PHYSICS 206a
HOMEWORK #13
SOLUTIONS

1. **What is the period of a pendulum with a bob mass of 1 kg, an initial angle of 0.1 radian, and a length of 3 meters?**

The period of a pendulum, in the small angle approximation, is given by $T = 2\pi\sqrt{\frac{L}{g}}$. If you get confused about whether the g goes on the top or the bottom (I can *never* keep that one straight!), do a dimensional analysis: The units of g are $\frac{meters}{second^2}$. Since we want a final answer in *seconds*, the g must be in the denominator. I find that it's easier to remember to do the dimensional analysis than it is to memorize the formula—I *never* trust myself to get it right!

The initial angle is small enough that we can use the small angle approximation. So, plugging in numbers, we get $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{3\,\text{meters}}{9.8\,\frac{\text{meters}}{\text{second}^2}}} = 3.47\,\text{seconds}$.

2. **For the pendulum in the previous problem, what is the circular frequency? What is the angular frequency?**

The circular frequency is just the reciprocal of the period. Keep these straight in your mind: The period is the amount of time it takes to go through one cycle. The circular frequency is the number of complete cycles it goes through in some fixed interval of time. Written mathematically, this is $\omega_{circular} = f = \frac{\text{number of cycles}}{\text{time}}$. We are free to pick the interval of time over which we want to count cycles, although we usually pick one second as the default. Imagine that we pick one period as the interval of time over which we count cycles. Well, by definition, in one period we have one cycle. So based on our formula above $\omega_{circular} = \frac{\text{number of cycles}}{\text{time}} = \frac{1}{T}$, as we stated at the beginning.

So, $\omega_{circular} = \frac{1}{T} = \frac{1}{2\pi}\cdot\sqrt{\frac{g}{L}} = \frac{1}{3.47\,\text{seconds}} = 0.288\frac{1}{s}$. The unit $\frac{1}{s}$ can be called "inverse seconds" or just read as "per second." It is also synonymous with a unit called a Hertz (abbreviated Hz)—which we will encounter again next semester. This is the total number of cycles (in this case, a fraction of a cycle) that the pendulum will go through in each second.

The angular frequency is basically the same thing, but it's a bit more difficult to describe conceptually. If we regard the pendulum's position as the shadow of a point on a circle rotating at a constant angular speed, the circular frequency is the total number of *radians* that circle will travel through each second. Clearly this is synonymous with the angular speed of the rotating circle. (I suppose, in principle, that one could do this in degrees, but I've never seen it done that way and hope never to do so!) While this may seem to be an unnecessary complexity, it actually makes calculations much easier: If we're interested in describing the position of the pendulum at any instant of time, rather than just describing how many complete cycles it may have gone through, use of the angular frequency is essential! Since one complete cycle has 2π radians, the angular frequency will be 2π times the circular frequency. This is $\omega_{angular} = \frac{2\pi}{T} = \sqrt{\frac{g}{L}} = 1.807\frac{1}{s}$.

The most confusing thing about the difference between angular frequency and circular frequency is that the unit is the same in both cases. One *can* read the circular frequency's unit as "radians per second" but this is not always done. One would be totally correct simply to read it as "per second," the same as for the circular frequency. Often, even the same symbol is used for the two frequencies! Errors due to this ambiguity are common even at a professional level. One simply has to know which frequency is being discussed based on context. Often, that simply can't be done unambiguously. Be careful! And, when in doubt, ask.

3. For the pendulum discussed in the previous problems, what is the equation giving the position (in the *x* direction) as a function of time? Take t=0 as the moment when the pendulum is released from its initial displacement. Be sure to include the phase!

The general form of this equation is $x = A\cos(\omega_{angular}t + \phi)$. Here, A is the "amplitude" of the oscillation—the maximum distance from the equilibrium position that the bob will swing to on each side of its oscillation. This can be found from the initial position at t=0 since, because of conservation of energy, the amplitude will be equal to the displacement from which the pendulum was originally released. Caution: We do *not* need to start the clock at the instant the pendulum was released! To find the amplitude, find position of the bob when it was released, not necessarily at t=0. In this case, we have $x_{initial} = A = L\theta = 3\,\text{meters} \times 0.1\,\text{radian} = 0.3\,\text{meters}$.

Now, to find the phase, we use the amplitude that we just found and take t=0 in the equation. This gives $x_{initial} = A\cos(\phi)$. Of course, we've just found that $x_{initial} = A$, so we must have $\cos(\phi) = 1$. This gives us $\phi = 0$. This won't always be the case! The phase allows us to start the clock at any point in the pendulum's oscillation.

