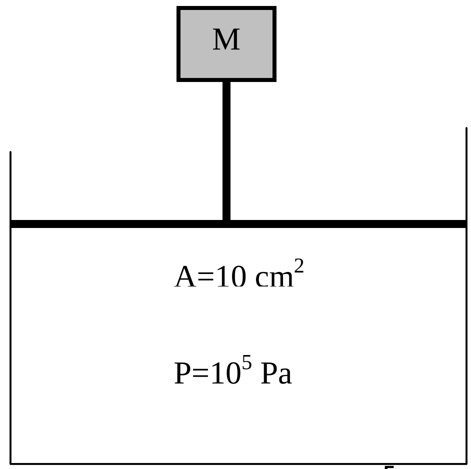
PHYSICS 206a
HOMEWORK #12
SOLUTIONS



1. A sample of gas has a pressure of 10^5 Pascals. (By the way: The atmospheric pressure at sea level is 101,325 Pascals.) If this gas is held in a cylinder that is capped with a piston with an area of 10 cm^2, as shown, what mass can be placed on the piston so that it is just supported by the gas? (Assume that the piston and cylinder are housed in a vacuum—i.e., ignore the pressure of surrounding air.)

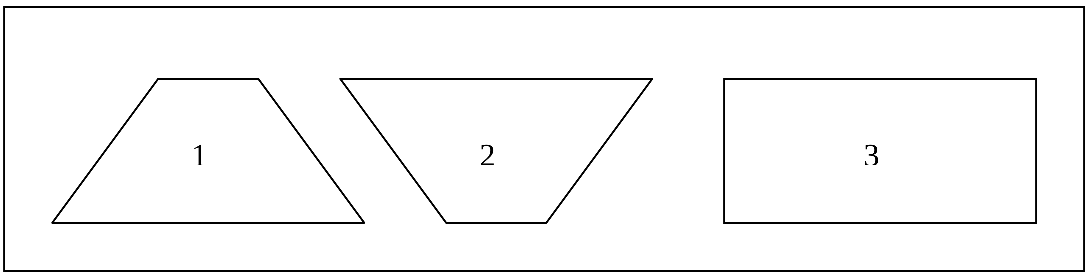
Since the mass isn't accelerating, the net force on it is zero. There are two forces that we can identify acting on the mass (draw a free body diagram to confirm this for yourself!): The force of gravity pulling it down and the force of the piston pushing it up. The force exerted by the piston comes from the gas in the cylinder. We know the pressure of the gas and its area, so the total force exerted on the piston by the gas is $F = PA$ from the definition of pressure ($P = \frac{F}{A}$). Setting these equal to each other, we have $mg = PA$ so

$$m = \frac{PA}{g} = \frac{10^5 \frac{\text{Newtons}}{\text{meter}^2} \times 10^{-3} \text{ meters}^2}{9.8 \frac{\text{meters}}{\text{second}^2}} = 10.2 \text{ kilograms}.$$

2. **Now, again consider the system in the previous question. Take the pressure of the surrounding air to be 101,325 Pa. If the mass to be supported is 13 kg, what pressure of gas must be used to fill the cylinder?**

Now the force on the piston pushing down is the combination of the weight of the mass and the pressure of the atmosphere times the area of the piston. If we put enough gas into the cylinder to create a pressure exactly equal to that of the atmosphere on the outside, we'll basically be back to the previous problem. So our pressure *inside* can be written $P = P_{atmosphere} + \frac{mg}{A}$. This can readily be solved to give

$$P = P_{atmosphere} + \frac{mg}{A} = 101325\ Pa + \frac{13\,\text{kilograms} \times 9.8\,\frac{\text{meters}}{\text{second}^2}}{10^{-3}\ \text{meters}^2} = 2.287 \times 10^5\ \text{Pascals}.$$



3. **The three vessels pictured above are all filled with water and are all sealed. Which one has the highest pressure at the bottom? Which one has the highest pressure at the top? Explain your answers!**

The pressure at the same height of each of the containers is exactly the same! According to Pascal's principle, the pressure at a given depth in an incompressible fluid in a sealed container is independent of the shape of the container or the location at that depth.

