PHYSICS 206a
HOMEWORK #10
SOLUTIONS

(Problems #11, 12, and 13 have been moved to the next assignment.)

3. Superman wants to stop the Earth from spinning.

> **a. Assuming a constant angular acceleration, what torque would he have to exert on it to stop it in one hour?**
>
> **b. If he exerts the torque by pushing on a point on the equator, what force does he have to exert?**
>
> **c. Rather than pushing on a point on the equator, he pushes on a point in Edwardsville, IL. Now what force does he have to exert?**

Angular acceleration, by analogy to regular acceleration, is the change in angular velocity divided by the time in which that change occurs. Mathematically, this is written $\vec{\alpha} = \frac{\Delta\vec{\omega}}{t}$. For this problem, we're unconcerned with the direction of the vectors, so let's treat them as scalars for simplicity. This allows us to discuss "speed" instead of "velocity." We're going from the angular speed found in problem #9 of assignment #9 ($\omega = \frac{\Delta\theta}{t} = \frac{2\pi}{86400\,\text{seconds}} = 7.272\times10^{-5}\,\frac{1}{s}$) to zero in one hour. So $\Delta\omega = -7.272\times10^{-5}\,\frac{1}{s}$ and we have $\alpha = \frac{\Delta\omega}{t} = \frac{-7.272\times10^{-5}\,\frac{1}{s}}{3600\,s} = -2.02\times10^{-8}\,\frac{1}{\text{s}^2}$. This is the angular acceleration.

Now, in the realm of rotating objects, torque serves in the same role as force does in the realm of moving objects: Torque is the thing that *makes* something start or stop rotating (or change the direction of its rotation). We have an analog for Newton's second law in the rotating realm: Instead of $\vec{F} = m\vec{a}$, we have $\vec{\tau} = I\vec{\alpha}$. (There is always some confusion here: I seem to have given you two "formulas" for the torque, one in which it's defined as a force times a moment arm and the other in which it's defined as a moment of inertia times an angular acceleration. What gives? Well, this is part of the danger of seeing these mathematical expressions as "formulas" rather than as what they really are: Statements about the relationships between entities. The way *I* think of these two equations is that one [the one in which torque is expressed as a moment arm times a force] is a *cause* of the torque while the other [the one in which torque is related to a moment of inertia times an angular acceleration] is the *effect* of the torque. Break yourself of the habit of thinking of these mathematical expressions as formulas into which quantities are to be plugged and commence thinking of them as descriptions of relationships written in the language of mathematics and you'll avoid a host of errors and open up a plethora of new insights!) We've just found the angular acceleration, so all that remains is to multiply this by the moment of inertia, *I*.

The moment of inertia of an object is the analog of mass: Mass is a measure of how difficult it is to change the *velocity* an object. Moment of inertia is a measure of how difficult it is to change the *angular velocity* of an object. It is more complex than mass since it depends first of all on the mass of the object and also on the geometry of the object and the axis about which the object is rotated. A single object does *not* have a single moment of inertia since we can rotate the object about any axis we choose. (Note that while torque can be determined from any origin we choose, giving the same answer for any choice, moment of inertia demands a particular choice of axis. Each one will be different.) The moment of inertia of a single blob of mass *m* sitting a distance *r* from an axis of rotation is $I = mr^2$. But an extended object, which has its mass distributed, will be different. For a *solid* sphere rotating about its center, the moment of inertia is $I = \frac{2}{5}mr^2$. (There is a table of such values in your text. On an exam, I will provide you with such a table. You need only memorize that the moment of inertia of a point mass a distance *r* from an axis is $I = mr^2$. Anything else, you can look up.)

