PHYSICS 206a
HOMEWORK #14
SOLUTIONS

As of the last class of the semester, you are able to do problems #3-14. Although we briefly touched on problem #15 in class, we did not spend enough time on it for you to understand it fully. Likewise, although you can do problems #11-13, I do not consider them “fair game” for the Final Exam and will not test you on them this semester. This leaves problems #3-10 and 14 which you should understand for the Finale. (I will leave off solutions to problems #1, 2, and 15 so we can do them next semester.)

1. Deferred to next semester.

2. Deferred to next semester.

3. **By hand (i.e., don’t use Excel or some other graphing program), sketch the following functions. Take the frequency to be $\omega = \pi \frac{1}{s}$ (that can be read “pi radians per second” or “180 degrees per second”). Make sure your plot extends over at least 4 seconds. Use whatever scale is convenient for the *y* axis.:**

   a. $y = 5 \times \sin(\omega t)$

   b. $y = 5 \times \sin(\omega t + \frac{\pi}{2})$

   c. $y = 5 \times \cos(\omega t)$

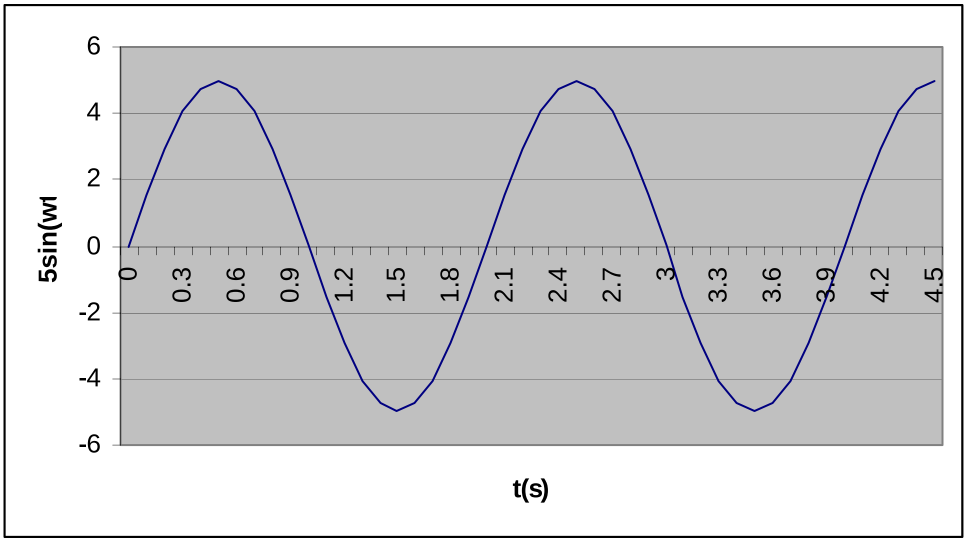
   d. $y = 5 \times \cos(\omega t - \frac{\pi}{2})$

   The most common question I have been asked regarding this one basically had to do with how to handle the ω in the argument of the sine or cosine function. The answer is so simple that I think many people just overlook it—they outsmart themselves. The procedure to use in all of these is this: We know the value of sine and cosine as it relates to the *entire* quantity between the parentheses (this is known as the “argument” of the function). In order to figure out the functional dependence on the time, *t*, we work backwards from this and figure out the values of *t* that will make the argument equal to some quantity for which we know the value of the sine or cosine.

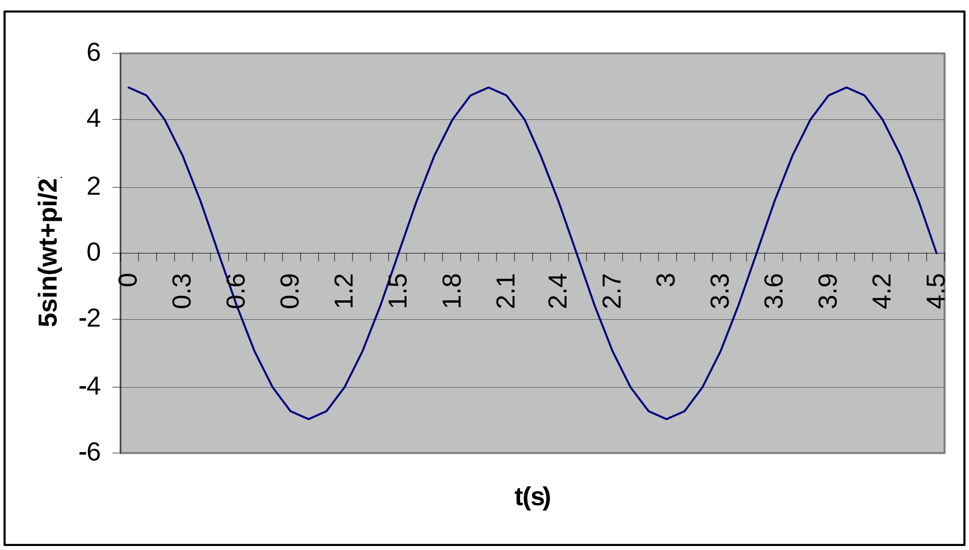
An example will clarify this. Let’s look at (a): $y = 5 \times \sin(\omega t)$ where $\omega = \pi \frac{1}{s}$. Now, I *know* that $\sin(\theta) = 1$ when $\theta = \frac{\pi}{2}$. In this case $\theta = \pi \times t$, so $\theta = \frac{\pi}{2}$ when $t = \frac{1}{2} s$. Thus $y = 1$ when $t = \frac{1}{2} s$. A similar approach can be used for other points.

I recommend starting with the t intercepts. I.e., the locations at which $y = 0$. These will occur whenever the argument is an integer multiple of π. Locate a few of these and the rest of the plot follows very easily.