4. **Molecules of N_2 (nitrogen) in the air have an average speed of $500\,\frac{\text{meters}}{\text{second}}$ at "room temperature" (about 300 K). (You will be calculating this value for yourselves in a later problem.) Consider a container holding nitrogen at 1 atmosphere of pressure. Assuming their collisions with the walls of the container are perfectly elastic, use the "impulse-momentum theorem" to determine the average number of molecules which impact a 1 cm^2 region of the wall of the container in one second. (Recall that the impulse-momentum theorem says that the total change in momentum in some interval of time is equal to the average force exerted multiplied by the time interval over which that force is exerted.)**

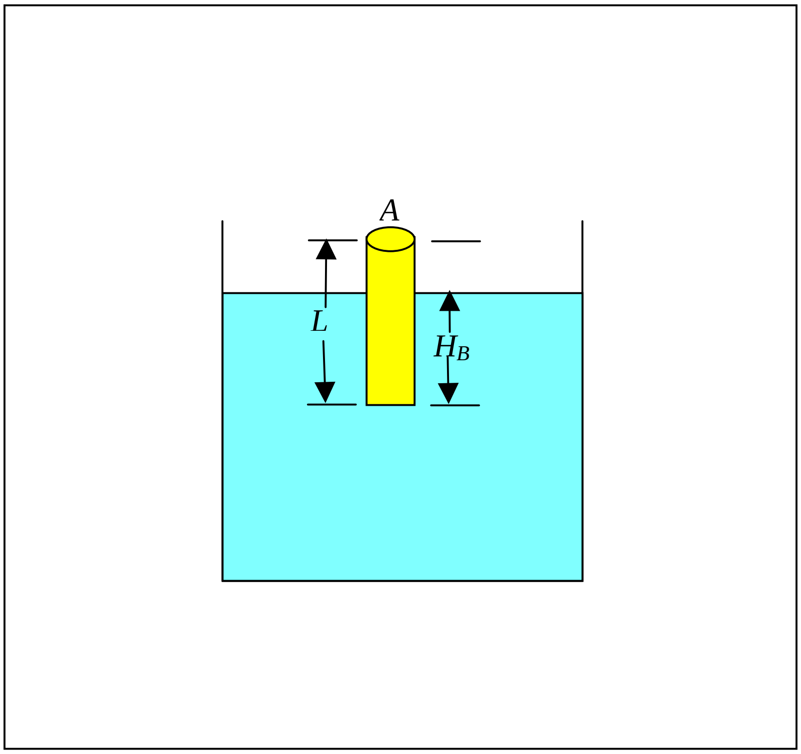
Way back in assignment #7 we discussed the momentum change of an object which bounces off a wall. We found that, for elastic collisions in one dimension, the change in the momentum of the object is *twice* the incoming momentum $\Delta p = 2p_{in}$. According to the impulse-momentum theorem, $\Delta p = Ft$. That is, the change in momentum of the molecule is the force exerted on it by the wall multiplied by the time over which that force acts. Now, if we were interested in a single collision, the time would be extremely short—probably less than 10^{-15} seconds or so. But we're *not* interested in a single collision. We are interested in the average force over a very large number of collisions occurring over a relatively large area. If we consider not one molecule but some large number *N* of molecules and ask for the average force on them, we can use the impulse-momentum theorem to write $N\Delta\overline{p} = Ft$ where $\Delta\overline{p}$ is the average change in momentum of the molecules. Now, *t* is not the time of one collision but the time of the *N* collisions—one second, in this case.

