PHYSICS 206a
HOMEWORK #11
SOLUTIONS

1. Using Newton's law of gravitation, derive the acceleration due to gravity experienced by an object of mass *m* at the surface of the Earth. Show that this acceleration is independent of the mass of the object.

Newton's law of gravitation says that the gravitational force experienced by an object of mass m_2 due to an object of mass m_1 is given by $F = G\dfrac{m_1 m_2}{r^2}$ where G is a universal constant and r is the distance between the centers of the masses. The force is always attractive and acts in the direction of a line drawn between the centers of the masses. You must memorize this! (Not just the formula—all the rest of it as well.) One very common error is to see the letter "*r*" in that equation and to assume that it is the radius of something. (Just one danger of formula hunting!) This is by no means necessarily the case! *r* is the distance between the centers of the two masses. That's it. Now, if we're talking about the force experienced by an object sitting on the surface of the Earth, then, indeed, *r* is the distance from the center of the Earth to its surface—its radius. But don't just assume that *r* is a radius!

Calling the mass of the Earth M_e and the mass of the object *m*, we get $F = G\dfrac{M_e m}{r^2}$. According to Newton's second law, the object will experience an acceleration due to this force if it is the only external force acting on the object. The relationship between the acceleration and the force is $F = ma$. Thus, we can write $G\dfrac{M_e m}{r^2} = ma$. A tiny bit of algebra gives $G\dfrac{M_e}{r^2} = a$. Thus the acceleration due to gravity is independent of the mass of the object being accelerated—as was theorized by Galileo.

Substituting in numbers for the surface of the Earth, we get $a = G\dfrac{M_e}{r^2} = 6.67 \times 10^{-11}\dfrac{\text{Newton} \cdot \text{meter}^2}{\text{kilogram}^2} \times \dfrac{5.98 \times 10^{24}\ kg}{\left(6.378 \times 10^6\ \text{meter}\right)^2} = 9.8\dfrac{\text{meters}}{\text{second}^2}$.
Obviously, this is the number we've been using all semester. Now you know where it comes from!

2. If the radius of the Earth changed and became ½ of its current value (but the mass stayed the same), what answer would you get for the previous problem?

I'm afraid the wording of this problem was a little ambiguous. What I *meant* was that the radius changes *and* that the object remained on the surface. If the

radius changes but the object stays at the old radius, *nothing* changes. If the radius changes and the object remains on the surface, the ratio of the old acceleration to the new acceleration will be given by $\dfrac{a_{old}}{a_{new}} = \dfrac{G\dfrac{M_e}{r_{old}^2}}{G\dfrac{M_e}{r_{new}^2}} = \dfrac{r_{new}^2}{r_{old}^2} = \dfrac{\left(\frac{1}{2}r_{old}\right)^2}{r_{old}^2} = \dfrac{1}{4}$. (How many of you forgot to square the ½?) So the new acceleration would be four times as great as the old acceleration—we'd have $g = 39.2\dfrac{\text{meters}}{\text{second}^2}$.

3. **Determine the acceleration due to gravity for an object on the surface of the Moon.**

   Here we need only repeat the analysis we did in problem #2 only using the numbers for the mass and radius of the Mooon: $a = G\dfrac{M_m}{r_m^2} = 6.67 \times 10^{-11}\dfrac{\text{Newton} \cdot \text{meter}^2}{\text{kilogram}^2} \times \dfrac{7.35 \times 10^{22}\,kg}{\left(1.737 \times 10^6\,\text{meter}\right)^2} = 1.6\dfrac{\text{meters}}{\text{second}^2}$, roughly 1/6 of the acceleration on the surface of the Earth.

4. **The "Clarke orbit" is named for science fiction writer Arthur C. Clarke. In 1945, Clarke suggested that there was an altitude above the surface of the Earth at which satellites would have the same angular speed as the Earth (*Wireless World*, October 1945, pages 305-308). Thus, a satellite placed in such an orbit would be "geosynchronous." That is, it would stay suspended over a single point on the globe. By setting the force of gravity equal to the centripetal force, find the radius of such an orbit. (Careful: Don't confuse the distance from the surface of the Earth with the radius of the orbit! They differ by the radius of the Earth.) There are currently over 300 satellites occupying such orbits!**

   Recall our discussion about centripetal acceleration: To say that an acceleration is centripetal is simply to say what direction it is. The force providing that acceleration can come from any appropriate source. In this case, the force of gravity is being used to create the centripetal acceleration. Since we do not know, initially, the distance from the center of the Earth to such an orbit, we *cannot* assume that the acceleration will be anything like what it is at the surface of the Earth—in fact, they're quite different. (Recall Newton's original derivation of the centripetal acceleration of the Moon.) We simply need to take the expression for the acceleration due to gravity and the needed centripetal acceleration and set them equal to each other.

