 is the location of the other car. To find the time at which $x_1 = x_2$, we must find expressions for the locations of each car as a function of time and then set those equal to each other. The time at which the two expressions are equal is the time at which one will catch the other.

Let's start with the one in front and let's call its location $x_1 = 0$ at $t = 0$. Now, here's a real trap that many people fall into: Remember that you have complete freedom to choose the origin of your coordinate system but that you can only do this *once*. Once we've decided that the origin of the coordinate system is the original location of the car in the front, we must describe all locations based on this decision. We cannot also say $x_2 = 0$ at $t = 0$, assuming a second coordinate system for the second car! That way lies disaster.

The location of the first car can now easily be stated: Since we know its speed and we know that this speed stays constant, we can say that the distance from the origin increases by 23 meters each second. Written mathematically, this is $x_1 = v_1 t = 23\,\frac{\text{meters}}{\text{second}} \times t$.

Now we do the same thing for the second car. Since it starts out 123 meters behind the first car at $t = 0$ and then travels at a different speed, we write $x_2 = x_2(0) + v_2 t = -123\text{ meters} + 26\,\frac{\text{meters}}{\text{second}} \times t$.

Setting the two equations equal to each other, we have

$$v_1 t = x_2(0) + v_2 t \quad \Rightarrow \quad t = \frac{x_2(0)}{v_1 - v_2} = \frac{-123\text{ meters}}{23\,\frac{\text{meters}}{\text{second}} - 26\,\frac{\text{meters}}{\text{second}}} = 41\text{ seconds}.$$

Now for the elegant way: We place the origin of our coordinate system *inside* the car in front. Remember that you can do this! This is very crucial: The thing which allows us to place the origin of our coordinate system anyplace we'd like implies that there's no such thing as absolute motion, only relative motion. Sure, it frequently makes sense to think of the Earth as fixed and motionless. That is convenient because everyone on Earth can then agree on the locations and motions of objects described relative to the surface of the Earth. But there is nothing requiring us to do this! We can just as easily decide that a car is motionless and that everything around it is moving. (One caveat here: We can do this for moving things, but if we do it for accelerating things, it gets very cumbersome and messy. It still can be done, but usually it's much easier to pick something that's not accelerating, even though it can still be moving. We'll talk about this next week.) So we place the origin of the coordinate system inside the car in front. Now, the equation describing its location is trivial: $x_1 = 0$ at all times.

The location of the second car is easily written down as well. $x_2 = x_2(0) + vt$. Notice that there's only one speed in this system—the relative speed of the car in the rear in the coordinate system fixed to the car in the front, so no subscript is necessary. So what is v? It's just $26\,\frac{\text{meters}}{\text{second}} - 23\,\frac{\text{meters}}{\text{second}} = 3\,\frac{\text{meters}}{\text{second}}$.

Now we can write $x_2(0) + vt = 0 \quad \Rightarrow \quad x_2(0) = -vt$.