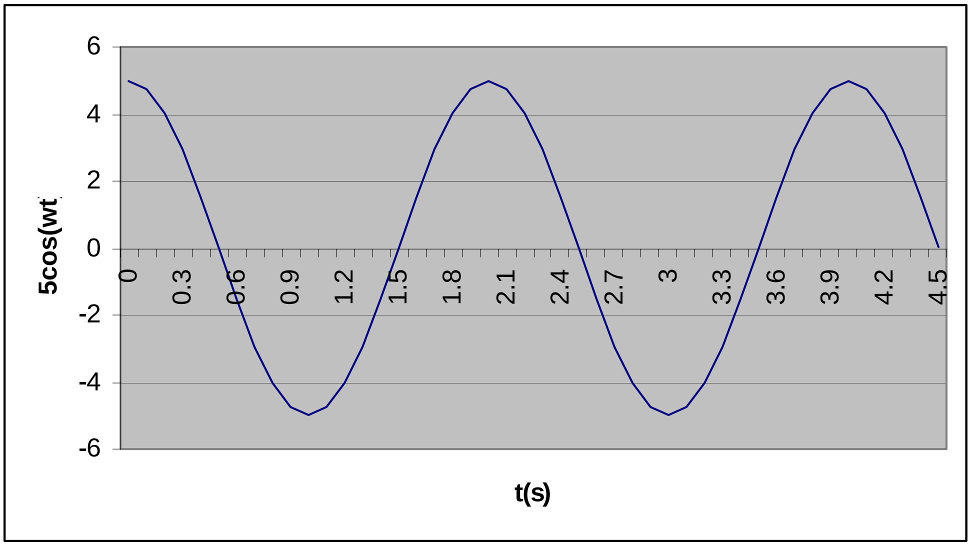
Of course, we’re not here just to plot things mechanically. We want to understand how a real event which is described by a particular function behaves. To get a deeper understanding of sinusoidal functions, I recommend taking a look at the unit for frequency. This is typically $\frac{1}{s}$ or, synonymously, “hertz.” Another way of reading this, however, is “per second.” We can translate $\theta = \pi \times t$ into English as “theta changes by pi radians per second.” Perhaps even more informative (but less useful for calculations) would be to rewrite this in terms of the *circular* frequency, f. Recall that $f = \frac{\omega}{2\pi}$, so using $\omega = \pi \frac{1}{s}$ we get $f = \frac{1}{2} \cdot \frac{1}{s}$. We read this in English as “the system goes through one half of a complete cycle every second.” (After a while working with things like this, seeing pi will make you automatically think “one half cycle.”) Using this, we can *very* quickly make the plot: Just draw a set of axes with t as the horizontal axis (the “abscissa”) and some appropriately-scaled vertical axis for y (the “ordinate”). Then draw a sine or cosine function that goes through one complete cycle two seconds. (a) has the appearance shown below (I know that you were told to do this by hand, but if I did that it would make it difficult to post the solutions on the web page, so I cheated and used Excel):



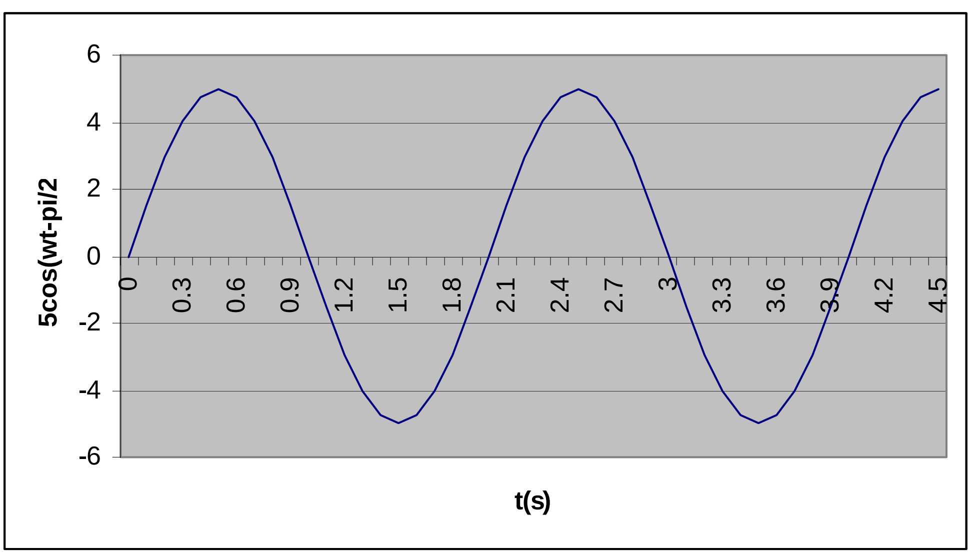
Now, for (b) we have to be a bit careful. Everything is the same as for (a) ***except*** we start the oscillations at a later point. Since π/2 is ***added*** to the argument, when t=0 (at the beginning), the thing that we're taking the sine of already is equal to π/2. This appears:



For (c), we switch functions altogether. Instead of sine, we use cosine. There’s an interesting result here that you should internalize. Here’s the plot:



Comparing this with the plot from above, we see a very important fact: $\sin(\omega t + \frac{\pi}{2}) = \cos(\omega t)$. This ***is not*** a coincidence! It comes from the definitions of the cosine and sine functions. The fact is that, for any angle θ, $\sin(\theta + \frac{\pi}{2}) = \cos(\theta)$. By extension, we can also say $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$. Thus, for (d) we have a plot identical to (a):