If we assume the earth is a solid sphere of mass $m = 5.97 \times 10^{24}\ kg$ and radius $r = 6.37 \times 10^6\ meters$, we find its moment of inertia to be $I = \frac{2}{5}mr^2 = 0.4 \times 5.97 \times 10^{24}\ kg \times \left(6.37 \times 10^6\ meters\right)^2 = 9.69 \times 10^{37}\ kg \cdot m^2$. Using this and the angular acceleration found previously (again, ignoring the direction of the vector and concentrating on its size only), we find the needed torque to be $\tau = I\alpha = 9.69 \times 10^{37}\ kg \cdot m^2 \times 2.02 \times 10^{-8}\ \frac{1}{s^2} = 1.96 \times 10^{30}\ Newton \cdot meters$.

Now, this links the torque to its effect. Let's link it to its cause: The torque we just found is presumed to be the result of a force exerted at the Earth's equator. We further assume that the force is exerted precisely perpendicular to the Earth's radius (if it weren't, we'd have to include the sine of some angle—all we care about is the component of the force that is perpendicular to the Earth's radius). Taking $r = 6.37 \times 10^6\ meters$ (I've seen different values given for this, don't be too concerned if you used a slightly different number), we have $\tau = rF$. (Again, we're ignoring the vector aspect very consciously. This is a risky thing, as you know! In this case, we can get away with it because we're not concerned with the direction of the result *and* we're assuming the force is perpendicular to the radius vector. Without these assumptions, we'd be making a horrible mistake by ignoring the vector nature of these quantities!) Using this, and the numbers found above, we have $F = \frac{\tau}{r} = \frac{1.96 \times 10^{30}\ Newton \cdot meters}{6.37 \times 10^6\ meters} = 3.08 \times 10^{23}\ Newtons$.

To find the force needed at Edwardsville, we use the radius at our latitude, as in problem #9 of assignment #9. This amounts to simply dividing the number above by the cosine of 39°. This gives

$$F = \frac{\tau}{r\cos(39°)} = \frac{1.96 \times 10^{30}\ Newton \cdot meters}{6.37 \times 10^6\ meters \times \cos(39°)} = 3.96 \times 10^{23}\ Newtons.$$

Notice that Superman has to push harder to stop the Earth at Edwardsville than he did when pushing at the equator. This is for exactly the same reason as why it is easier to open a door by pushing on a knob far from the hinge than to open the same door by pushing close to the hinge.

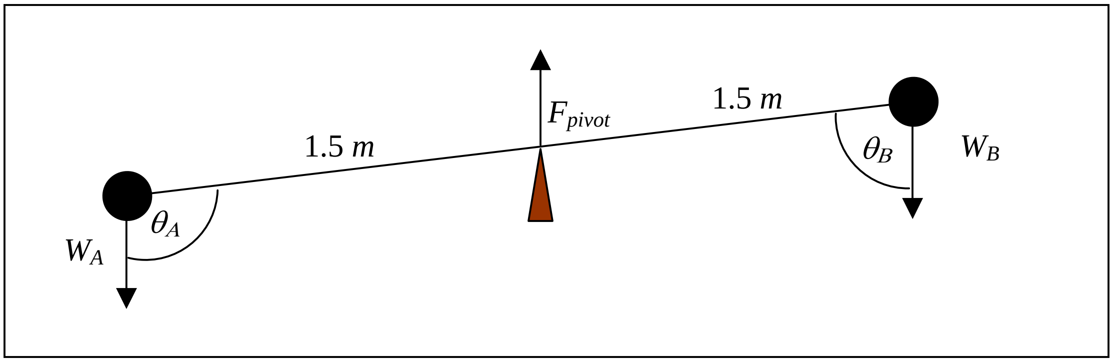
2. **The propeller on an airplane has a diameter of 223 cm and a mass of 26.8 kg. The desired rotation rate of the propeller is 2700 rpm. The engine provides a torque of $\tau = 550 N \cdot m$. How long does it take for the propeller to reach operating speed when the airplane's engine is started?**