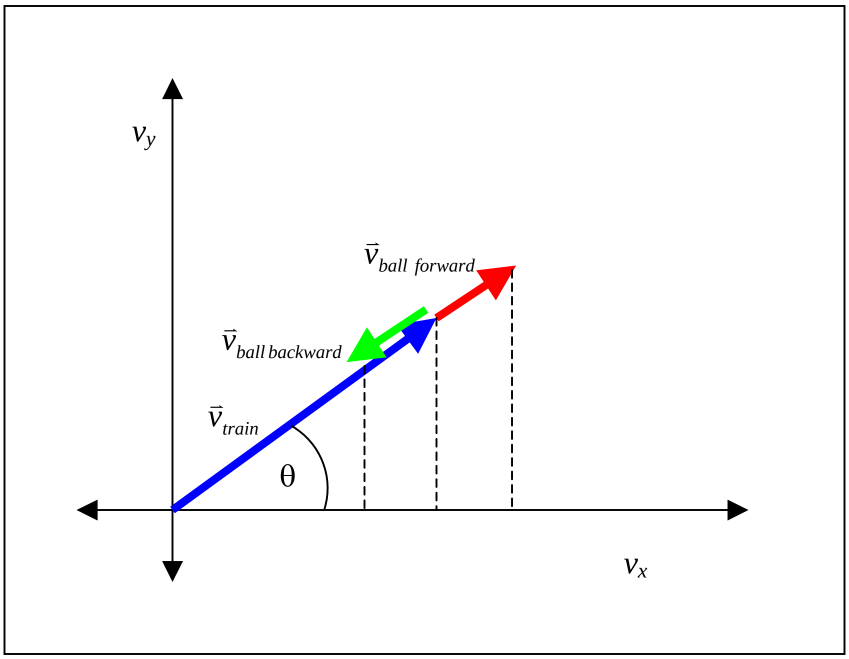
So

$$123\text{ meters} = 3\,\frac{\text{meters}}{\text{second}} \times t \quad \Rightarrow \quad t = \frac{123\text{ meters}}{3\,\frac{\text{meters}}{\text{second}}} = 41\text{ second}.$$

(By the way, the symbol "$\Rightarrow$" is read "implies" and is a strong statement of a logical relationship.)

Our two answers agree, as they must. The "brute force" solution is more reliable than the "elegant" solution. However, it takes more effort and time and, more importantly, implies a less-thorough understanding of the problem.

7. **Two men are playing catch in a train (hey, they're bored). They throw a ball back and forth with a speed of** $5\,\frac{\text{meters}}{\text{second}}$ **along the length of the train. From the perspective of an observer watching from the side of the tracks, the train has a velocity of** $18\,\frac{\text{meters}}{\text{second}}\hat{x}+18\,\frac{\text{meters}}{\text{second}}\hat{y}$. **From the perspective of the observer outside of the train, what is the velocity of the ball when it's going toward the front of the train? What is the velocity of the ball when it's going toward the back of the train? What is the speed of the ball?**



We once again have the situation in which we have a change of frames of reference. It would make sense to pick one of the axes for the train's frame as being parallel to the train, but this would make it *not* parallel to the coordinate system in which the problem was stated. There *is* a simple technique for dealing with going from one frame to another in which the axes are not parallel to those in the first frame, but that's a bit beyond us here. So we'll have to use the technique we've been using, but we'll have to be a little careful.

The train has a constant velocity, $\vec{v}_{train}$, which we are given. We are told that the ball travels along the length of the train, so we know that the components of the ball's velocity will have the same ratio as those of the train. That last sentence is a very powerful statement, so take a moment and make sure you understand what it said: For an observer who is not on the train, the train is traveling at an angle θ. Some of you noticed that this happens to be 45°, but that's not terribly important. What matters is that the angle implies that the $\hat{x}$ and $\hat{y}$ components of the velocity of the train have a ratio that is unique to that angle. (In the language of Trigonometry, that ratio is called the "tangent" of the angle.) *Anything* traveling at that angle will have the same ratio of its $\hat{x}$ and $\hat{y}$ velocity components. This is the heart of Trigonometry.

Knowing the components of the train's velocity tells us the angle at which it is traveling. In this case, we can readily find that this is 45° (see problem #2 for the technique to find this; I won't always be so nice as to give you a simple angle). As discussed above, the train is motionless in its own frame of reference, but its axis makes an angle of 45° with respect to the coordinate system that is fixed to it. Thus the ball's velocity when it is moving toward the front of the train is

$$\begin{aligned}\vec{v}_{ball\ forward} &= 5\,\frac{\text{meters}}{\text{second}}\cos(45°)\hat{x} + 5\,\frac{\text{meters}}{\text{second}}\sin(45°)\hat{y}\\ &= 3.54\,\frac{\text{meters}}{\text{second}}\hat{x} + 3.54\,\frac{\text{meters}}{\text{second}}\hat{y}\end{aligned}$$

and when it is moving toward the back of the train, it is

$$\begin{aligned}\vec{v}_{ball\ backward} &= -5\,\frac{\text{meters}}{\text{second}}\cos(45°)\hat{x} - 5\,\frac{\text{meters}}{\text{second}}\sin(45°)\hat{y}\\ &= -3.54\,\frac{\text{meters}}{\text{second}}\hat{x} - 3.54\,\frac{\text{meters}}{\text{second}}\hat{y}\end{aligned}$$