Since we know the pressure and the area, we can solve for the force. Likewise, we know the average speed of the molecules and their mass, so we can solve for the average change in momentum of the molecules. So, we can write

$$N = \frac{F}{\Delta\overline{p}}t = \frac{P \times A}{2mv}t = \frac{101325\,Pa \times 10^{-4}\ m^2 \times 1s}{2 \times 28\,amu \times 1.66 \times 10^{-27}\ \frac{kg}{amu} \times 500\,\frac{m}{s}} = 2.2 \times 10^{23}\ .$$

So there will be approximately 10^{23} collisions per second of nitrogen molecules with the 1 cm^2 section of the walls! It is the fact that this number is so huge that allows us to take the average without too much concern. The size of the region we're looking at or the time over which we're looking must be very tiny before the averaging doesn't give a very good answer.

5. A cylinder of wood is 17 cm long with a diameter of 3 cm. It has a density of $.7\dfrac{\text{gram}}{\text{cm}^3}$. If the piece of wood is placed in water and its long axis is oriented vertically, how high above the surface of the water is the top of the cylinder?



According to Archimedes' principle, the buoyant force on the stick will be equal to the weight of the water displaced by the stick. Let's think about this for a second. What this means is that the water will provide an upward force on the stick of some amount, call it F_B. Gravity pulls down on the stick with a weight $W = mg$. When we place the stick into the water, the upward force will grow as more and more of the stick becomes submerged. The water which used to occupy the volume now occupied by the stick is the "displaced water." It has some weight. Since the stick is less dense than the water, a volume of water less than the total volume of the stick will weigh the same as the stick. Read that last sentence through again. Make sure you understand it—it is the key sentence in this analysis. As we lower the stick into the water, the buoyant force will increase, according to Archimedes' principle. At some point, the weight of the displaced water will be the same as the weight of the stick. This is the point at which the stick will be able to float. We *could* lower the stick even further, but then we'd have to push down actively to get it to stay under. The buoyant force would be *greater* than the weight of the stick beyond the floating point. We're not interested in that situation now, however. If we just let go of the stick, the stick will float and the buoyant force will be identical to the weight of the stick.

The volume of a cylinder is $V = \pi r^2 h$ where h is the length (height) of the cylinder. This can also be written $V = \pi r^2 h = Ah$, where A is the cross sectional area of the cylinder ($A = \pi r^2$). Thus, the total weight of the cylinder of wood is $W_{wood} = \rho_{wood} LAg$ where ρ_{wood} is the density of the wood and g is the strength of gravity at the surface of the Earth, as usual. The weight of the displaced water is $W_{water} = \rho_{water} H_B Ag$ where ρ_{water} is the density of the water and H_B is the height below water of the cylinder. When the cylinder is floating, these two are equal to each other, so we have $\rho_{water} H_B Ag = \rho_{wood} LAg$. A bit of algebra gives $H_B = \frac{\rho_{wood}}{\rho_{water}} L$ or, using the numbers given, $H_B = 0.7L = 0.7 \times 17\,cm = 11.9\,cm$.

Of course, the question asked for the height *above* water of the top of the cylinder, so we just subtract the height below water from 17 cm and get $H_A = 5.1cm$. (That's just a detail. If you missed that aspect on an exam I probably wouldn't dock you anything.)

6. **Now, the cylinder in the previous problem is placed in a tank of oil, with a density of $0.8 \frac{\text{gram}}{\text{cm}^3}$. How high above the oil will the top of the cylinder be found in this case?**

We can just use the equation found in the previous problem with the density of the oil substituted for the density of the water: $H_B = \frac{\rho_{wood}}{\rho_{oil}} L$. This gives $H_B = \frac{0.7}{0.8} \times 17\,cm = 14.875\,cm$ for a height above water of $H_A = 2.125\,cm$.

7. **A cube is to be built of steel sheets. (The cube will be hollow and filled with air when finished.) The density of the steel is $\rho = 7.9 \frac{\text{gm}}{\text{cm}^3}$. The sheets of steel are ½ cm thick. What is the minimum width of the cube (i.e., the length of one side) such that the cube will float in water?**

Ships are routinely built of steel, we know this. But we also know that steel is far more dense than water. A chunk of steel placed in water *will* sink. So how does a steel ship manage to stay afloat? Well, Archimedes' principle says that the buoyant force experienced by an object will be equal to the weight of the fluid displaced by the object. If we can take steel and ensure that it displaces a greater volume than of the steel itself, the buoyant force can be made arbitrarily large.