   We have $G\dfrac{M_e}{r^2} = \dfrac{v^2}{r}$. Now, we use the angular speed relation $v = \omega r$ to rewrite this as $G\dfrac{M_e}{r^2} = r\omega^2$. This gives $G\dfrac{M_e}{\omega^2} = r^3$. Now, the definition of the angular speed is the angle traveled divided by the time it takes to travel it $\omega = \dfrac{\Delta\theta}{t}$. The requirement for

our satellite to be geosynchronous is that it travel through one full circle in one day, so $\Delta\theta = 2\pi$ and $t = 1\,\text{day} = 86400\,\text{seconds}$. This gives $G\dfrac{M_e}{4\pi^2}t^2 = r^3$. (See the next problem for some interesting insight into this intermediate result.)

Solving for r, we have $r = \sqrt[3]{G\dfrac{M_e}{4\pi^2}t^2}$. Substituting numbers, we get

$$r = \sqrt[3]{G\frac{M_e}{4\pi^2}t^2} = \sqrt[3]{6.67\times10^{-11}\,\frac{\text{Newton}\cdot\text{meter}^2}{\text{kilogram}^2} \times \frac{5.98\times10^{24}\,\text{kilogram}}{39.48} \times (86400\,\text{seconds})^2}$$

$$= \sqrt[3]{7.54\times10^{22}\,\text{meter}^3} = 4.22\times10^7\,\text{meters}.$$

This is 35,800 km from the surface of the Earth. In English units, this is about 26,000 miles from the Earth's center or 22,300 miles above the surface of the Earth.

These satellites are used primarily for weather and communications. If you've got a satellite antenna for your television, the reason you are able to keep your dish antenna pointed in only one direction is that the satellite it is receiving a signal from is in a Clarke orbit. If you go out at night and look in the direction the antenna is pointed, you very well might be able to see the satellite! Of course, it will just look like a dim star, but it will remain in one position as the background stars move across the sky in the course of the night.

5. **Generalize the result you found in the previous problem: By using gravity as the thing providing an orbiting object with its needed centripetal acceleration, prove Kepler's third law. What is the constant of proportionality? I.e., Kepler's third law says that $T^2 = kR^3$ where *T* is the period of the orbit, *R* is the mean radius of the orbit, and *k* is some constant. Find *k*. You should assume the orbit is circular.**

Note the equation we got in the previous problem along the way to the final answer. At one point we found that $G\dfrac{M_e}{4\pi^2}t^2 = r^3$. This was found simply by assuming that an object is traveling in a circular path and that the centripetal force needed to keep it in that path is provided by gravity. Thus, this equation is true in general—it is true for any object in an "orbit," not just satellites of the Earth and not just those in geosynchronous orbits. A small manipulation gives $t^2 = \dfrac{4\pi^2}{GM}r^3$. Note that I've dropped the subscript on the mass since this is true for any object being orbited, not just the Earth. For a different object (e.g., the sun) we'd just substitute the appropriate mass. The entire factor $\dfrac{4\pi^2}{GM}$ is a constant for a given system, so we can replace it by a single symbol for simplicity. This gives $t^2 = kr^3$ where $k = \dfrac{4\pi^2}{GM}$. This is what was to be found. (If you really want a number, just substitute the appropriate values for a given system in this equation.)

6. **Some years ago, a (crackpot) theory circulated which predicted doom for the Earth due to the gravitational effect of the planets all lining up. This effect was known as "The Jupiter Effect"—which was the title of the book in which this prediction was made. The Earth did not, in fact, suffer any of the ill effects predicted. Calculate the force of gravity exerted by Jupiter on a person on the surface of the Earth when the two are closest. Jupiter's mean distance from the Sun is $7.79 \times 10^{11}\,m$ and its mass is $1.9 \times 10^{24}\,kg$. The Earth's mean distance from the sun is $1.49 \times 10^{11}\,m$. What mass of object would have a weight on the surface of the Earth equivalent to the force of gravity of Jupiter on this person?**

The force of gravity exerted on one object by another depends only on the two masses and the distance between their centers. Let's call the mass of the person m_p and the mass of Jupiter m_J. I didn't give you the Earth-Jupiter distance, but I gave you the distance from each of these to the sun. The distance from Jupiter to the Earth is just the distance from Jupiter to the sun minus the distance from the Earth to the sun. (This is *only* true when they are at "closest approach." At other times, more complicated geometry would be necessary.) I made a real booboo in this problem: I mistyped the mass of Jupiter! It should have been given as $1.9 \times 10^{27}\,kg$ —a factor of 1000 different from what I gave you. I do apologize for this. (Note that this is about 300 times the mass of the Earth!) I'll do the calculation using the correct numbers. Please take a look at your solution and think about how it would change with these numbers inserted appropriately. Let's take the mass of the person to be 100 kg (pick whatever value you think reasonable or just call it m).