I intentionally didn't tell you anything about the geometry of the propeller. There are a variety of ways you can visualize this. The easiest is to think of the propeller as a rod being rotated around its midpoint. The moment of inertia of such a rod is $I = \frac{1}{12} ML^2$. We are given the torque (the *cause* of the angular acceleration), so all we need is to find its effect. We have $\tau = I\alpha$. Recalling that $\alpha = \frac{\Delta\omega}{t}$ (again neglecting the vector aspect since we're unconcerned with direction in this case), we see that we can write $\alpha = \frac{\tau}{I}$ so $t = \frac{\Delta\omega}{\alpha} = \frac{I\Delta\omega}{\tau}$. From here it's just plug-and-chug except for one point: I gave you the propeller's final rotation rate in revolutions per minute. We really need it to be in radians per second to be meaningful. Since there are 2π radians in each revolution and there are 60 seconds in each minute, $2700 \text{ rpm} = \frac{2700 \times 2 \times \pi}{60s} = 282.7 \frac{rad}{s}$. We can now plug this into the above equation and get $t = \frac{I\Delta\omega}{\tau} = \frac{26.8kg \times (0.223m)^2 \times 282.7 \frac{rad}{s}}{12 \times 550 N \cdot m} = 0.057s$.

Frankly, that's a ridiculous number! I always *try* to give you realistic numbers on the problems I assign. I don't know where the difficulty crept in, here: I used manufacturer specifications for all the numbers. I will do some more research on this to see where I may have provided an incorrect input. Nevertheless, the answer above is consistent with the inputs. Only the plausibility of the assumptions is in question.

3. **Two children sit on the ends of a see-saw. The distance between the children is 3 meters. Child "A" has a mass of 22 kg and child "B" has a mass of 31 kg. The pivot of the see-saw is halfway between them. They begin with child "B" in the air and child "A" on the ground. At this time, the see-saw makes an angle relative to the horizontal of 0.2 radians. What is the magnitude (size) of the net torque on the see-saw?**

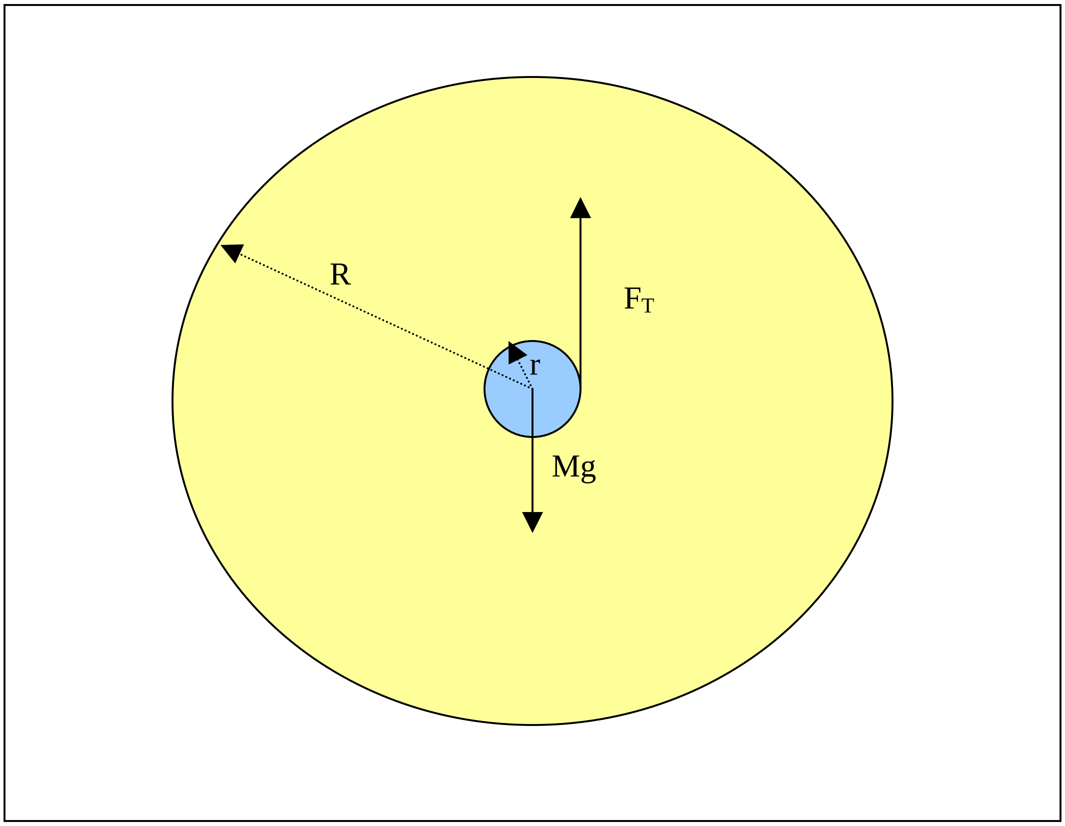
This is just like problem #7 and 8 of the previous assignment, with a twist (pun intentional): Now the forces do not act perpendicular to the rod. Recall that the definition of torque is *NOT* distance times force. It is distance times force in a direction perpendicular to the vector connecting the origin (which we are free to choose) to the location where the force acts. That vector is called the "moment arm." (We can also think of this as distance in a direction perpendicular to the force times the force—they're equivalent statements. Personally, I find it much easier to visualize the distance as a fixed quantity and then to consider only the part of the force perpendicular to that distance.) So we must decompose our forces into components that are parallel to the moment arm and components that are perpendicular to the moment arm. Fortunately, this always gives us (treating only the size of the torque and ignoring its direction for now—we'll deal with the direction in a little while) $\tau = dF\sin(\theta)$, where θ is the angle between the force and the moment arm. Our situation is pictured below.