Again, these are the velocities of the ball in the frame of the train. But we want the velocities in the frame of the observer. You're now experts in this, so it's trivial at this point. The hard part was finding the velocity in the train's frame. The velocity in the observer's frame is just the velocity in the train's frame plus the velocity of the train in the observer's frame. This is $18\,\frac{\text{meters}}{\text{second}}\hat{x}+18\,\frac{\text{meters}}{\text{second}}\hat{y}$, so we have

$$\begin{aligned}\vec{v}_{ball\ forward-earth} &= \left(3.54\,\frac{\text{meters}}{\text{second}} + 18\,\frac{\text{meters}}{\text{second}}\right)\hat{x} + \left(3.54\,\frac{\text{meters}}{\text{second}} + 18\,\frac{\text{meters}}{\text{second}}\right)\hat{y}\\ &= 21.54\,\frac{\text{meters}}{\text{second}}\hat{x} + 21.54\,\frac{\text{meters}}{\text{second}}\hat{y}\end{aligned}$$

and

$$\begin{aligned}\vec{v}_{ball\ backward-earth} &= \left(-3.54\,\frac{\text{meters}}{\text{second}} + 18\,\frac{\text{meters}}{\text{second}}\right)\hat{x} + \left(-3.54\,\frac{\text{meters}}{\text{second}} + 18\,\frac{\text{meters}}{\text{second}}\right)\hat{y}\\ &= 14.46\,\frac{\text{meters}}{\text{second}}\hat{x} + 14.46\,\frac{\text{meters}}{\text{second}}\hat{y}\end{aligned}$$

8. **An object is traveling $5\,\frac{\text{meters}}{\text{second}}\,\hat{x}$. It undergoes an acceleration of $-1.2\,\frac{\text{meters}}{\text{second}^2}\,\hat{x}$. What is its velocity after 3 seconds? What is its velocity after 5 seconds?**

This is an easy one, but with some real depth at the bottom. Recall that acceleration is the rate of change of velocity. In this case, we're working in one dimension, so we can ignore the vector aspect of both acceleration and velocity. (Don't get too comfortable with this situation: We will very shortly start working with accelerations that affect both components of the velocity. Indeed, we will very soon see accelerations which *only* affect the direction of the velocity without affecting its size at all.)

Since the acceleration has an opposite sign to that of the velocity, the acceleration is in an opposite direction to the velocity. At the beginning of the process, this means that the object will slow down. Sooner or later, if the acceleration persists, the object will stop. It will then reverse direction and start speeding up. A very crucial point here and one which you absolutely ***MUST*** understand is that the acceleration can remain constant, even as the velocity becomes zero and then starts growing again. This is particularly difficult for many people to grasp when considering the instant at which the speed is zero. Just as in the problem #5 in this assignment, the change in a quantity is distinct from the quantity itself. The fact that the velocity is zero does *not* mean that the acceleration is zero. Nor does the fact that the acceleration is negative mean that the object is (necessarily) slowing down. In this case, we see that the velocity gets smaller and smaller, eventually becoming zero and then changes direction and starts getting bigger and bigger. All without the acceleration *ever* changing size or sign.