Consider the cube in this problem. The cube is sealed. It is *hollow*. So the volume of the cube is significantly different than the volume of the steel sheets that are used to make it. By making the cube big enough, we can make it float.

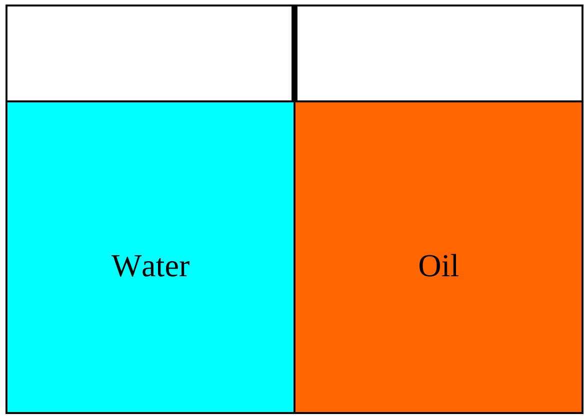
Let's begin by finding the volume of the cube. Let's call the width of the cube L (this is the quantity that we are looking for). The volume of the cube will be $V = L^3$. Now, here's the key step in the whole problem: We want to make sure the cube floats. What does the word "float" mean in this context? It means that the buoyant force experienced by the object is exactly equal to the weight of the object. Let's imagine that we've got the cube sitting in some water, completely immersed. If we were to make the cube bigger, it would displace more water and the buoyant force would increase. If we were to make the cube smaller, the buoyant force would decrease. Note that this has nothing to do with the weight of the cube! The buoyant force only depends on the volume of the cube and the weight of the water, not on the weight of the cube. Now, we want to make the cube exactly the right size so that the buoyant force is equal to the cube's weight. So, using Archimedes' principle, $L^3 \rho_{water} g = m_{cube} g$. We're stuck unless we can find the mass of the cube.

The cube is made out of sheets of steel. They're all identical and we'll need six of them to make the cube. So the mass of the cube is equal to six times the mass of one side. We can say $m_{cube} = 6 \times m_{side}$. (Let's ignore the mass of the air inside the cube. It will have only a tiny effect.) What is the mass of a side? Once again, this is the volume of one side of the cube times the density of the steel, which you were given. If we can find the volume of the side, we'll be done.

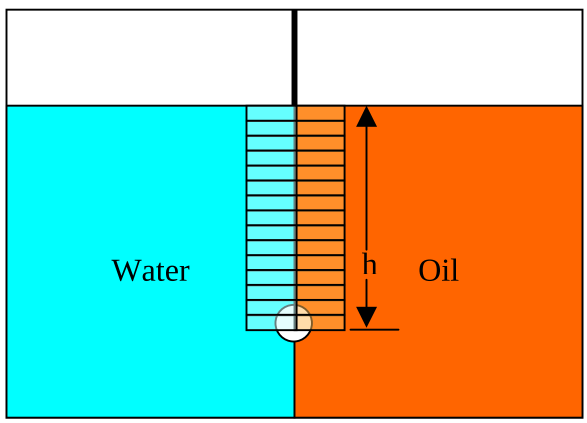
The side of the cube is a square with some thickness. We are told the thickness, but let's just call it t for now. The volume of a side of the cube is just $V_{side} = L^2 t$ so the mass of the side is $m_{side} = \rho_{steel} L^2 t$. This allows us to find the mass of the entire cube: $m_{cube} = 6 \times m_{side} = 6 \times \rho_{steel} L^2 t$.

