PHYSICS 206a

HOMEWORK #1

SOLUTIONS

Note: I reiterate some of the things stated on this assignment: A full presentation of reasoning is essential to getting credit on problems in this class! I do not expect your solutions to look like mine. I want them to be demonstrations of your minds at work, however. This is also true for exams. Any answer presented as simply a number (or even an algebraic expression) with no explanation is ***wrong***! An answer which is a mere cut-and-paste from a website is worse than wrong—it is plagiarism. “Right,” in this context, means that ***you*** have reasoned correctly. I don’t need the factual, numerical answer—I already know the answers to the problems you will see in this course. You are here to learn how to reason, not how to look things up.

1. **Between the M.U.C. and the library is a roughly circular patch of grass surrounding “the Rock” (if you’re new here, ask someone, you’ll have no trouble finding it). How many blades of grass are in this patch?**

Remember that good reasoning is the name of the game. “Correctness” is quite irrelevant. There are a number of ways to do this first question reasonably. The best way I can think of to find an answer that’s close to the true number of blades of grass without actually counting all of them would be to count the number of blades in several sample areas (say, a square inch or so) taken at different points on the field and then to average the counted number of blades per square inch thus found to reduce errors caused by local variations in grass density and then to multiply by the number of square inches in the whole field to find the total number of blades of grass. This is still a time-consuming procedure, and I don’t have much time, so I’ll just guess at a density and work from there.

I would guess that in the patch there are around a hundred blades of grass in an average square inch. This is probably far from correct, but I doubt that the number is as low as ten or as high as a thousand, so I’m in the ballpark.

The next step in the problem is to find out how many square inches of grass there are in the patch. If we consider the patch to be a circle, the area is given by a formula known to most of you (also, it's in the front of your textbook): $Area = \pi \times r^2$ where π is the shorthand way of writing the number 3.14 etc. and *r* is the radius of the circle (half its diameter).

Now, I estimate the diameter of the patch to be 20 feet. So its radius is 10 feet and its area is:

$$Area = \pi \times r^2 = 3.1 \times (10\ feet)^2 = 3.1 \times 10\ feet \times 10\ feet = 3.1 \times 100\ feet^2 = 310\ feet^2$$

(Note the use of the units in this: Multiplying *feet* by *feet* gives us *feet*^{2} or "*square feet.*" In this course, all quantities must be accompanied by appropriate units. Lack of units makes the answer wrong.)

There's one more thing that needs to be done, though. Since I've guessed at the number of blades of grass *per square inch*, I need to convert the area of the patch to square inches. I know that there are 12 inches in a foot. Thus ***each*** of the measurements must be multiplied by 12. This is a very common source of error: A square foot is ***NOT*** 12 square inches! It is 12×12 square inches. $12 \times 12 = 144$, so a square foot is 144 square inches. (I intentionally did not use the metric system in this problem just to give some practice to this cumbersome unit conversion.) From this, I find the area of the patch to be:

$$Area = 310\ feet^2 = 310\ feet^2 \times 144 \frac{inches^2}{feet^2} = 44640\,inches^2 = 4.46 \times 10^4\,inches^2$$

Finally, we get the number of blades of grass by multiplying:

$$Total\ number = \frac{Number}{Area} \times Total\ area = \frac{100\,Blades}{inch^2} \times 4.46 \times 10^4\ inches^2 = 4.46 \times 10^6\ Blades$$

That’s more than four *million* blades of grass!

Now all of this was an exercise in estimation. We mustn't take the final answer *too* seriously. This is an example of what computer programmers call **GIGO**: **G**arbage **I**n-**G**arbage **O**ut. My guess about the number of blades of grass in a square inch isn’t very good. Neither is my guess about the size of the field. But it's a pretty good start if I need this number for some reason. Certainly it didn't take anywhere near as much effort as it would have if we wanted a truly reliable number. The effort put into a piece of research should not be independent of the expected return from that research!

2. **Assuming that human beings have inhabited the earth for about 1 million (i.e. 10^6) years, estimate the number of generations that have lived since the first humans.**

This one’s a lot easier than the last one. One “trick” to help get on the right path is to understand that Math is a language. One can never truly translate something written in one language into another but there are frequently points of contact. Some mathematical expressions “translate” into English. I’ll try to point these out as the semester progresses. One important case is the word “per.” “Per” means “divided by.” I use this in the following:

Assuming twenty years per generation, all we need to do is divide years by years per generation:

$$Generations = \frac{Total\ Years}{\frac{Years}{Generation}} = \frac{1\times10^6\ Years}{20\frac{Years}{Generation}} = 5\times10^4\ Generations$$

That’s fifty thousand generations of human beings.