When the acceleration is constant, we can safely use the definition of the average acceleration to solve problems. Recall that this is $\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$ —in words "the acceleration is the change in velocity over the time in which the change occurs." So, to find the total change in the velocity, we simply multiply $\Delta \vec{v} = \vec{a}\Delta t$. So the *change* in the velocity after three seconds is $\Delta v = -1.2\frac{\text{meters}}{\text{second}^2} \times 3\,\text{seconds} = -3.6\frac{\text{meters}}{\text{second}}$. Once again, this is the *change* in the velocity (note that I left the arrows off since we're only in one dimension, so I could say "speed" with accuracy here). To find the velocity after this time, we add this number to the original velocity $v = 5\frac{\text{meters}}{\text{second}} - 3.6\frac{\text{meters}}{\text{second}} = 1.4\frac{\text{meters}}{\text{second}}$.

Repeating the above procedure with a time of 5 seconds, we get $\Delta v = -1.2\frac{\text{meters}}{\text{second}^2} \times 5\,\text{seconds} = -6\frac{\text{meters}}{\text{second}}$. And, adding this to the original velocity, we have $v = 5\frac{\text{meters}}{\text{second}} - 6\frac{\text{meters}}{\text{second}} = -1\frac{\text{meter}}{\text{second}}$. This would be in a direction opposite to that of the original velocity.

9. **An airplane lands with a speed of $53\,\frac{\text{meters}}{\text{second}}$. It accelerates to a stop at a constant rate. (Remember: We don't use the word "decelerate" in this class.) If its acceleration is $-1.2\,\frac{\text{meters}}{\text{second}^2}$, how far does it travel before coming to a stop?**

Once again, there are at least two different ways of doing this. As before, I'll do it the hard-but-straightforward way first. Then, I'll show you the elegant way.

We begin by using the techniques discussed in the previous problem to figure out how long it will take for the plane to stop—i.e., to reach a speed of zero. (Once again, we have a one-dimensional problem, so we don't need to use vectors.) Since the acceleration is constant, we can write $a = \frac{\Delta v}{\Delta t}$ so $\Delta t = \frac{\Delta v}{a}$. Now, what is the change in the speed? Since we're going from some speed to zero, the change is the whole thing, $-53\,\frac{\text{meters}}{\text{second}}$. Note the use of the minus sign! This is a common source of error. Since the speed gets smaller, the change is negative. Plugging this in, we have $\Delta t = \frac{\Delta v}{a} = \frac{-53\frac{\text{meters}}{\text{second}}}{-1.2\frac{\text{meters}}{\text{second}^2}} = 44.17\,\text{seconds}$. From here, it's just "plug and chug"—provided you know what to plug into! In class, we derived an expression for the final position of an object which is accelerating at a constant rate—even if it has some initial speed and/or position. The formula (which you are expected to commit to memory!) is $x = x_0 + v_0 t + \tfrac{1}{2}at^2$. In our case, the initial position is zero (we *declare* that it is zero, since we have the ability to set our origin anywhere we want, why not put it somewhere convenient?). The initial speed is $53\,\frac{\text{meters}}{\text{second}}$. So, we have

$$x = v_0 t + \tfrac{1}{2}at^2 = 53\frac{\text{meters}}{\text{second}} \times 44.17\,\text{second} + \tfrac{1}{2}\left(-1.2\frac{\text{meters}}{\text{second}^2}\right) \times \left(44.17\,\text{second}\right)^2.$$

Be very careful to use the right signs in this—the velocity is positive and the acceleration is negative! Sticking numbers in, we get $x = 1170\,\text{meters}$.

Now, for the elegant method. The technique is *very* powerful, but your brain had better be ready for it: Run the clock backwards! Imagine the plane starting from a stop but moving backwards to accelerate up to the $53\,\frac{\text{meters}}{\text{second}}$. The time is the same. We use a different origin for our coordinate system in this case, choosing it at the place where the plane "starts" (i.e., where its speed is zero). Also, our initial speed is now zero, so we can simply write $x = \frac{1}{2}at^2 = \frac{1}{2}\left(-1.2\frac{\text{meters}}{\text{second}^2}\right) \times (44.17\,\text{second})^2 = -1170\,\text{meters}$. Be careful to drop the minus sign, however: We've just calculated the distance traveled while accelerating backwards. But that's not what we were asked for.