We have $F = G \dfrac{m_J m_p}{\left(r_{Jupiter-Sun} - r_{Earth-sun}\right)^2}$. Substituting numbers, we have

$$F = G \frac{m_J m_p}{\left(r_{Jupiter-Sun} - r_{Earth-sun}\right)^2}$$
$$= 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \times \frac{1.9 \times 10^{27}\,kg \times 100\,kg}{\left(7.79 \times 10^{11}\,m - 1.49 \times 10^{11}\,m\right)^2}$$
$$= 3.19 \times 10^{-5}\,N$$

For an object to have this weight on Earth, it would need to have a mass given by $mg = 3.19 \times 10^{-5}\,N$, which yields $m = \dfrac{3.19 \times 10^{-5}\,N}{9.8 \dfrac{m}{s^2}} = 3.26 \times 10^{-6}\,kg$. This is the mass of a small grain of sand.

While this force *is* very small, notice that it is still large enough that it doesn't seem unreasonable to expect that we could measure it. Indeed, a measurement of the effect of this force is quite feasible. It's small, but not completely negligible! Jupiter (and all the other planets in the solar system) *does* have an effect on Earth. Usually, we can ignore it, but it would be important if we were doing calculations to predict, for example, long-term climatological effects.

7. **What is the difference in the force experienced by a cubic meter of water due to the Moon's gravitational pull when the Moon is on the same side of the Earth as the water and when the Moon is on the opposite side of the Earth as the water? (I.e., about 12 hours later.) What is the force of the Earth's gravity on that quantity of water (i.e., its weight)?**

Recall that tides are caused, primarily, by the gravitation of the moon pulling water away from the Earth. Of course, the Moon "loses" the competition, but it has a significant effect nonetheless. The Moon has an average distance from the center of the Earth of $r_m = 3.844 \times 10^8\,m$. The radius of the Earth is $r_E = 6.378 \times 10^6\,m$. (Notice that I've used a similar symbol, r, for two very different things here: Newton's law of gravitation specifies that we need the distance between the centers of the masses under consideration. In this case, I *also* need a radius in order to find that distance. Don't fall into the trap of thinking that r means "radius." In this context, it *might* mean radius. But it also might just mean the distance between the centers of a pair of objects. You will need to determine, from context, which of these is appropriate.) Thus, when the water is on the same side of the Earth as the Moon is we will have $R_{close} = r_m - r_E$. When the water is on the opposite side of the Earth from the Moon we will have $R_{far} = r_m + r_E$.

Using these expressions, the difference in force experienced by the water between the two locations is

$$\Delta F = GM_{moon} m_{water} \left(\frac{1}{R_{close}^2} - \frac{1}{R_{far}^2} \right)$$

$$= GM_{moon} m_{water} \left(\frac{1}{(r_m - r_E)^2} - \frac{1}{(r_m + r_E)^2} \right)$$

$$= GM_{moon} m_{water} \left(\frac{1}{1.43 \times 10^{17}\,m^2} - \frac{1}{1.53 \times 10^{17}\,m^2} \right)$$

Now, the mass of a cubic meter of water is 1000 kg. So we have

$$\Delta F = GM_{moon} m_{water} \left(\frac{1}{1.43 \times 10^{17}\,m^2} - \frac{1}{1.53 \times 10^{17}\,m^2} \right)$$

$$= 6.67 \times 10^{-11}\,\frac{N \cdot m^2}{kg^2} \times 7.35 \times 10^{22}\,kg \times 1000\,kg \times \left(\frac{1}{1.43 \times 10^{17}\,m^2} - \frac{1}{1.53 \times 10^{17}\,m^2} \right)$$

$$= 2.24 \times 10^{-3}\,N$$

This is roughly equivalent to adding one cubic centimeter of water to the sample with the Moon at its farthest location and removing one cubic centimeter of water from it when the Moon is at its closest approach.

For comparison, the force exerted by the Earth on our cubic meter of water is $W = mg = 1000 kg \times 9.8 = 9800 N$. So the Moon's effect is about 0.2 parts per million on the weight of the water. Still, this is sufficient to cause the tides which are essential to life on Earth!