Finally, we can insert this back into our original expression to get $L^3 \rho_{water} g = m_{cube} g = 6 \rho_{steel} L^2 t g$. A bit of algebra gives

$$L = \frac{6 \rho_{steel} t}{\rho_{water}} = \frac{6 \times 7.9 \frac{\text{gm}}{\text{cm}^3} \times \frac{1}{2}\text{cm}}{1.0 \frac{\text{gm}}{\text{cm}^3}} = 23.7\text{cm}.$$



8. **A tank of oil and a tank of water are separated by a single wall, as shown. If a hole is drilled in the wall separating the tanks, which direction will fluid flow? I.e., will the water get into the oil tank or will the oil get into the water tank, or will they both stay put? Explain!**



To understand this, we must understand *why* there is a pressure at a certain depth in a fluid. Imagine that you've divided a column of fluid into layers, as shown in the picture. The top layer supports only its own weight, so the pressure in that layer is very small—let's call it zero, roughly (we could make this exact by using techniques that are very familiar to those of you who have had calculus). The next layer down supports the top layer, so the pressure is somewhat higher. The layer below that must support the first two layers and so the pressure in that layer is higher still. Now, continue this process all the way down a depth *h*. At the bottom of the column, the total force being exerted by the lowest layer is the weight of the full column. This force is $W = \rho A h$. Since pressure is force divided by the area over which the force is acting, we can say $P = \frac{W}{A} = \rho h$. Thus the fluid with the higher density will have the greater pressure at a given depth. Water has a higher density than oil. (This is a common source of error: Oil is more "viscous" than water. That is, it's thicker and more difficult to move through. So it feels heavier than water: If you stick your hand in a bucket of oil, more oil will stick to it when you pull it out than if the bucket had been filled with water. We interpret this to mean that the stuff itself is heavier, even though the perceived heaviness is just due to more stickiness. Oil floats on water—as many Alaskan seabirds and seals can attest thanks to the noble Physics demonstration performed by the Exxon corporation in 1989. This is because it is *less* dense than water, despite being more viscous.) Therefore, a column of oil will weigh less than an identical column of water.

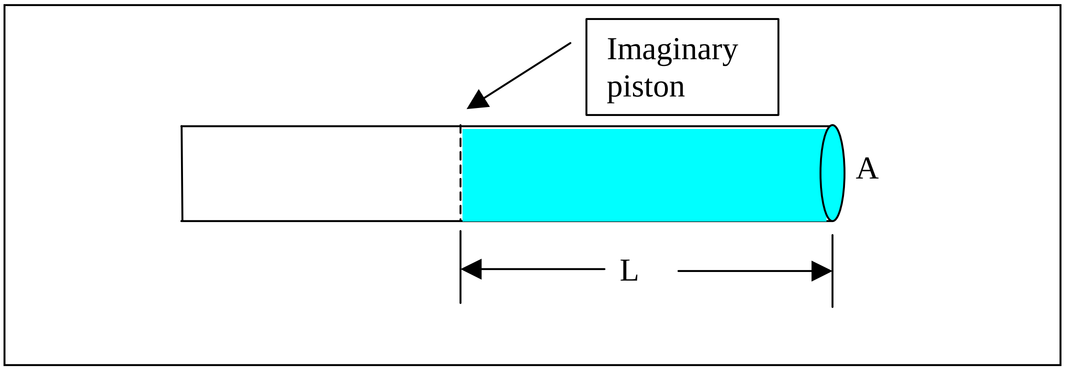
If a hole were drilled in the wall between the tanks, water would flow from its tank into the oil. This would continue until the level of the liquid in the oil side had risen far enough and the level in the water side had fallen far enough that the pressures were the same on both sides.

Personal history note: Before coming to SIUE, I worked with a device that was held in a large tank that looked very much like the one in the picture above. The water side held 15,000 gallons while the oil side held 8,000 gallons. We had to be very careful that, when emptying or filling one of the fluids, we'd empty or fill the other tank at the same rate. Otherwise, a pressure difference between the two could have resulted in the wall between them breaking!