For the sake of symmetry, let's pick the pivot as the origin for this problem. (As a variation, I recommend that you give it a shot with one of the children as the origin.) We can now look at the torques due to each of the two children individually. These are $\tau_A = m_A gd\sin(\theta_A)$ and $\tau_B = m_B gd\sin(\theta_B)$. In both cases, the distance, *d*, is 1.5 meters. Since the see-saw makes an angle of 0.2 radians with the horizontal, the angles are $\theta_A = \frac{\pi}{2} + 0.2$ and $\theta_B = \frac{\pi}{2} - 0.2$. (Those of you who are paying attention will note that $\sin\left(\frac{\pi}{2} + 0.2\right) = \sin\left(\frac{\pi}{2} - 0.2\right)$. Let's choose not to use this fact in this case. Normally, we should take advantage of it, however.) Inserting numbers, we get $\tau_A = m_A gd\sin(\theta_A) = 22\,kg \times 9.8\frac{m}{s^2} \times 1.5\,m \times \sin(\frac{\pi}{2} + 0.2) = 316.95\,Newton \cdot meters$ and $\tau_B = m_B gd\sin(\theta_B) = 31\,kg \times 9.8\frac{m}{s^2} \times 1.5\,m \times \sin(\frac{\pi}{2} - 0.2) = 446.62\,Newton \cdot meters$.

Now, although we are neglecting direction in this, we cannot completely ignore it. Note that one of the torques found above should be positive and the other negative. It really doesn't matter which one is which as long as we're neglecting the direction of the net, but we might as well do it right. (The way to see that it must be this way is to note that both of the forces are pointed in the same direction. However, the vector from the origin [the pivot, in this case] to child "A" points in the opposite direction to the vector from the origin to child "B". Thus the torques *must* have opposite signs.) Using the Right Hand Rule, as demonstrated in the solution to problem #1, we see that child "A"'s torque points out of the page while child "B"'s torque points into the page. Again using out of the page as positive, we get a final torque of $\tau = \tau_A + \tau_B = 316.95\frac{kg \cdot m^2}{s^2} - 446.62\frac{kg \cdot m^2}{s^2} = -129.67\,Newton \cdot meters$. This indicates that the rotation induced by this torque will be *clockwise*.