So why did this work? Coincidence? Magic? Nope, actually we set it up to work and if we'd been smarter it would have been obvious (this is frequently the case with mathematics). Let's look at our original position formula $x = v_0 t + \frac{1}{2}at^2$. We assumed that $-v = at$. Again, why the minus sign? Because we were slowing down! Sticking this into the position formula gives

$x = v_0 t + \frac{1}{2}at^2 = (-at)t + \frac{1}{2}at^2 = -at^2 + \frac{1}{2}at^2 = -\frac{1}{2}at^2$. This actually *proves* the conjecture I used above: That the system running in reverse retraces its steps. The distance traveled in the process of accelerating to a certain speed is half the distance the object would have traveled had it moved at that final speed for the entire length of time, without accelerating. Recall that this was one key step in our derivation of the formula in the first place. So we've just come full-circle.

Try wrapping your head around this technique. It can really save a lot of work, but it can be a trap if you don't keep your wits about you!

10. **An oil tanker is traveling at $11\frac{\text{meters}}{\text{second}}$ when it notices an iceberg 2.8 kilometers in front of it. Its maximum acceleration is $.02\frac{\text{meters}}{\text{second}^2}$ (whether forward or backward). Will it strike the iceberg? If so, how fast will it be going when it strikes? If not, how far from the 'berg will it be when it stops?**

A lot of people have a lot of trouble with this problem. The source of the problem, in many cases, and the solution to it are one in the same. The most powerful problem solving strategy, in general, is, in this case, absolutely essential: The problem must be broken up into pieces. It cannot be solved in a single step, but must be solved one step at a time. Let's begin with the first question: Will the ship strike the iceberg?

As with other questions, this one is clarified by restating it in mathematical language. What we're really asking is "will the ship still have a speed that is greater than zero when its location is the same as that of the iceberg?" To answer this, we calculate how far the ship would travel, assuming it's trying as hard as possible to stop, if the 'berg weren't there. This, once again, uses the equation $x = x_0 + v_0 t + \frac{1}{2}at^2$. Again, we also have to start by figuring out how long the stopping would take. As before this is $\Delta t = \frac{\Delta v}{a}$. Sticking the appropriate numbers in, we have $\Delta t = \frac{\Delta v}{a} = \frac{-11\frac{\text{meters}}{\text{second}}}{-.02\frac{\text{meters}}{\text{second}^2}} = 550\,\text{seconds}$.

Again, we take the origin to be at the initial position of the ship. Let's go ahead and use the trick we worked out in the previous problem—we can either run the clock backwards and take $x = \frac{1}{2}at^2$ for 550 seconds or take $x = \frac{1}{2}v_0 t$ for 550 seconds. Caution: Never use a formula blindly! If you don't truly understand where either of these comes from, go ahead and use the full form $x = x_0 + v_0 t + \frac{1}{2}at^2$. This will always be correct (if the acceleration is constant). It's only a little bit more work.

I think $x = \frac{1}{2}v_0 t$ is easiest, so let's use that. Plugging in numbers, we have $x = \frac{1}{2}v_0 t = \frac{1}{2}\times 11\frac{\text{meters}}{\text{second}}\times 550\,\text{seconds} = 3025\,\text{meters}$. Since the iceberg is 2800 meters from the ship to begin with, the ship will clearly strike it before coming to a stop.

So now we need to know how fast the ship will be going when it strikes the iceberg. This is where most people lose it. You must gain some facility for working equations in both directions: Just because a formula is set up to give you, say, "x" as a function of "t," you shouldn't lose sight of the fact that what it's really doing is giving you an *overall* relationship between "x" and "t." You can (almost) always flip things around! In this case, we know the distance to the iceberg. We need to find the time it will take to get there in order to calculate the speed it has when it hits. (There *is* another method. I'll show you this at the end.) So let's use our full formula—we can't use the shortened form in this case since that's only useful when the object comes to a stop!