An interesting side note to this is that the work done on the water, lifting and then dropping it, has a dissipative piece. There is energy lost every time a wave slaps on the shore. This energy is being extracted from the Moon's orbital energy (rotational kinetic energy). So the moon is slowly moving away from the surface of the Earth—each year, the moon is a few cm farther away than in the previous year. (This motion has been used to "prove" that the Earth cannot be as old as scientific measurements indicate it to be. Unfortunately, the folks who promulgate this "proof" made an arithmetic error and their numbers are all wrong as a result. I once found over 1,000 web sites quoting the erroneous result! Sometimes a little slip on your calculator can have far reaching effects.)

8. **A space shuttle has a mass of 80,000 kg. What is the difference in the rotational kinetic energy that needs to be given to the shuttle if it is launched from a point on the equator or the north pole in order to get it into a circular orbit? Assume the height of the orbit is 700 km above the surface of the Earth.**

This is more tricky than difficult. The first thing you need to realize is that the speed of an object in orbit is determined exclusively by the height of the orbit. All orbits at a given height take the same amount of time. This can be seen from the solution to the Clarke orbit problem and the one on Kepler's law. (I've caught some of you assuming that all orbits are "geosynchronous" like the Clarke orbit. This is not true!) As soon as we specify the height of the orbit, we have specified the gravitational acceleration. Since the gravitational acceleration is the centripetal acceleration, only one speed will allow an object to remain in orbit at a particular height. So the shuttle will have the same rotational kinetic energy once it gets *into* orbit regardless of whether it started at the pole or the equator. Any difference between the rotational kinetic energy needed to be given to the shuttle between the two launch locations is due to whatever differences in rotational kinetic energy they have while sitting on the ground.

Now, an object at a pole has virtually no rotational kinetic energy. Its rotational radius (the distance from the rotational axis to the mass) is nearly zero compared to what it is at the equator. However, an object at the equator has a fair amount of rotational kinetic energy: It's moving around in a circle with a radius of 6.378×10^6 meter (the radius of the Earth) once per day. So the shuttle's rotational kinetic energy when sitting at

$$K.E._{rotational} = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} m r^2 \omega^2$$

the equator is

$$= \tfrac{1}{2} \times 8 \times 10^4 \, kg \times \left(6.378 \times 10^6 \text{ meter}\right)^2 \times \left(\frac{2\pi}{86400 \text{ seconds}}\right)^2$$

$$8.6 \times 10^9 \text{ Joules}$$

So, a space shuttle sitting at the equator already has 8.6 billion Joules of the energy it needs to get into orbit. Every little bit helps! This is one of the reasons why launch pads are placed as close to the equator as possible.

9. It is frequently stated that astronauts are in "zero g" when in orbit around the Earth. What is the net force of gravity on a 80 kg astronaut in a 700 km orbit? Why does the astronaut float?

It is crucial that you understand that gravity doesn't get turned off! Neither does it get cancelled out. The gravitational force on astronauts in orbit is darn near the same as that which they experience at the surface of the Earth. To see this, let's use the result we obtained for the acceleration at the Earth's surface in problem #1:

$$a = G\frac{M_e}{r^2} = 6.67 \times 10^{-11}\frac{\text{Newton} \cdot \text{meter}^2}{\text{kilogram}^2} \times \frac{5.98 \times 10^{24}\ kg}{\left(6.378 \times 10^6\ \text{meter}\right)^2} = 9.8\frac{\text{meters}}{\text{second}^2}.$$

Now, add 700 km to the radius. This gives

$$a = G\frac{M_e}{r^2} = 6.67 \times 10^{-11}\frac{\text{Newton} \cdot \text{meter}^2}{\text{kilogram}^2} \times \frac{5.98 \times 10^{24}\ kg}{\left(7.078 \times 10^6\ \text{meter}\right)^2} = 7.96\frac{\text{meters}}{\text{second}^2}$$

a change of about 19%. Now, multiply by his/her mass to find the astronaut's weight: $W = m \times 7.96\frac{m}{s^2} = 80\,kg \times 7.96\frac{m}{s^2} = 636.9$ Newtons. Compare this to the 784 Newtons he/she would weigh on the surface of the Earth.

So why do these people float? It's because they have enough forward velocity that the acceleration toward the center of the Earth that they experience is simply sufficient to make their path curve. In fact, the forward velocity is carefully "tuned" so that their path just curves enough so that they move in a circle. (Actually, an ellipse, which may be quite eccentric. But we'll neglect the extreme cases for now.) The key thing is that *everything* in orbit at the same height is moving at the same speed and accelerating at the same rate. (Reread that one a couple of times.) So everything from the shuttle itself to the astronaut to tiny specs of dust in orbit at that height are moving at the same speed and accelerating at the same rate—they're all following the same path. So, although they might have very high speed and although they're accelerating almost as much as they would if they jumped off a building on Earth, none of them is moving *relative* to each other! They all seem to float as though there were no gravity at all.

Problems #10 and #11 have been moved to the next assignment.