4. A yoyo has a total mass of 185 grams and a radius of 6 cm. Consider the yoyo to be a perfect solid disk. The hub has a radius of 3 millimeters. The yoyo begins with its string completely wound around the hub. (Neglect the thickening of the hub due to the string.) When the string is held and the yoyo allowed to fall by "unwinding," what total downward acceleration will the yoyo experience? (Hint: Determine the angular acceleration and then calculate the amount of string played out due to the rotation. Remember that linear acceleration and angular acceleration are related, in this case.)



Sorry for all the arrows in the picture! We'll need them, so try to keep them all straight in your mind.

The first thing to realize in this problem is that, as implied by the hint, the downward acceleration is directly linked to the angular acceleration. This is because the yoyo goes down by unwinding string. The total amount of string that is unwound from the hub at any instant is a direct measure of how far the yoyo has traveled (we do need to assume that the string doesn't slip around the hub; if we wanted to include that slippage, we'd need to add friction to the problem). So, if we find the rate at which the yoyo is spinning, we can determine how fast it's falling. The yoyo experiences an angular acceleration which results in a linear acceleration (because of the string). The angular acceleration is caused by a net torque, so we must use the angular analog of Newton's second law, $\vec{\tau} = I\vec{\alpha}$, to find the angular acceleration.

The next step is to remember that torque problems start out as force problems. (That's an important sentence. Reread it.) To solve a torque problem, you should begin with a free-body diagram, just as you would with a force problem. The crucial difference is that with a force problem, it really doesn't matter where you draw the arrows. With a torque problem, the size of the arrows, their directions, *and* their locations carry meaning, so you must be more careful.

There are two forces acting in this problem: The tension on the string and the force of gravity. Newton's second law didn't stop being true all of a sudden: If we knew F_T we'd be done with the problem right now. But we *don't* know F_T (it is *not* the same as the weight of the yoyo—if it were, the yoyo wouldn't go down). All we know is the mass of the yoyo, so we'll have to work with this. Since $\vec{\tau} = \vec{d} \times \vec{F}$ and there are two forces but we only know one of them, we'd better put the origin at the place where one of the forces acts—this is the point where the string meets the hub. This means that the only force creating a non-zero torque is the one due to the weight of the yoyo. Since the two forces (the weight and the tension) point in exactly opposite directions (if they didn't, the yoyo would accelerate *away* from the person holding it, not just up and down) $\vec{d}$ and $\vec{F}$ are perpendicular to each other and we can just multiply them without using the sin(θ) term. This gives $\tau = r\mathrm{M}g$.

Time for a little review: Newton's second law is a statement of cause and effect. In the rotational case, this is written $\vec{\tau} = I\vec{\alpha}$ with the *cause* being the torque and the *effect* being the angular acceleration. Try to think about it this way to avoid confusion. I've had many conversations with students who are dazed by the fact that we seem to have two equations for τ. The source of the confusion is linked to your desire for *formulas*. You see an equation that has a τ to the left of the "=" sign and you think "oh, that's a formula for torque." WRONG! Seeing it written that way simply means that it's an equation involving torque. It can be turned into a formula for any of the quantities in the equation. The equation has meaning that you should seek to understand. Do not simply memorize and regurgitate these things or you'll find yourself confused and in trouble!