We have $x = x_0 + v_0 t + \frac{1}{2}at^2$. Let's put our origin at the location where the ship starts to slow down. Let's set $x = 2800\,\text{meters}$ and subtract this distance from both sides of the "=" sign. This gives us $\frac{1}{2}at^2 + v_0 t - 2800\,\text{meters} = 0$. I hope you will recognize this as a quadratic equation where *t* is the variable to be solved for. For this, we us the "quadratic formula." If your memory of this has dimmed, you *must* review the topic. We will have many occasions to use the formula this semester and I expect you to know both what it is and how to use it. (I'll be happy to give you a little review during office hours.)

Solving for *t* using the quadratic formula, we have $t = \frac{-v_0 \pm \sqrt{v_0^2 + 2a\times 2800\,\text{meters}}}{a}$. (Note the sign: Since the 2800 meters is negative, our "–4*ac*" term becomes positive.) From here, we just plug in numbers to get $t = \dfrac{-11\,\dfrac{\text{meters}}{\text{second}} \pm \sqrt{121\,\dfrac{\text{meters}^2}{\text{second}^2} - .04\,\dfrac{\text{meters}}{\text{second}^2} \times 2800\text{ meters}}}{-.02\,\dfrac{\text{meters}}{\text{second}^2}}$. Careful!!! Note that the acceleration is *negative*! Keep the signs straight or you'll really botch things. We're slowing down, so the acceleration has a sign opposite to that of the velocity. Stick the numbers into your calculator and you'll get $t = 550\text{ seconds} \pm 150\text{ seconds}$. Now, as is almost always the case with the quadratic formula, we have two answers. Frequently, we must pick one and throw away the other. This calls for a bit of judgment: Does one answer make sense and the other not? In this case, yes. We know from our previous analysis that the ship will strike the iceberg if it travels for 550 seconds. So our answer had better be less than 550 seconds. We pick the negative root. The ship will strike the iceberg in 400 seconds. (What does the other root represent? Think about it. This is the time at which it would again be at the location of the iceberg [the equation doesn't know about the horrible sinking of the ship] after coming to a stop naturally and turning around.)

Almost done! All that's left is to find the speed of the ship when it's been slowing down for 400 seconds. From our definition of acceleration, we know that $\Delta v = a\Delta t$, so we can immediately solve for the change in speed $\Delta v = a\Delta t = -.02\,\dfrac{\text{meters}}{\text{second}^2} \times 400\text{ seconds} = -8\,\dfrac{\text{meters}}{\text{second}}$. Since the ship's initial speed is $11\,\dfrac{\text{meters}}{\text{second}}$, its final speed, when it strikes the iceberg, is $3\,\dfrac{\text{meters}}{\text{second}}$.

Now, there's an easier way to do this. Let's call the speed of the ship at the moment it sees the iceberg V_1. This, of course, is $11\,\dfrac{\text{meters}}{\text{second}}$ but let's just call it V_1 for generality. Let's call the speed of the ship when it hits the iceberg V_2. Now, still taking the original location of the ship to be at the origin ($x_0 = 0$), we use our formula for the location of an accelerating object $d = \frac{1}{2}at^2 + V_1 t$. But, since the acceleration is constant, we know that the change in speed is the acceleration times the time—that's the *definition* of the change in speed, after all. So $V_2 - V_1 = at$. We solve this for the time $t = \dfrac{V_2 - V_1}{a}$. Now, for convenience, we rewrite our location expression by noting that there's a common factor of t that can be factored out. $d = \frac{1}{2}at^2 + V_1 t = \left[\frac{1}{2}at + V_1\right]t$. Since we've come up with an expression for t that uses the speeds and acceleration, we substitute this everywhere there is a t. $d = \left[\frac{1}{2}at + V_1\right]t = \left[\frac{1}{2}a\left(\dfrac{V_2 - V_1}{a}\right) + V_1\right]\left(\dfrac{V_2 - V_1}{a}\right)$. Now, cancel the acceleration inside the bracket and combine terms.