4. **Again by hand, plot the following functions as a function of x. Again take the frequency to be $\omega = \pi \frac{1}{s}$. Take $k = \frac{1}{3} cm^{-1}$. Make sure that your plot extends over at least 12 cm.**

a. $y = 5 \times \sin(kx - \omega t), \quad t = 0s$

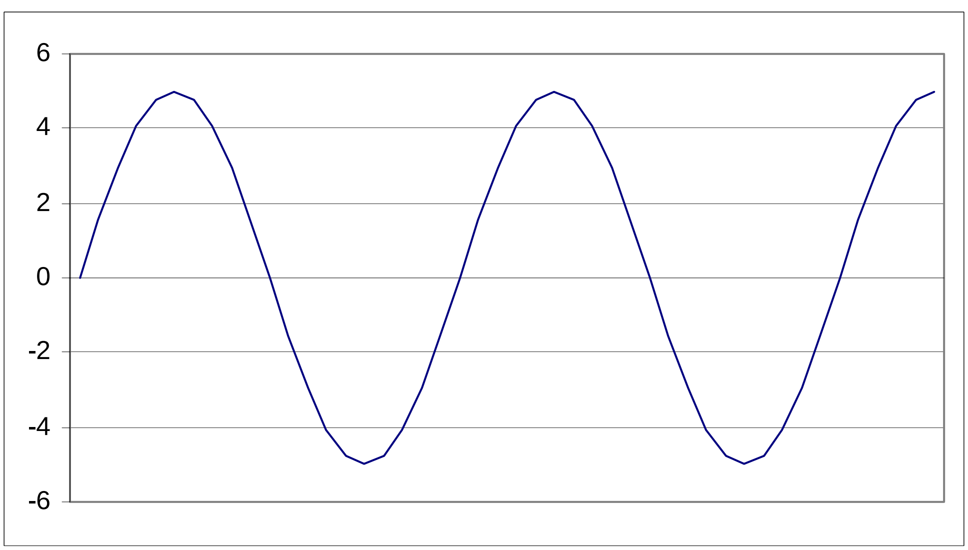
b. $y = 5 \times \sin(kx - \omega t), \quad t = \frac{1}{2}s$

c. $y = 5 \times \sin(kx - \omega t), \quad t = 1s$

d. $y = 5 \times \sin(kx - \omega t), \quad t = 1.5s$

This is precisely the same as the previous problem except that I've given you the phase as something that depends on time and the functions are to be plotted as functions of distance rather than time. Plot these up and then discuss them a bit. Also, despite having gotten some nice practice doing this sort of thing in the previous problem, let's go ahead and do at least the first couple in some detail.

Let's begin with (a). We *know* the overall shape of this will be a sine wave. What we don't know, off the bat, is where things like the x intercepts and the maxima and minima of the wave will occur. So let's just sketch the wave without putting numbers in on the x axis. We have (The y axis should exactly touch the graph at its beginning. The



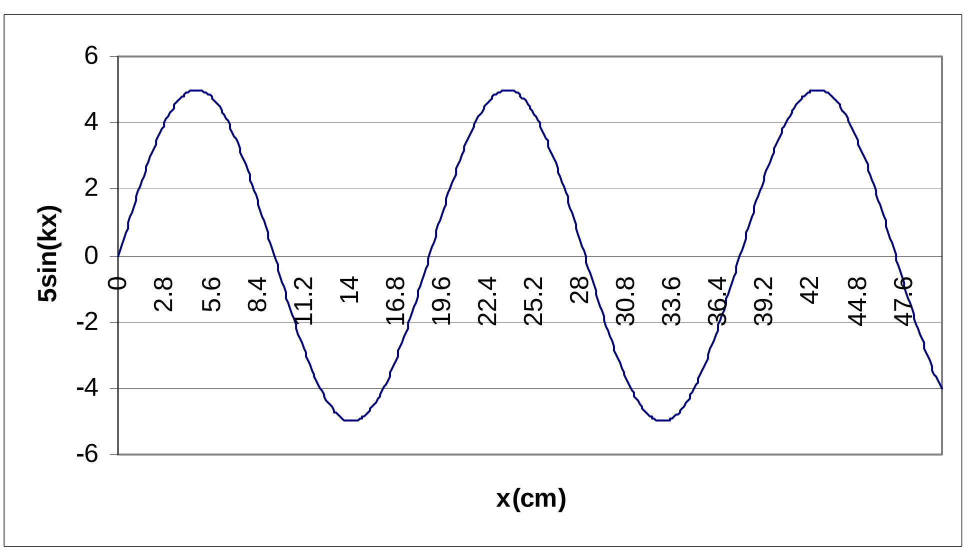
gap that you see is an artifact of the plotting program—very annoying!)

Now, notice that the plot crosses the x axis at several locations. If we can figure out the values of x at which these happen, we'll be well on our way to finishing the plot. At the outset, we do not know what these x values are. But we know how the sine function varies with its argument. Consider the function $y = \sin(u)$. We know that we will have $y = 0$ when $u = 0$ and when $u = \pi$ and when $u = 2\pi$ and so on. Well, what we have called u is just ***everything*** in the parentheses of our functions.

For (a), we can say $u = kx - \omega t$. We are told in the problem that $t = 0$ for this case. So we have $u = kx$. The first place where the function equals zero (i.e., where it crosses the x axis) will be at $u = 0$. This allows us to solve for x trivially—we can say that $x = 0$ for the first place where the graph crosses the x axis.

The next place at which this happens will be when $u = \pi$. So we can write $kx = \pi$. Again a pretty trivial piece of algebra allows us to say that $x = \frac{\pi}{k}$ for this point. We can now substitute in the value given for k, $k = \frac{1}{3} cm^{-1}$, and solve for x explicitly: $x = \frac{\pi}{k} = \frac{\pi}{\left(\frac{1}{3} cm^{-1}\right)} = 3\pi\, cm$. It is not incorrect to go ahead and write this as (approximately) $x = 3 \times 3.14 cm = 9.42 cm$, but it is not necessary to do so.