3. **How many piano tuners are there in New York City?**

This is one of the most famous Fermi problems. If you search on the internet (and too many of you probably already did this while working on this assignment), you will find numerous web sites which discuss it. Please use your imagination to come up with alternate methods!

I'm going to use the technique suggested by a student who took this class from me when I lived in Albuquerque to do this problem. New York City has about ten million people. Albuquerque has about half a million (500,000) people. I'll assume that about the same percentage of people have pianos in Albuquerque as in New York (this is probably not true, but it gets us into the ballpark) and that they all need to be tuned with about the same frequency. I'll also assume that Piano tuners in both places tune the same number of pianos per day. What all this means is that both places need the same ratio of piano tuners to population and so we can simply scale piano tuners (forgive the pun) the same way the population scales:

$$\frac{\textit{Population of New York}}{\textit{Population of Albuquerque}} = \frac{\textit{Piano Tuners in New York}}{\textit{Piano Tuners in Albuquerque}} = \frac{1\times10^7}{5\times10^5} = 20.$$

So there are (about) twenty times as many *piano tuners* in New York as in Albuquerque because there are (about) twenty times as many *people* in New York as in Albuquerque and people in Albuquerque need (about) as many piano tuners per person as people in New York do. A quick check of my Albuquerque Yellow Pages©®™, yields (about) twenty piano tuners. So New York City has (about) $20\times20=400$ piano tuners. (There are *lots* of other ways to approach this problem! Someday, I'll check a St. Louis Yellow Pages and rework the numbers. [This is made difficult by the ambiguity in the definition of "St. Louis"—there it's not clear what area is actually represented by a given phone directory.] The technique is what's important, though.)

4. **An atom is approximately 10^{-8} cm in diameter.**
   a. **What is its approximate volume?**
   b. **How many would fit in a box of volume 1 cm^3 (that is, "1 cubic centimeter")?**
   c. **What is your volume in cm^3?**
   d. **How many atoms can fit in you?**

Since the size of the atom was given as a diameter, the implication was that you approximate the atom as a sphere. The volume of a sphere is (this could have been found from a variety of sources, such as inside the front cover of your textbook, for those of you who do not know it offhand): $V = \frac{4}{3}\pi R^3$, where π is the constant 3.14 etc., and *R* is the radius of the sphere. Be careful when you are given a diameter and need a radius. You have to divide by 2 before cubing or squaring (raising to the third or second power), otherwise you get a big error. Dividing 10^{-8} cm by 2 gives $5\text{x}10^{-9}$ cm. So the volume is:

$$V = \frac{4}{3} \times 3.14 \times \left(5 \times 10^{-9}\ cm\right)^3 = \frac{4}{3} \times 3.14 \times \left(125 \times 10^{-27}\ cm^3\right) \approx 4 \times 125 \times 10^{-27}\ cm^3 = 5 \times 10^{-25}\ cm^3$$

Note that the $\approx$ symbol means "approximately" and I used it because I approximated π by 3.00 (remember **GIGO**).

Now we know the approximate volume of the spherical atom and we need to know how many would fit into a 1 cm^3 box. This question is very simple, but it really does a good job distinguishing people who have been trained to think mathematically from those who haven't. The first thing to do is to assume that the atoms completely fill the box. A quick look at a jar of marbles will show you that spheres can't be packed together in such a way that they fill all the available volume. There's always space between the spheres. Still, we can approximate them as filling the available volume for a first shot (how thoroughly they actually fill a given volume is characterized by a thing known as a "packing fraction" which gets into some pretty hairy mathematics). If I have a volume to fill and I want to know how many objects of a lesser volume will fill it, all I need to do is divide the large volume into chunks the size of the smaller volumes:

$$\textit{Number of Small Things} = \frac{\textit{Volume of Large Thing}}{\textit{Volume of Small Things}}.$$

In this case, the large thing is a box with a 1 cm^3 volume and the small things are the atoms with a $5\times10^{-25}\ cm^3$ volume. So: $\textit{Number of Atoms} = \frac{1cm^3}{5\times10^{-25}\ cm^3} = \frac{1}{5}\times10^{25} = 2\times10^{24}$ atoms will fit in the box.