Thus we can write $x = A\cos(\omega_{angular}t)$. So, our final equation is $x = 0.3\,\text{meters} \times \cos(1.807\frac{1}{s} \times t)$.

4. If the pendulum in the previous problems has its bob replaced with one of twice the mass, what happens to the frequency?

The frequency is unaffected by the mass! In a practical sense, having a larger mass makes pendula less susceptible to external influences like stray breezes and dust specs landing on them, but the mass has no effect on frequency, *per se*.

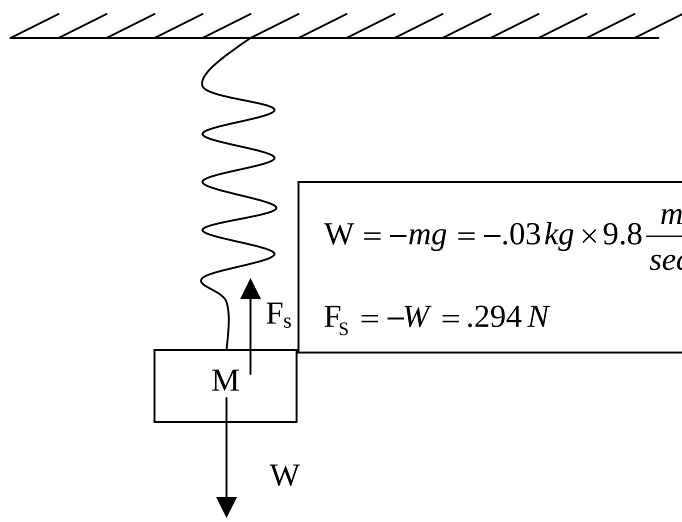
5. If the pendulum in the previous problems has its shaft (the "string") replaced with one of twice the length, what happens to the frequency?

The shaft length, on the other hand, has a direct influence on the frequency of the pendulum. For problems like this, I prefer to work with ratios—the question asks "what happens to the frequency?" which is a roundabout way of asking for a comparison rather than a direct number. Working with the angular frequency (it doesn't really matter which one—they both depend on the string length the same way), let's call the new (longer-string) angular frequency ω_{new} and the old one ω_{old}. This gives $\frac{\omega_{new}}{\omega_{old}} = \frac{\sqrt{\frac{g}{L_{new}}}}{\sqrt{\frac{g}{L_{old}}}} = \sqrt{\frac{L_{old}}{L_{new}}} = \sqrt{\frac{1}{2}} = 0.707$. Since $\omega_{old} = 1.807\frac{1}{s}$, we have $\omega_{new} = 0.707 \times 1.807\frac{1}{s} = 1.278\frac{1}{s}$.

6. **A spring has a constant** $k = 7\,\dfrac{\text{Newtons}}{\text{meter}}$**. The spring is oriented vertically and a mass of 30 grams is hung from it.**