From here it's easy, the rest of the places where the function will cross the x axis are $x = 6\pi\, cm, 9\pi\, cm, 12\pi\, cm\ldots$ and so on. This allows us to label the x axis of the graph

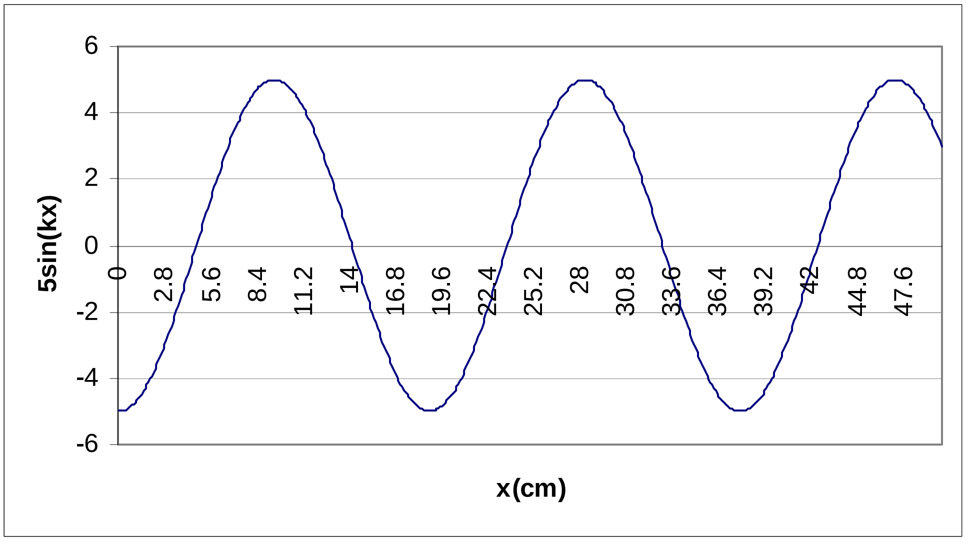


easily, giving:

(I extended the plot well beyond 12 cm just for giggles.)

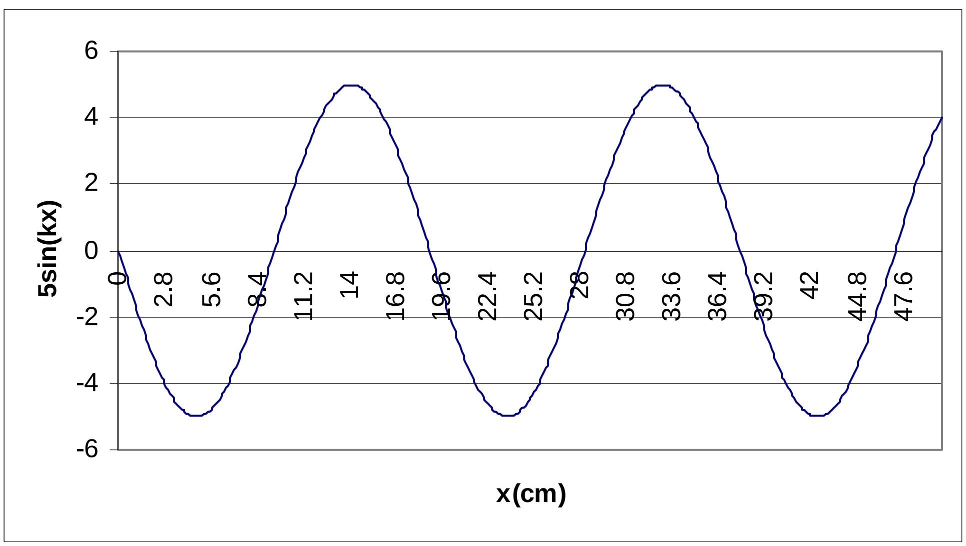
Now, (b) is the same function one half second later. We'll plot this systematically in a moment, but let's ***think*** about it first. The equation that we are plotting is that of a traveling wave. The overall shape isn't changing. The wave is ***propagating*** as time passes, however. So we would expect to get the exact same shape only shifted somewhat to the right (or left, perhaps). How much will it be shifted? Well, the frequency is $\pi\frac{1}{s}$. That is, as discussed in problem #1, the pattern will shift by ½ cycle (pi is one half a cycle—180° for those of you who aspire to become ancient Babylonian priests) each second. We are plotting the wave ½ second after the previous plot, so we would expect it to be shifted by ¼ cycle. Let's see if this is the case.

Using the technique of (a), we write $u = kx - \omega t$. We then substitute the values we are given for ω and t. This gives $u = kx - \omega t = kx - \frac{\pi}{2}$. (Note the similarity of this to part (b) of problem #1.) Again, we recognize that the function will cross the x axis whenever u is an integer multiple of pi. So our first crossing will be at $kx - \frac{\pi}{2} = 0$ which gives $x = \frac{\pi}{2k} = \frac{\pi}{2 \times \frac{1}{3} cm^{-1}} = \frac{3}{2}\pi\, cm$. We can use the same procedure (setting $u = \pi, 2\pi, 3\pi \ldots$) for subsequent crossings of the x axis. Alternatively, we can realize that we found, in part (a), that the zeros repeat every $3\pi\, cm$. The place where the function "starts" (it doesn't really start there, of course, but we started our analysis with that point) doesn't affect the distance between crossings, just what their overall value will be. Thus we can save a lot of algebra and just add $3\pi\, cm, 6\pi\, cm, 9\pi\, cm \ldots$ to $\frac{3}{2}\pi\, cm$ to get the other crossings. This gives $x = \frac{9}{2}\pi\, cm, \frac{15}{2}\pi\, cm, \frac{21}{2}\pi\, cm \ldots$ Plotting this, we have