$$d = \left[\tfrac{1}{2}a\left(\frac{V_2 - V_1}{a}\right) + V_1\right]\left(\frac{V_2 - V_1}{a}\right) = \left[\frac{V_2 - V_1}{2} + V_1\right]\left(\frac{V_2 - V_1}{a}\right) = \left[\frac{V_2}{2} - \frac{V_1}{2} + V_1\right]\left(\frac{V_2 - V_1}{a}\right)$$

$$= \left[\frac{V_2}{2} + \frac{V_1}{2}\right]\left(\frac{V_2 - V_1}{a}\right) = \left[\frac{V_2 + V_1}{2}\right]\left(\frac{V_2 - V_1}{a}\right) = \frac{V_2^2 - V_1^2}{2a}$$

So, finally, we can write $V_2^2 - V_1^2 = 2ad$. (This is a useful relation to add to your toolbox.) We can use this to find the speed of impact directly. We take $d = 2800\ \text{meters}$ and do a tiny bit of algebra. This allows us to write $V_2^2 = V_1^2 + 2ad = \left(11\,\frac{\text{meters}}{\text{second}}\right)^2 - 2\times .02\,\frac{\text{meters}}{\text{second}^2}\times 2800\ \text{meters}$. This gives an impact speed of $3\,\frac{\text{meters}}{\text{second}}$ —the same as using our other method.

Note that we could have used this method from the beginning to do the whole problem in one step—*provided* that we'd gone through the effort to derive the necessary formula first. Using an "equation from heaven" is risky! Unless you've seen the derivation yourself, you never know what assumptions were at its core. Once you understand its innards, you can use it when appropriate. This is a good one to add to your toolbox!

11. **A police car passes a car traveling in the exact opposite direction. The police car is traveling at** $32\,\frac{\text{meters}}{\text{second}}$ **and the officer notices that the oncoming car is traveling at** $35\,\frac{\text{meters}}{\text{second}}$**. The police car begins braking at the exact instant the two cars pass each other. The police car accelerates to a stop at a rate of** $4\,\frac{\text{meters}}{\text{second}^2}$ **and then accelerates at the same rate until the police car catches the speeder. (All speeds and accelerations in this problem were given relative to the ground.)**

   a. **How long after the two cars initially pass each other will it be before the police car catches the speeder?**

   b. **When the police car catches the speeder, what will its speed be, relative to the ground?**

   c. **When the police car catches the speeder, what will its speed be relative to the speeder?**

   d. **Is the acceleration stated in this problem for the braking portion of the motion a realistic value?**

   e. **Is the acceleration stated in this problem for the "pursuit" portion of the motion a realistic value?**

Alright, let's do this one step at a time. The acceleration of the police car is a constant. Think about that for a second: I didn't say that it's constant until the car stops and is then constant again until the speeder is caught. I said that it's a constant, overall. The police car is initially moving in one direction, let's call it the $-\hat{x}$ direction. (Why did I decide to call the initial direction of the police car's velocity negative? Well, it really doesn't matter, but I know that the final velocity will be in the opposite direction and it's just cleaner, by a smidge, if I have that be a positive quantity. As with so many things in this course, it doesn't matter as long as you're consistent!) It brakes and reverses direction. When braking, its acceleration is in the positive $\hat{x}$ direction. It then speeds up to catch the speeder, again accelerating in the positive $\hat{x}$ direction. So the direction of the acceleration is always positive $\hat{x}$ and we are told that the magnitude of the acceleration is the same both when speeding up and slowing down. One approximation we'll make in this is that we will ignore the time the officer spends fiddling with the gear-shift lever going from forward to reverse. This may or may not be reasonable—we can check on that at the very end. If the time spent moving the lever, which I'd guess is roughly one second, is very small compared to the other times of interest in the problem, we can safely ignore it. So the acceleration is a constant vector for the entire problem and we can use some math that we've previously developed.