9. **Again consider the tanks in the previous problem. Assume the hole is drilled 2 meters above the bottom of the tanks. What depths, h_w and h_o (for "water" and "oil" respectively), must the tanks be filled to without any flow occurring?**

Note that the 2 meter datum is something of a red herring: The only distance that matters in doing the pressure calculation is the height above the hole. The 2 meter height *is* a bit important, though: As water flows into the oil tank, it will sink to the bottom. If so much water flows into the oil tank that it actually pools to a depth that takes it above the hole, calculating the pressure becomes a bit more difficult. So let's assume that all the water that makes it into the oil tank just sinks to a level below the hole. There will be no flow when the two pressures are equal. Based on the results of the previous few questions, this will occur when $\rho_w h_w = \rho_o h_o$. Thus, we need $h_w = \frac{\rho_o}{\rho_w} h_o$ to prevent flow.

10. 2 gallons of water per minute flow through a pipe with a diameter of 5/8 inch. What is the speed at which the water is flowing?



The key to solving this problem is to visualize it correctly. The way that I find works best is to imagine the pipe as having a piston in it that is pushing the liquid out, as shown above. In some sense this is true: The water acts as its own piston. The crucial thing is to recognize that a volume of pipe equal to 2 gallons empties out in one minute. This volume will be the area of the pipe, A, times the length of the pipe which empties in one minute, L. This is $\mathrm{V} = \mathrm{AL}$.

Now, since the speed of the liquid is the same as the speed of our imaginary piston, we realize that the piston will move a distance L in a time *t*. This means that the piston's speed is $v = \frac{\mathrm{L}}{t}$. But $\mathrm{L} = \frac{\mathrm{V}}{\mathrm{A}}$ so $v = \frac{\mathrm{L}}{t} = \frac{\mathrm{V}}{\mathrm{A}t}$. The volume is 2 gallons and the time under consideration is one minute. All that remains is to find the area.

Here's where the metric system comes in very handy. In the English (or "Imperial") system which includes inches and gallons, there is no clear relationship between lengths and volumes. How many cubic inches are there in a gallon, for example? It's a tough problem! The metric system has a fixed relationship between volumes and lengths. Since a volume is always created by a product of three lengths, the unit for a volume is just a product of the unit for length cubed. Finding the area of the pipe in square inches while knowing the volume in gallons just wouldn't help much in finding the length emptied. On the other hand, knowing the area in square centimeters and knowing the volume in liters (1000 cubic centimeters in one liter) would give us the length immediately. So let's convert the diameter to centimeters and the volume to liters.

There are 2.54 centimeters in one inch. So our radius is $r = \frac{Diameter}{2} = \frac{\frac{5}{8}\,\mathrm{inch} \times 2.54\,\frac{\mathrm{centimeter}}{\mathrm{inch}}}{2} = 0.79375\,\mathrm{centimeter}$. This gives an area of $\mathrm{A} = \pi r^2 = 1.98\,cm^2$.

Gallons to liters is best just looked up in a table somewhere (check the internet). There are 3.785 liters in one gallon. So the volume that passes out of our pipe in one minute is $\mathrm{V} = 2 \times 3.785\,\text{liters} = 7.57\,\text{liters}$.

Putting all these together (and using the fact that there are sixty seconds in one minute), we have

$$v = \frac{\mathrm{L}}{t} = \frac{\mathrm{V}}{\mathrm{A}t} = \frac{7.57 \times 10^3\,cm^3}{1.98\,cm^2 \times 60\,s} = 63.7\,\frac{\text{centimeter}}{\text{second}} = 0.637\,\frac{\text{meter}}{\text{second}}.$$

11. If a connector is added to the pipe in the previous problem which increases the diameter to 7/8 inch, at what speed will the water flow through the enlarged section? Which section will have a higher pressure: The large one or the small one?