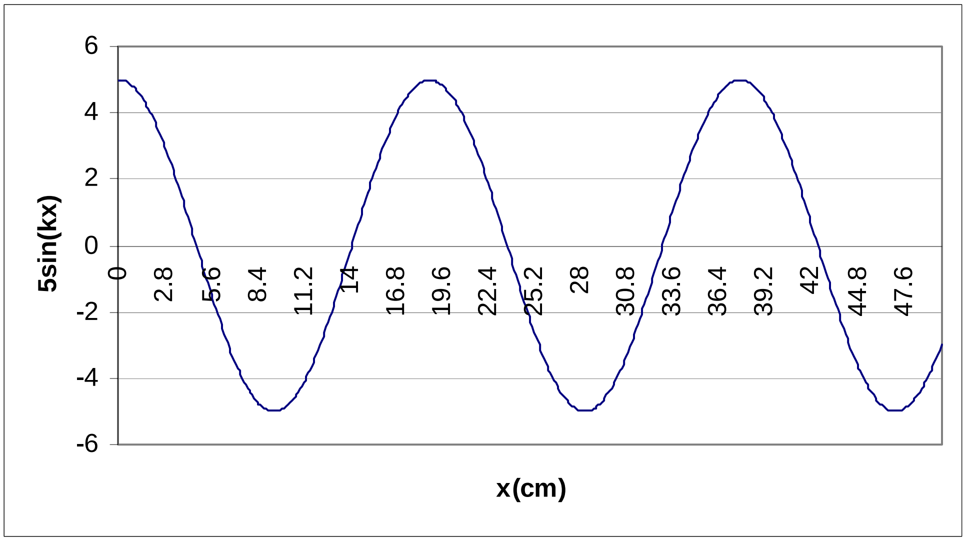


As expected, the function is shifted to the right by ¼ cycle compared to the plot we produced in (a).

Continuing this process, we perform the exact same operations with the other two values of t which were given. For $t = 1s$ we get



And taking $t = 1.5s$ we get



Imagine that these four plots are four frames in a movie. Each successive plot shows the wave at a slightly later time. Notice how it “moves” to the right as time passes. This is what we mean when we say that the wave is propagating.

5. **What is the speed of the wave you plotted in problem #4?**

We have a very general equation that relates the speed with which any wave propagates to its frequency and wavelength. This is $\lambda f = v$. We weren't told either the wavelength or the circular frequency, but we were told the wavenumber (k) and the *angular* frequency (ω). A little algebra is all that is necessary to turn these into what we need.

Using the definition of the wavenumber, we have $k = \frac{2\pi}{\lambda}$ so $\lambda = \frac{2\pi}{k}$. Similarly, we know that $\omega = 2\pi f$ so $f = \frac{\omega}{2\pi}$. Multiplying these together, we get $v = \lambda f = \frac{2\pi}{k} \times \frac{\omega}{2\pi} = \frac{\omega}{k}$. Substituting in the numbers provided in the problem, we have $v = \frac{\omega}{k} = \frac{\pi \frac{1}{s}}{\frac{1}{3} cm^{-1}} = 3\pi \frac{cm}{s}$.

Note that the relation $v = \frac{\omega}{k}$ is true in general and is just another way of saying $\lambda f = v$.

6. **Briefly (a paragraph or two) distinguish between bulk motion and wave motion. Also distinguish between longitudinal waves and transverse waves.**

A wave is a disturbance that moves. Since what's "waving" is the amount of disturbance, it does not necessarily involve motion of material. "Bulk motion," on the other hand, involves an actual relocation of "stuff" from one place to another.

A fine example of this is "the wave" as performed by the attendees of a baseball game (as observed by a student in this class). As one group of people in the arena starts sitting down, another group of people nearby them starts to stand up. This coordinated behavior continues, column after column, all the way around the stadium. When viewed from elsewhere (like on the television at home), it is clear that *something* is working it's way around the stadium. But no person actually travels along with the disturbance. The people in the stadium just stand up briefly and then sit down again. The thing that is traveling is the message "stand up now." If a cluster of standing people actually ran around the stadium, it would be bulk motion. (Notice that this would happen at a very different speed than the speed of the wave.)

Bulk motion involves motion of material. Wave motion involves only motion of information.

7. **A very long guitar string has a linear density of $4.466\,\frac{\text{grams}}{\text{meter}}$. It is strung with a tension of 97 Newtons. If it is plucked, with what speed will the disturbance travel down the length of the string?**

This is simply application of the formula $v=\sqrt{\frac{F_T}{\mu}}$. We are given μ: $\mu = 4.466\,\frac{\text{grams}}{\text{meter}}$. We are also given F_T. From here, it's just "plug and chug":

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{97\,\text{Newtons}}{4.466\,\frac{\text{grams}}{\text{meter}}}}=\sqrt{\frac{97\,\text{Newtons}}{4.466\times10^{-3}\,\frac{\text{kilograms}}{\text{meter}}}}=147.4\,\frac{\text{meters}}{\text{second}}.$$

8. **A string which has a linear density of $7\,\frac{\text{grams}}{\text{meter}}$ is suspended from the ceiling. A mass of 1 kg is hung from the end of the string. The string is 2 meters long. A fly is sitting on the string near where it attaches to the ceiling. The fly's wings beat with a frequency of 197 Hz. What is the length of the waves generated in the string by the fly? (Neglect the contributions of the weight of the string and the weight of the fly to the tension of the string.)**

Since we are neglecting the weights of the fly and the string, the tension on the string is just the weight of the hanging mass. This is $W = mg = 1kg \times 9.8\frac{meters}{second^2} = 9.8\,Newtons$. Thus, the speed of propagation of the wave is

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{9.8\,\text{Newtons}}{7\,\frac{\text{grams}}{\text{meter}}}}=\sqrt{\frac{9.8\,\text{Newtons}}{7\times10^{-3}\,\frac{\text{kilograms}}{\text{meter}}}}=37.4\,\frac{\text{meters}}{\text{second}}.$$

The disturbance occurs with a frequency of 197 Hz, so to find the wavelength we use the fact (true for ***all*** waves) that $\lambda f = v$. This gives

$$\lambda = \frac{v}{f} = \frac{37.4\,\frac{\text{meters}}{\text{second}}}{197\,\text{Hz}} = 0.19\,\text{meters}\ .$$

9. **A string with a length of 10.0 m and linear density $25\,\frac{\text{grams}}{\text{meter}}$ is suspended from the ceiling. An object of mass 0.200 kg is hung from the string. What is the speed with which a disturbance in the string will propagate near (a)the bottom of the string, (b)midway up the string, and (c)near the top of the string? (Do *not* neglect the contribution of the weight of the string to the tension in the string!)**