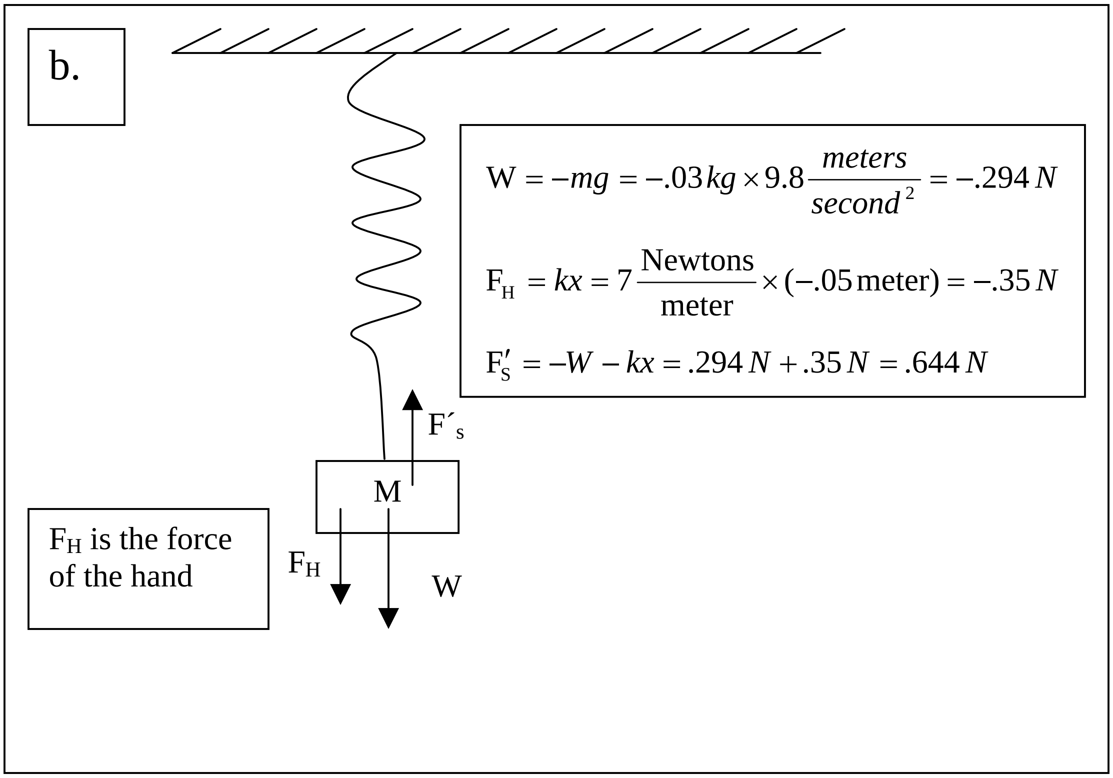
- **a.** **Draw a free-body diagram for the mass when it at rest at its equilibrium position. Give the sizes of all the forces in your diagram.**
- **b.** **The mass is pulled down 5 cm by someone's hand and held in that position. Now draw a free-body diagram for the mass. Give the sizes of all the forces in your diagram.**
- **c.** **The spring is released. How high up does it go?**
- **d.** **At the top of its travel, just when it has stopped and is about to begin moving down again, draw a free body diagram for the mass.**

a.



$$\mathrm{W} = -mg = -.03\,kg \times 9.8\,\frac{meters}{second^2} = -.294\,N$$

$$\mathrm{F_S} = -W = .294\,N$$



Initially, before the mass was attached, the spring had some unstretched length. When the mass was attached, the spring stretched out enough that the spring's force was exactly equal to the force of gravity on the mass. The mass's position after this stretch is called the "equilibrium position." All future motion of the mass is measured relative to this position because at this position the net force on the mass is zero. We from here on, if we like, we can pretend that gravity doesn't exist and that the force exerted by the spring at its equilibrium position is zero. Since these two cancel, they might as well not be there. This is shown in the figure for **a**.

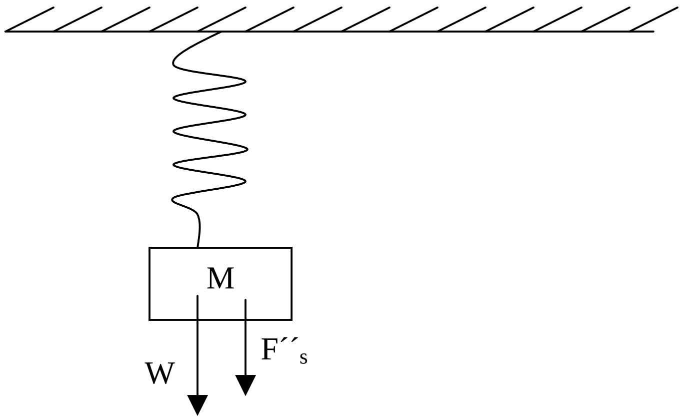
When the spring is stretched by 5 cm from its equilibrium position, the force exerted by the hand must be exactly the same as the additional force exerted by the spring due to its stretch. This is found from Hooke's law: $F = -kx$, where x is the stretch from the equilibrium position. This is shown in the figure for **b**.

We can use conservation of energy to see that the mass will rise the same distance *above* the equilibrium position as it was originally pulled down below the equilibrium position. Here is where confusion comes in: You might well be tempted to include gravitational potential energy in the problem. *Don't*! Recall that the potential energy in the system represents work done on the system that is still available to the system. Since the spring is pulling up on the mass with a force that is equal to the weight of the mass plus the force needed to resist the hand, the force of gravity might as well not be there at all. The work that the hand did pulling the mass down was just the force needed to move it from its equilibrium position times the distance from the equilibrium position that the mass was pulled. (Notice that the force is not constant so we cannot simply write $P.E. = Fx$.)