We now replace the $1cm^3$ box with one with the same volume as you. So we're going to have to figure out the volume of a human being. The key step in figuring out your volume is to come up with an appropriate approximation. Approximating the shape of your body by some relatively simple geometric shape allows a simple formula to be used, although I can think of several, equally valid methods. I'll do this by approximating myself as a cylinder six feet tall and one foot in diameter. Other reasonable approaches would be to approximate oneself as a cube, a sphere, or a parallelepiped (a brick). One could also break one's body into sections and approximate each one separately: the head as a sphere, the fingers as cylinders, etc. That approach would be overkill for this problem! Anyway, the volume of a cylinder is $V_{cylinder} = \pi R^2 h$ where h is the height of the cylinder and R is its radius. In my case this becomes (remember to change my height and diameter to centimeters [there are 2.54 centimeters in one inch] and divide the diameter by 2):

$$V_{Jack} = \pi\times\left(6\ inches\times2.54\ \frac{cm}{inch}\right)^2\times72\ inches\times2.54\ \frac{cm}{inch}$$

$$\approx 3\times(15\ cm)^2\times183\ cm = 3\times225cm^2\times183cm \approx 1.2\times10^5\ cm^3$$

Now all that remains to be done is to multiply this answer by the answer found in part (b) to get the total number of atoms:

$$\textit{Number of Atoms in Jack} = 2\times10^{24}\ \frac{atoms}{cm^3}\times1.2\times10^5\ cm^3 = 2.4\times10^{29}\ atoms.$$

5. **How many kernels are on an ear of corn?**

I think an ear of corn is best modeled as a cylinder. An ear of corn is about two inches in diameter and is about 12 inches long. Now, the kernels occur in rings around the ear. With a diameter of two inches, the circumference will be six inches (this is obtained by using the relation $c = 2\pi r = 2 \times 3 \times 1 inch = 6 inches$; it would be absolutely ridiculous to say that the circumference is $6.28 inches$ based on the approximation that $\pi = 3.14$—our estimate of the diameter of the ear isn't precise to within 5% so there is no point in using a 5% precision in our estimate of pi either). Now, kernels of corn are oblong. I'll estimate that a kernel is about ½ inch in the direction around the circumference of the ear and about half that in the direction along the length of the ear. So we will have $\frac{6 inches}{\frac{1}{2}\frac{inch}{kernel}} = 12 kernels$ per ring. Since each ring is one kernel wide, each ring will be about ¼ inch wide. So, taking the length of the ear to be 12 inches, there will be $\frac{12\ inches}{\frac{1}{4}\frac{inch}{ring}} = 48\ rings$ on each ear. Multiplying these two, we have $\frac{kernels}{ear} = 48\frac{rings}{ear} \times 12\frac{kernels}{ring} \approx 600\ \frac{kernels}{ear}$. (Note that I used a round value of 600 rather than the value of 576 that my calculator gave me when I multiplied 12×48. Why? Well, our estimate has so many guesses in it that we're only going to be correct to within 50% or so, at best. Frankly, I should have just kept it to an order of magnitude estimate and said 1000! Keeping that "576" leads one to believe that the value is 576, not 577 or 575. Ridiculous! These are known as "drama digits"—digits kept on a number which lead someone to infer that the number has a greater precision than it actually does. This criticism certainly applies to some of the answers I presented in some of the problems in this set. I'm trying to be gentle in my introduction of principles here!)

In order to check my estimate, I went to the web site maintained by the Iowa Corn Grower's Association, http://www.iowacorn.org/cornuse/cornuse_20.html
According to these folks, who have taken rigorous data rather than using a pure estimation method, there are

between 500 and 1200 kernels on an ear of corn with 800 being a typical value. Note that these folks have a *need* for a precise value, so it was worth the time and effort (and cost!) for them to obtain such a value. We don't have that same need, so our estimation, which agrees delightfully with their numbers, is completely justified. Don't put more effort into a calculation than your application requires!