Here's an annoying little piece of gear-shifting for you! All last semester, when we worked on problems involving tension, I repeated to you that the tension in a string is the same everywhere. Now, out of nowhere, I've gone and changed the rules! ☺

Any particular point on the string has to exert an equal and opposite force to the force acting on it—this is Newton's third law. The force acting on a particular point along a string hanging straight down is that of *any* mass hanging from it. The string doesn't know whether the mass is that of, say, a brick hanging 20 meters away or a brick hanging 2 cm away. Nor does it know whether the weight is that of the brick or that of the string itself. Any string ***above*** the point in question will exert only an upward force on the point (you can't push on a rope!). So the tension will be precisely the *total* weight below the point. In this case, this is the weight of the hanging mass plus the weight of the string below the point.

If we begin measuring height, h, at the bottom of the string, he have $m_{string} = \mu h$. Thus, the weight supported by the string a distance h above the bottom will be $W(h) = m_{total}\,g = (m_{object} + m_{string})g = (m_{object} + \mu h)g$. This is the same as the tension force. Thus we can write the speed of the wave at any given point as $v(h) = \sqrt{\frac{F_T(h)}{\mu}} = \sqrt{\frac{(m_{object} + \mu h)g}{\mu}}$.

(a) At the bottom of the string, the only weight is that of the hanging object. ($h = 0$), so

$$v(h)=\sqrt{\frac{F_T(h)}{\mu}}=\sqrt{\frac{(m_{object})g}{\mu}}=\sqrt{\frac{.2\,\text{kg}\times 9.8\,\frac{\text{meters}}{\text{second}^2}}{25\,\frac{\text{grams}}{\text{meter}}}}=\sqrt{\frac{.2\,\text{kg}\times 9.8\,\frac{\text{meters}}{\text{second}^2}}{.025\,\frac{\text{kg}}{\text{meter}}}}=8.85\,\frac{\text{meters}}{\text{second}}$$

(b) Midway up the string, we use $h=5\,\text{meters}$ which gives

$$v(h)=\sqrt{\frac{F_T(h)}{\mu}}=\sqrt{\frac{(m_{object}+\mu h)g}{\mu}}$$

$$=\sqrt{\frac{\left(.2\,\text{kg}+5\,\text{meters}\times .025\,\frac{\text{kg}}{\text{meter}}\right)\times 9.8\,\frac{\text{meters}}{\text{second}^2}}{.025\,\frac{\text{kg}}{\text{meter}}}}$$

$$=\sqrt{\frac{(.2\,\text{kg}+.125\,\text{kg})\times 9.8\,\frac{\text{meters}}{\text{second}^2}}{.025\,\frac{\text{kg}}{\text{meter}}}}=11.3\,\frac{\text{meters}}{\text{second}}$$

(c) At the top of the string, we use $h=10\,\text{meters}$

$$v(h)=\sqrt{\frac{F_T(h)}{\mu}}=\sqrt{\frac{(m_{object}+\mu h)g}{\mu}}$$

$$=\sqrt{\frac{\left(.2\,\text{kg}+10\,\text{meters}\times .025\,\frac{\text{kg}}{\text{meter}}\right)\times 9.8\,\frac{\text{meters}}{\text{second}^2}}{.025\,\frac{\text{kg}}{\text{meter}}}}$$

$$=\sqrt{\frac{(.2\,\text{kg}+.25\,\text{kg})\times 9.8\,\frac{\text{meters}}{\text{second}^2}}{.025\,\frac{\text{kg}}{\text{meter}}}}=13.3\,\frac{\text{meters}}{\text{second}}$$

10. **A 64 cm long guitar string has a linear density of $4.466\,\frac{\text{grams}}{\text{meter}}$. At what tension must it be strung if it is to have a fundamental frequency of 110 Hz? (This is the tone identified by musicians as "A".)**

A guitar string has harmonics where the frequency of the string's vibration forms a standing wave. The condition for a standing wave on a string anchored at both ends is $n\lambda = 2L$ where n is any positive integer and L is the length of the string between the anchors. The fundamental has $n = 1$. Thus, taking the length to be 64 cm, we have $\lambda = 2L = 1.28\,\text{meters}$.

Now, we know the wavelength and the frequency (which is given), so we can calculate the speed of propagation of the wave. This may seem odd, since this is a standing wave. How can it propagate? Recall, however, that a standing wave is created by a propagating wave and all of its reflections. If the reflections line up with each other, a stable, standing wave will emerge. But each of the reflections *is* propagating. They just add up to make something that stands still.

The propagation speed is related to the wavelength and frequency via $\lambda f = v$, so we have $\lambda f = v = 1.28\,\text{meters} \times 110\,\text{Hz} = 140.8\,\frac{\text{meters}}{\text{second}}$. This, in turn, is related to the tension via $v = \sqrt{\frac{F_T}{\mu}}$. We are given the linear mass density as $\mu = 4.466\,\frac{\text{grams}}{\text{meter}}$ so we have (after a smidge of algebra)

$$F_T = \mu v^2 = 4.466 \times 10^{-3}\,\frac{\text{kilograms}}{\text{meter}} \times \left(140.8\,\frac{\text{meters}}{\text{second}}\right)^2 = 88.5\,\text{Newtons}.$$

11. **A man blows across the mouth of a bottle of soda shortly after the bottle has been opened, creating a flute-like sound. The space in the bottle above the liquid is initially filled with carbon dioxide due to the fizz in the soda. He sets the bottle aside and comes back several days later and tries blowing again. Now, the CO_2 has been replaced by air. (Treat the bottle as a pipe. The reality is that the odd shape of a bottle changes the resonant frequencies dramatically, as we saw/heard in class. Neglect this for this problem.) (By the way, this would be tough to do in reality—the man's breath would really screw things up. Neglect that effect in the problem.)**

a. **By what factor will the frequency of the "toot" change?**

b. **If the height of the gas is 10 cm, what was the frequency of the tone with the CO_2?**

c. **What is the frequency with the air?**

d. **What are the wavelengths in the two cases?**

This is just a single-end-closed wind instrument, within the approximation of the problem. The liquid in the bottle forms the closed end. Thus, the length of the resonator is $L = 10\,cm$. The resonance condition for this kind of pipe is $\lambda = \frac{4L}{n}$ where *n* is any *odd* integer. (Personally, I prefer to write this a different way. I prefer to write $\lambda = \frac{4L}{(2m+1)}$ where *m* is *any* integer, even or odd. You are welcome to use either form of this relation, just don't mix them up!) Thus, from the resonator condition, we know the wavelength.