So, because we can ignore gravity, it becomes obvious that at the bottom of its travel the kinetic energy of the mass is zero (it's stopped) and at the top of its travel it is also zero. All energy is potential. The potential energy of the spring is $P.E. = \frac{1}{2}kx^2$. This is independent of whether *x* is positive or negative, so the mass travels 5 cm up past the equilibrium position.

For part **d.**, there are actually *two* possible answers. This is why: As stated above, the spring starts out unstretched, before the mass is hung from it. Let's say that when the mass is hung on the spring it stretches a distance x_0 to reach its equilibrium position. Now, unlike a string, a spring can either pull *or* push. Our spring will *pull* until it has been compressed to its unstretched length, then it will begin to push. Since gravity is pulling on the mass continuously, after the mass passes the *equilibrium* position the net force on the mass will be downward. However, the spring will still be pulling upward until it has traveled up a height x_0. After that, the spring will begin pushing. It is crucial for you to realize that the difference between the two regimes is totally irrelevant to the motion of the mass. So far as the mass is concerned, only the deviation from the equilibrium position is relevant. However, to get our free-body diagram correct, we need to determine whether the initial pull (5 cm) is either greater than x_0 or less than x_0. If it's greater than x_0, the force of the spring on the mass will be downward at the top of its trajectory. If the initial pull is les than x_0, the force of the spring will be upward at the top of its trajectory. Since the weight of the mass is .294 Newtons, using Hooke's law we have $x_0 = \frac{W}{k} = \frac{.294\,\text{Newtons}}{7\,\frac{\text{Newtons}}{\text{meter}}} = 4.2\,cm$. Thus, x_0 is less than the initial pull and we can conclude that at the top of the trajectory the spring will be *pushing* on the mass. Since the difference between the unstretched length and the equilibrium position is 0.8 cm, we can conclude that the downward force exerted by the spring is $F = -kx = -7\,\frac{\text{Newtons}}{\text{meter}} \times .8\,cm = -.056\,N$. This is pictured on the next page.

d.



$$W = -mg = -.03\,kg \times 9.8\,\frac{meters}{second^2} = -.294\,N$$

$$F_S'' = -.056\,N$$

7. **For the system in the previous problem, what is the frequency of oscillation of the mass?**

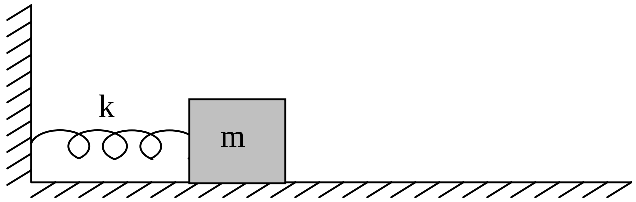
Just a formula (memorize it!): $\omega = \sqrt{\frac{k}{m}}$. In this case, this gives $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{7\,\frac{\text{Newtons}}{\text{meter}}}{.03\,\text{kilograms}}} = 15.28\,Hz$. (This is the angular frequency.)

8. A molecule of carbon monoxide vibrates as though the two atoms in it were held together by a spring. The spring constant in this situation is $k = 465.5\frac{\text{Newtons}}{\text{meter}}$. Find the <u>circular</u> frequency of the vibrations of this molecule. Note: Because the two masses move together rather than one mass moving while the other end of the spring stays fixed (as in our usual setup) you must use the "reduced mass" for this system. This is $\mu = \frac{m_1 m_2}{m_1 + m_2}$ where the masses are those of the carbon and oxygen atoms, respectively. Just use μ anywhere you would normally use *m* in this calculation.

This is basically just a one-liner except for the need to convert masses. I just wanted you to see that masses on springs aren't limited to the contrived environment of classroom demonstrations. The numbers I've used are real-world quantities. We once again use $\omega = \sqrt{\frac{k}{m}}$, however we want the circular frequency rather than the angular frequency. (The people who study molecular dynamics use circular frequency, by convention.) Also, we have to substitute μ for *m*, as stated in the problem. Thus, we have $\omega_{circular} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$.