6. **An atom weighs approximately 10^{-27} kg.**
   a. **How many atoms are there in you?**
   b. **Does this number agree with your answer to problem 4, part (d)? If there is a significant difference between the answers, can you think of a reason for the discrepancy?**

Once again, we need to divide. Being on the heavy side makes the arithmetic in this problem easier: I mass about 100 kg (by the way, how many of you caught the error in the phrasing of the question? of course an atom does not "weigh" 10^{-27} kg since kilograms are units of mass not weight; I did this so as not to confuse you since we haven't had a chance to talk about the difference between weight and mass—sometimes clarity must win over correctness). Using this, I get:

$$Number\ of\ \ Atoms\ in\ Jack = \frac{Mass\ of\ \ Jack}{Mass\ of\ \ One\ Atom} = \frac{100\,kg}{10^{-27}\ kg} = 10^{29}\ atoms\ .$$

Remember, though, that atoms come in a range of masses going from one to around three hundred (what unit is used for this is a topic that we'll discuss later in the course; as a preview: it's called an Atomic Mass Unit [ah, these physicists are so clever with their names!], or AMU for short) while their diameters vary only a little (if that doesn't sound strange to you, think about it a bit more). The mass given is really on the very low end. A factor of ten higher would have been a bit better.

7. **How many dogs are there in the United States? Based on this, estimate:**
    a. **How many 1 lb. cans of dog food are sold each year?**
    b. **How much money is spent on dog food each year?**

There are roughly 100 million (10^8) households in the US. I guess that the ratio of dogs to households is about one in four. This gives twenty five million dogs ($2.5x10^7$). This may not be terribly reasonable, but I think that only about one tenth of the households that I know have dogs at all, but many of those that do have a dog have more than one. Anyway, I think a typical dog eats a pound of dog food a day, but probably only about 20% of the households with dogs feed them canned food (the people I know tend to be from a poorer segment of the population, so I may be skewed in this). 20% of $2.5x10^7$ is five million. This is the number of cans of dog food used per day. There are 365 days in a year, so multiply five million by 365 to get 1.8 billion ($1.8x10^9$) cans of dog food.

I don't live with a dog, but based on the price of cat food in cans, I'd guess that dog food costs about a dollar per can. Thus the American public spends 1.8 billion dollars per year on dog food.

Is this right? Certainly not. But how wrong is it? Maybe I'm off by a factor of 10—maybe the canned dog food industry does 18 billion dollars in business per year, but I kinda doubt it. If I needed a very precise answer, I could put more effort into getting it. But if all I need is something to get me going, a rough estimate like this is good enough. This sort of first step is essential to Science!

8. **How many miles does an average person walk in a lifetime?**

I can walk about two miles in a day without really noticing that I've done a lot of walking. When I walk as much as five miles in a day, I really know it. Using myself as typical (I'm not, but I'm all I've got and I'm not so far off the mark that I need to worry about it; if I worked for the Post Office as a letter carrier, I think using myself as a standard in this problem would be inappropriate) and realizing that I do a lot of walking that's spread out over the course of a day, I'll estimate that I walk about three miles per day. Assuming that I'll live the three score and ten (70) years promised me by the Bible, I will live 365x70=25550 days. Multiplying this by three miles per day, I estimate that I will walk 76650 miles in my lifetime. I'm glad I bought new shoes recently!

9. **Compute the following *without* using a calculator:**

- **a.** $10^2 \times 10^3$
- **b.** $10^2 \div 10^3$
- **c.** $10^2 + 10^3$
- **d.** $\dfrac{10^{35} \times 10^{80}}{10^{12} \times 10^{-3}}$
- **e.** $\dfrac{\left(6 \times 10^{23}\right) \times \left(1 \times 10^{-19}\right)}{2 \times 10^3}$

Remember, when multiplying powers of a number, add the exponent; when dividing, subtract. When we add *different* powers of a number there is nothing that can be done but write the whole thing out if possible. So the answers are:

a) $10^2 \times 10^3 = 10^{2+3} = 10^5$

b) $10^2 \div 10^3 = 10^{(2-3)} = 10^{-1} = 0.1 = \dfrac{1}{10}$

c) $10^2 + 10^3 = 100 + 1000 = 1100 = 1.1 \times 10^3$

This one deserves a bit of comment: In this case, we could easily have bent the rules on scientific notation a little and rewritten the sum as $10^2 + 10^3 = 1 \times 10^2 + 10 \times 10^2 = 11 \times 10^2$ or some other variation of that. This would have allowed us to keep the convenience of scientific notation while adding numbers. The requirement that numbers in scientific notation be written as a factor that is greater than zero and less than ten is merely a convenience. This is a tool which you should feel free to modify to your convenience.