For an incompressible fluid, the continuity equation says that the product of area and speed is conserved. That is, in any two regions of a pipe $A_1 v_1 = A_2 v_2$. Thus, to find the speed in another region if we know the speed in one region we just need to divide to get $v_2 = \frac{A_1 v_1}{A_2}$. This gives

$$\begin{aligned} v_2 &= \frac{A_1 v_1}{A_2} = .637\,\frac{\text{meter}}{\text{second}} \times \frac{\pi r_1^2}{\pi r_2^2} = .637\,\frac{\text{meter}}{\text{second}} \times \frac{r_1^2}{r_2^2} \\ &= 0.637\,\frac{\text{meter}}{\text{second}} \times \frac{\left(\frac{5}{8}\right)^2}{\left(\frac{7}{8}\right)^2} = 0.637\,\frac{\text{meter}}{\text{second}} \times \frac{25}{49} = 0.325\,\frac{\text{meter}}{\text{second}} \end{aligned}.$$

The pressure will be *higher* in the section in which the area is higher and where the speed is lower.

12. A scientist has a one-liter container of oxygen and two one-liter containers of hydrogen. All three containers are at the same pressure which is 15 pounds per square inch. They are also at the same temperature which is 400 Kelvin (quite hot). He mixes all three containers together and, after he recovers from the injuries he sustains in the ensuing explosion, he repeats the experiment using a stronger container. He finds that he has a container full of water vapor at the end. If the water vapor is allowed to cool to 400 Kelvin and the size of the container is adjusted (perhaps by means of a piston) so that the pressure of the water vapor is 15 pounds per square inch, what will be the volume of the container of water vapor? (This sounds more complicated than it actually is. Try drawing some pictures to help yourself understand what's going on.)

(This is almost precisely a restatement of the experiments of an early chemist named Joseph Louis Gay-Lussac, which were pivotal for the acceptance of the notion of atoms.) The key concept here is that we can ignore the volume of a gas that's actually taken-up by the atoms or molecules of which the gas is composed because the molecules themselves are so tiny compared to the average distance between them. That is, if I have two gases, one composed of atoms and the other of molecules that are several times the size of the atoms, the fact that the molecules are bigger makes no difference at all. The volume of a certain number of atoms or molecules of a gas at a given pressure and temperature is independent of the size of those atoms or molecules. What's important is the distance between them.

In the experiment described, the scientist starts with three equal volumes of individual particles: Molecules of either Hydrogen or Oxygen. They are all at the same temperature and pressure so they must each have the same number of molecules. That's important. All three samples have the same volume, the same pressure, and are at the same temperature, so they must each have the same number of molecules. It doesn't matter that Oxygen molecules are somewhat larger than Hydrogen molecules.

So, the scientist starts out with a total of three volumes of "stuff." He mixes these together and the Hydrogens and Oxygens combine in the ratio of two Hydrogens to each Oxygen to form water. Now what the scientist has is not three times some particular number of atoms, but one times that number of molecules since each molecule is made out of three of the original atoms. But the size of the molecule doesn't matter. All that matters is the fact that the scientist now has the same number of molecules, in total, as he originally had of molecules in any one of the original containers. The pressure is the same, as is the temperature, so the volume must be the same as **one** of the original three containers.

13. If I have a sample of a gas at 310 K at a pressure of 100,000 Pa, what is the volume, V, it occupies?

From the ideal gas law $PV = Nk_BT$ we get $V = \frac{Nk_BT}{P}$. I actually intended to tell you how many molecules there were in the sample, but this somehow didn't make it into the posted assignment! Sorry. The solution is simply the equation above with numbers plugged in, but, lacking the number of molecules, we just can't take it any further.

For completeness, however, let's assume that we have one mole of molecules. This is $N = 6.02 \times 10^{23}$ molecules. Inserting numbers, we have

$$V = \frac{Nk_BT}{P} = \frac{6.02 \times 10^{23} \times 1.38 \times 10^{-23} \frac{\text{joule}}{\text{kelvin}} \times 310\,\text{kelvin}}{1 \times 10^{5} \frac{\text{newton}}{\text{meter}^2}}.$$

$$= 2.57 \times 10^{-2}\, m^3 = 25.7\,\text{liters}$$