Now, the frequency and the wavelength are related to each other via the propagation speed of the wave. This relation is $\lambda f = v$. As stated before, this is true for *all* waves. (Recall that, at resonance, we have a standing wave in the "tube" created by a wave and all of its reflections adding up. This is why the speed matters.) Thus, the frequency for a given "mode" (value of *n*) is $f = \frac{v}{\lambda} = \frac{v}{\left(\frac{4L}{n}\right)} = \frac{nv}{4L}$, where *n* is any odd integer.

Using this, the ratio of frequencies expected with two different media filling the bottle is $\frac{f_1}{f_2} = \frac{\frac{nv_1}{4L}}{\frac{nv_2}{4L}} = \frac{v_1}{v_2}$.

The wavelength will be the same in both cases since this is determined *solely* by the geometry of the resonator (the bottle/tube), as stated above.

Using some numbers (these can be found in your textbook on page 484), the speed of sound (at 0° C) in carbon dioxide is $v_{CO_2} = 259\,\frac{m}{s}$ and that in air is $v_{air} = 331\,\frac{m}{s}$. Thus, the ratio of frequencies is $\frac{f_{air}}{f_{CO_2}} = \frac{v_{air}}{v_{CO_2}} = \frac{331\,\frac{m}{s}}{259\,\frac{m}{s}} = 1.278$. Stated as a factor, we can say that the frequency in air 1.278 times the frequency in carbon dioxide.

The wavelengths are fixed by the geometry: The possible wavelengths are the same whatever gas fills the bottle, $\lambda_n = \frac{4L}{n} = \frac{40\,cm}{n}$ (note that I've subscripted the lambda to keep it clear that there isn't just one wavelength possible but a whole set). Each of these will correspond to its own frequency which *will* depend on the gas, as we saw above. These are $f_n = \frac{v}{\lambda_n} = \frac{nv}{40\,cm}$. So, for carbon dioxide, the possible frequencies are $f_n = \frac{nv_{CO_2}}{40\,cm} = n\frac{259\,\frac{m}{s}}{40\,cm} = n \times 647.5\,Hz$. (Note the unit in this: My solution came out with a unit of $\frac{1}{s}$ and I could have used this—it wouldn't have been the least bit incorrect. But the hertz, Hz, is conventional. The two are totally synonymous.) For air, these become $f_n = \frac{nv_{air}}{40\ cm} = n\frac{331\frac{m}{s}}{40\ cm} = n \times 827.5\ Hz$.

12. **A spider makes a web across the inside of an organ pipe which is open at both ends. The web is fine but still creates a small dissipative force (drag) for any air molecules moving at its location. The web is built at the exact midpoint along the length of the pipe. Which overtones (harmonics) of the pipe will be most affected by the presence of the web? Which overtones (harmonics) will be least affected by the web? Explain.**

The key here is that the web won't affect pressure but will affect motion. So the pressure variation due to a wave at the location where the web is can be pretty much anything. (There would be a problem if there were a pressure *difference* between the two faces of the web. In that case, the web could be damaged. But a spider web is so thin and porous that no significant pressure difference is likely.) However, any mode that has a nonzero amplitude for its displacement variation will experience a dissipative force. Those modes will not build up significantly. On the other hand, any mode that has a node (say *that* three times fast!) in its displacement variations at the location of the spider web will simply not experience the web at all. Those modes will be unaffected.

Since the open end of the pipe results in an antinode for the displacement variations at the end of the pipe, we have the requirement of a node at the middle and an antinode at the end. (Alternatively, you can view this as changing a both ends open pipe to a pipe with one end closed but of half the length.) Any mode which can fit an odd integer number of quarter-wavelengths in the distance between the end and the middle of the pipe will meet these criteria. Thus we have $n\frac{\lambda}{4} = \frac{L}{2}$ (with only odd values of n). This gives $\lambda = \frac{2L}{n}$ with n odd. Notice that this is exactly the same as the resonance condition for the original tube, with both ends open, except for the constraint that n be odd. Thus, we conclude that the spider web will limit our organ pipe to its odd harmonics. The even harmonics will be weak if they exist at all.

13. **An organ pipe is open at one end. It is designed to play a frequency of 440 Hz. If the pipe is designed to operate at a temperature of 23° C, what frequency will it play if the temperature of the air is reduced to 8°C? (Neglect expansion of the pipe itself.)**

This problem is essentially identical to problem #1. It is troubling to some people that you don't have an explicit formula for the speed of sound in air as a function of temperature. That is, you know that $v = const.\times\sqrt{T}$, but you don't know how to find the constant. The neat thing is that you don't need to! For one thing, knowing the speed of sound at *any* temperature will allow you to find the constant (for the particular medium under discussion, of course; the constant *does* depend on the particular substance being considered) without doing the fundamental calculation of it. But this is rarely necessary because we can usually find an answer in terms of a ratio which will eliminate the constant from the calculation altogether. Let's do it both ways.