This result shows us another powerful feature of scientific notation as well: Compare the exponents on the two numbers in the sums. One is a 2 and the other is a 3. They are very close to each other, so the numbers being added are of a very similar order of magnitude. Now, imagine that I'd asked you to perform the sum $x = 10^9 + 10^3$. Yes, you could do it. But there's something else that you should get out of this. The exponents are *very* different. we are adding two numbers that are different by six orders of magnitude. You would be quite justified in writing $x = 10^9 + 10^3 \approx 10^9$ in this case—just ignore the 10^3. In fact, since *everything* that we do in Physics is an approximation, the "wiggly equals" sign ("$\approx$" which is read "is approximately equal to") is almost redundant. The crucial thing is to realize *that* you have made an approximation and to remember *what* the approximation was. Sometimes, an approximation can come back to haunt you—you realize, down the road, that the approximation wasn't a good idea and you have to go back and fix it. In that case, knowing what has been approximated away is essential. Don't be afraid to approximate in this course (indeed, failure to use estimation and approximation will adversely impact your performance in this course), but always be aware when you are doing it and be prepared to justify it!

d) $$\frac{10^{35}\times 10^{80}}{10^{12}\times 10^{-3}} = \frac{10^{(35+80)}}{10^{(12-3)}} = \frac{10^{115}}{10^{9}} = 10^{(115-9)} = 10^{106}$$

e) $$\frac{\left(6\times 10^{23}\right)\times\left(1\times 10^{-19}\right)}{2\times 10^{3}} = \frac{6\times 1}{2}\times\frac{10^{23}\times 10^{-19}}{10^{3}} = 3\times\frac{10^{(23-19)}}{10^{3}}$$
$$= 3\times\frac{10^{4}}{10^{3}} = 3\times 10^{(4-3)} = 3\times 10^{1} = 3\times 10 = 30$$

9. a) The number googol is defined as 10^{100}. If a sheet of paper can hold 100 lines of 100 characters each, how many lines are needed to write down 1 googol?

b) 1 googolplex is defined as ten to the power googol (i.e. 10^{googol}). How many sheets of paper would be needed to write down 1 googolplex without using scientific notation?

This one is just for fun. There is no physical quantity that is so large that one needs even a single googol to express it. (Actually, I can think of one—but only one.) In fact, about the largest number I've ever seen anyone deal with in real life is around 10^{70} (except for some mathematicians for whom a googol is unbelievably small).

a) 1 googol is a one followed by 100 zeroes so one line of one sheet of paper will suffice for writing down the zeroes and the first space of the next line will be needed because of that darned 1.

b) There are 1 googol of zeros in a googolplex, plus a one in front which we will ignore. (Think about it: 10^4 is a one with 4 zeros. 10^{10} is a one with 10 zeros. By extension, 10^{whatever} is a 1 with *whatever* zeros after it.) Each page can hold 100 rows of 100 characters each. Multiplying rows times the number of characters in each row, we have 10,000 characters or 10^4 characters per sheet of paper. To see how to proceed, let's think of a situation that's a *little* bit more reasonable.

A googolplex has a googol of zeros. Let's imagine a number that has a lot of a zeros but not quite so many as googolplex does. Let's imagine a number that has 50,000 zeros. In other words, $10^{50,000}$. If we wanted to figure out how many pages it would take to write this down, it would be rather straightforward: We have 50,000 zeros, total, and we can write 10,000 of them on each page. So it would take five pages.

To solve this problem we used a procedure that we understood implicitly. A useful problem-solving trick is to take a procedure that we understand implicitly and then to figure out, explicitly, what we did so that we can generalize it. In this case, we figured out that the total number of pages is the total number of *zeros* divided by the number of zeros per page. Let's call the total number of zeros x and the total number of pages n. We can say that $n = \frac{x}{10,000} = \frac{x}{10^4}$. You can easily check that this works in the example of $x = 50000$ that we used. Now, let's just substitute googol (since there are googol zeros in a googolplex) where we had 50000 before. This gives $n = \frac{x}{10^4} = \frac{10^{100}}{10^4} = 10^{96}$ sheets of paper. Not only is this more sheets of paper than we could ever get, if we tried to write this down the universe would literally not be big enough to hold all the paper necessary even if we could find it!.

This problem was intended to give you an idea of the power of scientific notation: We can quite easily write down and work with numbers that are so large that we'd have no hope of dealing with them if we didn't use this system. It's worth thinking about.