Since the acceleration is constant, we can jump right in and use our full formula for the position of an accelerating object: $x = x_0 + v_0 t + \frac{1}{2}at^2$. Here's where you see the power of this equation: We're basically done at this point except for some algebra. All the complex speeding up and slowing down is already built into the equation. It's truly remarkable! But we do need to put the right numbers into it.

Let's take the initial position to be $x_0 = 0$. This will make life easier. We are told that the initial velocity is $v_0 = -32\,\frac{\text{meters}}{\text{second}}$ (we can leave the explicit direction out since, once again, we're in a one-dimensional problem) and that $a = 4\,\frac{\text{meters}}{\text{second}^2}$. (Once again, note the signs! Get these wrong and you're pretty well doomed.) We're left with only two other symbols in the equation: Time and final position. We're trying to find the time, so we'll need to know the final position. Well, here we have the exact same situation as in problem #5 and we'll use the exacts same method: Since the police car is catching the speeder, we'll take the time at which the two are at the same position to be the time at which the speeder is caught. Thus we have $x = v_{speeder}t$. We substitute this into the equation along with the condition that $x_0 = 0$ and we get $(v_{speeder} - v_0)t = \frac{1}{2}at^2$. Oh, boy! We don't have to use the quadratic formula! Why not? well, there's a factor of *t* on both sides, which we can cancel. This gives $t = \frac{2(v_{speeder} - v_0)}{a} = \frac{2\left(35\,\frac{\text{meters}}{\text{second}} + 32\,\frac{\text{meters}}{\text{second}}\right)}{4\,\frac{\text{meters}}{\text{second}^2}} = 33.5\,\text{seconds}$. (Be sure to confirm that the units work out!) (Note that our decision to ignore the time needed to move the shift lever adds about a 3% error to our time calculation. Not a big deal, in this case, but maybe important in some other application.)

To find the speed at the time we just calculated we, again, use the fact that the acceleration is constant and use $\Delta v = at$. This gives $v_{final} - v_0 = at = 4\,\frac{\text{meters}}{\text{second}^2} \times 33.5\,\text{seconds} = 134\,\frac{\text{meters}}{\text{second}}$. Be careful at this stage: This is the *change in* the velocity. We want the final velocity. Here's another place where getting the sign wrong will really hurt. We have $v_{final} = 134\,\frac{\text{meters}}{\text{second}} + v_0 = 134\,\frac{\text{meters}}{\text{second}} - 32\,\frac{\text{meters}}{\text{second}} = 102\,\frac{\text{meters}}{\text{second}}$. (This is over 228 mph, so we've clearly exceeded the bounds of reason with this one, by the way.)

Relative to the speeder, this is simply $v_{relative} = v_{final} - v_{speeder} = 102\,\frac{\text{meters}}{\text{second}} - 35\,\frac{\text{meters}}{\text{second}} = 67\,\frac{\text{meters}}{\text{second}}$.

Now, is the acceleration while slowing down and speeding up realistic? Well, let's put it into units for which you have some intuitive sense. $4\,\frac{\text{meters}}{\text{second}^2}$ corresponds to a change in speed of 9 mph ("miles per hour") per second. This would be "zero to sixty" in a little under seven seconds. I think this is reasonable for the speed-up phase of the travel. Of course a real car would not be able to *maintain* that acceleration for more than a few seconds. After a while, a realistic car would simply not be able to accelerate at the same rate any longer—certainly long before getting to 228 mph!

Automobile braking accelerations tend to be somewhat higher than speeding-up accelerations, so this number is on the small side for that, but not by as much as you might think. I did a quick calculation with some realistic numbers (we'll do this later in the semester, after we've studied friction) and found a reasonable braking acceleration rate of about 12 mph per second. Only about 30% off from what we used.