Now, let's figure out the reduced mass. The mass of a carbon atom is 12 amus (atomic mass units) while that of an oxygen atom is 16 amus. This gives a reduced mass of $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{16 \times 12}{16 + 12} = \frac{48}{7} = 6.86\, amu$. Converting, we have $\mu = 6.86\, amu \times 1.66 \times 10^{-27}\, \frac{kg}{amu} = 1.14 \times 10^{-26}\, kg$. This gives,

$$\omega_{circular} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}} = \frac{1}{2\pi}\sqrt{\frac{465.5\frac{\text{Newtons}}{\text{meter}}}{1.14 \times 10^{-26}\ \text{kg}}} = 3.22 \times 10^{13}\ Hz\,.$$



9. A 2 kg mass rests on a frictionless, horizontal surface. It is attached to a spring connected to a wall, as shown above. It is pulled 11 cm from its equilibrium position before being released. The spring constant is $k = 15\frac{\text{Newtons}}{\text{meter}}$. What is the speed of the mass as it passes through the equilibrium position?

There are two ways of doing this. The easiest is to use conservation of energy, but I haven't given you the tools to use that yet. So we'll just use the formula derived in class. I'll do it using conservation of energy next, just to show you how it's done.

In class, we showed that the speed of an object oscillating under a Hooke's Law type force (Simple Harmonic Motion) is given by $v = -A\omega\sin(\omega t + \phi)$. Now, we could go through a lot of work to find out what the time is in this situation. Or, we could be clever. I like being clever! The quantity in parenthesis, $(\omega t + \phi)$, is called the "argument" of the sine function. Sine doesn't care how the argument got to any particular value—is time zero and there's some phase, or is phase zero and there's some time, or is there some combination of phase and time? It doesn't matter. If we look at the equation for the *position* of the mass, $x = A\cos(\omega t + \phi)$, we see that the mass will pass through the equilibrium position ($x = 0$) when the argument is equal to $\frac{\pi}{2}$. (Or $\frac{3\pi}{2}, \frac{5\pi}{2} \ldots$, it repeats every π radians.) So all we need to do is say $(\omega t + \phi) = \frac{\pi}{2}$ and then find that $v = -A\omega\sin(\omega t + \phi) = -A\omega$ at that instant. Now all we need to do is find A and ω.

Since we know that the mass was initially pulled back 11 cm, we can state that $A = 0.11m$. For the frequency, we just use the formula, $\omega = \sqrt{\frac{k}{m}}$. Inserting numbers into this, we get $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{15\frac{newtons}{meter}}{2\,kg}} = 2.74\frac{1}{s}$. Thus $v = 0.11m \times 2.74\frac{1}{s} = 0.30\frac{m}{s}$.

Using conservation of energy, this is essentially identical to the problems we did with gravity when a mass was dropped from some height. The biggest difference is that the potential energy is represented by a different equation. Let's do it in detail, however.

Our conservation of energy expression is (as always) $P.E.+K.E.=constant$. Our mission is to determine the constant. Any problem that can be solved using conservation of energy will have *some* point at which we know enough to determine the constant. In this case, we can find this from the initial condition: The spring was initially stretched a known distance. Since the potential energy stored in a spring compressed or stretched by a length *x* is given by $P.E.=\frac{1}{2}kx^2$ and we know both *k* and *x*, we can find the potential energy at the beginning. When the spring is initially stretched, its speed is zero, so there is no kinetic energy at that point. So our constant is just the initial potential energy. Thus, we have at *any* time $P.E.+K.E.=\frac{1}{2}kx_{original}^2$.

We could easily use the equation that we just found to determine the speed of the mass at any position in its oscillation. This is an incredibly powerful technique! However, all we need is the speed at the instant the mass passes through the equilibrium position. At that point, the potential energy is zero. Why? Well, because the force is zero at that point. We have *selected* it to be the origin of our coordinate system. This is the point at which $x=0$. So our conservation of energy equation gives us $K.E._{equilibrium}=\frac{1}{2}kx_{original}^2$. Now, I warned you of a trap back when we were doing gravitational potential energy and that same trap is still present: The conservation of energy equation is $P.E.+K.E.=constant$ and this is the form you should use. Do *not* just write $\frac{1}{2}mv^2=\frac{1}{2}kx^2$! This happens to be true in this one, simple case. It is *not* true in general.

Here, we have $\frac{1}{2}mv_{equilibrium}^2=\frac{1}{2}kx_{original}^2$. And now it's just a bit of algebra to get $v_{equilibrium}=\sqrt{\frac{kx_{original}^2}{m}}$. Inserting numbers, we have $v_{equilibrium}=\sqrt{\frac{15\frac{N}{m}\times(0.11m)^2}{2kg}}=0.30\frac{m}{s}$, as before.

Problems #10 and #11 have been moved to the next assignment