First, let's find the constant for air. We know that at 0° $v_{air} = 331\frac{m}{s}$. Of course, we must use the *absolute* (Kelvin) temperature. So we have $v_{air} = 331\frac{m}{s} = const.\times\sqrt{T} = const.\times\sqrt{273K}$. We can immediately solve this for the constant to get $const. = \frac{331\frac{m}{s}}{\sqrt{273K}} = 20.03\frac{m}{s\cdot K^{\frac{1}{2}}}$. (Don't worry about that funky unit. It really *is* an odd one, but that's not relevant. Don't let it get to you.)

Now we *could* use this constant to find the speed of sound in air at any temperature using $v = const.\times\sqrt{T}$. But that won't be necessary. Here's where those of you who have broken yourselves of the old habit of obsessing over numbers really get their payoff! Let's see how.

We know that it is *always* true that $\lambda f = v$. We also know that the wavelength of sound in a resonant system is determined solely by the geometry of the system. (Make sure that you ***really*** understand that last sentence! It is absolutely essential.) So λ will be the same at both temperatures. (Note that this would not be the case if we wanted to include considerations of thermal expansion of the pipe itself. One thing that's really interesting is that the effect of thermal expansion is to make λ bigger with higher temperature but the effect of the speed of sound's variation is to make the frequency for a given wavelength bigger as well. These two effects cancel somewhat, but not completely. The net is that organ's [and other wind instruments] are less sensitive to temperature variations than the results of this problem would lead you to conclude. For your own entertainment, I recommend working out the full problem—you have all the tools needed to calculate the final frequency including both the variation of the speed of sound and the change in the length of the pipe.) So we can take the ratio of the frequencies at the two temperatures and a whole lot of stuff will cancel. We have:

$$\frac{f_{8^\circ}}{f_{23^\circ}} = \frac{\frac{v_{8^\circ}}{\lambda}}{\frac{v_{23^\circ}}{\lambda}} = \frac{v_{8^\circ}}{v_{23^\circ}}$$

And using the dependence of speed on temperature, this gives

$$\frac{f_{8^\circ}}{f_{23^\circ}} = \frac{v_{8^\circ}}{v_{23^\circ}} = \frac{const.\sqrt{8^\circ C}}{const.\sqrt{23^\circ C}} = \frac{\sqrt{8^\circ C}}{\sqrt{23^\circ C}} .$$

Of course, we ***must*** change the temperature to Kelvins before continuing. This gives

$$\frac{f_{8^\circ}}{f_{23^\circ}} = \frac{\sqrt{8^\circ C}}{\sqrt{23^\circ C}} = \frac{\sqrt{281K}}{\sqrt{296K}} = 0.974 .$$

Of course, $f_{23^\circ} = 440Hz$, as stated in the problem. So $f_{8^\circ} = 0.974 \times 440Hz = 428.6Hz$. Notice that we never actually needed the speed, ***per se***, at all!

14. **Guitars are built such that the twelfth fret (usually a bar of metal on the neck of the guitar) is exactly midway along the lengths of the strings. Guitarists have learned that touching the string *very gently* at this fret will force a node to occur at that point. A guitar string which is 64 cm long with a linear density of** $4.466\,\frac{\text{grams}}{\text{meter}}$ **is strung with a tension of 88.5 Newtons. If a guitarist applies gentle pressure on the twelfth fret under such a string, what frequency will it produce?**

This problem is essentially identical to problem #2: Once again, we depend on the fact that the wavelength depends only on the geometry of the system. We introduce a selective loss into the system that damps any energy that finds its way into certain modes while leaving other modes untouched (mostly). By touching a guitar string, the player forces a node to exist at that location. Thus the only modes that can exist are those in which there is a node at the midpoint of the string. Effectively, we have cut the string in half.

This suggests two different ways of approaching the solution: We can either use the full solution for a string of length $\frac{L}{2}$ or we can keep the length and figure out which modes have nodes in the middle. Frankly, I find the former method to be easier, so let's do the latter one first.

We have a string of length L. Since the ends are anchored, the allowed modes (those which can support a resonance) will be the ones which have nodes at the ends. By definition, a sinusoidal wave (one described by a sine or cosine function) will have a node every half-wavelength. Thus we require that an integer number of half wavelengths fit into one string length: $n\frac{\lambda}{2}=L$.

Another way of looking at this is to note that the resonance requirement is that one *full* cycle (one full wavelength) be gotten through in one roundtrip of the wave. This perspective leads us to $n\lambda = 2L$. These are identical statements and either perspective is equally valid.

Now, here's a point where I must admit to not understanding exactly what's going on (I know several Physicists who are also musicians and are trying to make measurements of this for me): When the string is lightly tapped, only the *n=2* mode for the full-string (this would be *n=1* for the half string) is excited. Normally, a large number of modes are excited (i.e., vibrating) to a greater or lesser extent. This gives sounds a richness—a complexity to the *timbre* of the sound. Typically, the lowest possible *n* mode is vibrating with the highest amplitude while higher *n* valued modes have much lower amplitudes. But they are present. In this case, only the *n=2* mode vibrates. I'll let you know when I find out what's up. (If the string were fully pressed, the *n=2* mode would also be the dominant mode. But there would be small amplitudes in all even-numbered modes.)

In any event, we take *n=2* and get $\lambda = L$. Now, the problem asks for the frequency. Here we use $\lambda f = v$ to get $f = \frac{v}{\lambda}$. To find the speed of propagation, we use $v = \sqrt{\frac{F_T}{\mu}}$. Combining these gives $f = \frac{v}{\lambda} = \frac{\sqrt{\frac{F_T}{\mu}}}{\lambda} = \frac{\sqrt{\frac{F_T}{\mu}}}{L}$. Substituting numbers gives, finally $f = \frac{\sqrt{\frac{F_T}{\mu}}}{L} = \frac{\sqrt{\frac{88.5 \text{ Newtons}}{4.466 \times 10^{-3} \frac{\text{kilograms}}{\text{meter}}}}}{.64 \text{ meter}} = 220\ Hz$, recalling that $1\ Hz = 1 \frac{1}{\text{second}}$.

15. Deferred to next semester.