So, we've found the *cause* of the angular acceleration: $\tau = r\mathrm{M}g$. The cause is linked to the effect by $\tau = I\alpha$ (once again just concerning ourselves with the size and ignoring the direction for now). These give $I\alpha = r\mathrm{M}g$. This can be solved for the angular acceleration to give $\alpha = \frac{r\mathrm{M}g}{I}$. Now we encounter a problem: We picked the place where the string meets the hub of the yoyo as the origin. We really should use the moment of inertia about that point. But that is beyond the level of this class. So we will just make the approximation that the moment of inertia of a disk rotated about a point *close* to the center is *approximately* the same as the moment of inertia of a disk. I'll use a different technique at the end of this solution to show you what it looks like if we do it exactly. The moment of inertia of a disk rotated about its center is $I = \frac{1}{2}\mathrm{MR}^2$. We use this to find the total angular acceleration: $\alpha = \frac{rMg}{\frac{1}{2}MR^2} = \frac{rg}{\frac{1}{2}R^2}$. (Note: I've noticed a disturbing fraction of you latching onto the *FALSE* definition $I = \frac{1}{2}MR^2$. This is true for a solid disk or cylinder rotated about its axis. It is *not* true for all shapes! Be careful to use the correct moment of inertia for a particular situation.)

If we express the angular acceleration in $\frac{\text{radians}}{\text{second}^2}$, we can write $a = r\alpha$ (notice which radius is being used—that of the hub, not the entire yoyou; make sure you understand why), so $a = \frac{r^2 g}{\frac{1}{2}R^2}$. Sticking in the numbers, we have $a = 4.9\frac{cm}{s^2}$.

Now for an alternate solution that will yield the exact answer, without approximating the moment of inertia. To do this we need to take the origin at the center of the yoyo—an axis for which the moment of inertia is well known exactly. This means we're left with F_T as an unknown. But all is not lost. From the torque equation, we have $\tau = rF_T$ and since $a = r\alpha$ (as we discussed above), we can write $\alpha = \frac{a}{r}$. Now we write the rotational form of Newtons second law $\tau = I\alpha$ and put these three pieces together to write $rF_T = I\frac{a}{r}$. Or, $F_T = I\frac{a}{r^2}$. Now, Newton's second law (the regular form) says $Mg - F_T = Ma$, so we can solve for F_T: $F_T = Mg - Ma$. This can be substituted into our acceleration expression found from the torques to give $Mg - Ma = I\frac{a}{r^2}$. A bit of algebra on this to solve for a gives $Mg = I\frac{a}{r^2} + Ma = a\left(M + \frac{I}{r^2}\right)$ which yields $a = \frac{Mg}{\left(M + \frac{I}{r^2}\right)} = \frac{Mg}{M + \frac{\frac{1}{2}MR^2}{r^2}} = \frac{g}{1 + \frac{\frac{1}{2}R^2}{r^2}} = \frac{r^2 g}{r^2 + \frac{1}{2}R^2}$. Notice that our two answers are, as we predicted, very similar. If r is very much less than R, the addition of an r^2 in the denominator makes insignificant difference. However, if r starts to get big enough that it's a significant fraction of R, we'd better use this form. (Remember that our first answer was correct except for the fact that we used the wrong moment of inertia. If we'd used the accurate moment of inertia for a disk rotated around a point away from the center by a distance r, which is $I = Mr^2 + \frac{1}{2}MR^2$, we would have gotten the precise answer using that method.)

5. **Consider again the yoyo described in the previous problem. Use conservation of energy to determine the angular speed of the yoyo after it has descended ½ meter.**

Energy is conserved in this problem. Thus we can write the condition for conservation of energy $P.E.+K.E.=constant$. After the yoyo has descended by ½ meter its *P.E.* has decreased by $\Delta P.E.=\mathrm{M}gh$. Its *K.E.* must have increased by the same amount. But here there's a difference between this problem and ones we've done before: The *K.E.* can go to two different places. It can go to the overall motion of the yoyo moving downward, called "translational kinetic energy," or it can go into the spinning of the yoyo, called "rotational kinetic energy." Do *NOT* presume that it goes into the two 50/50! We'll need to think about how it's distributed rather than just assuming that it's divided up "fairly." Fortunately for us, since the string requires that the downward motion is exactly related to the spinning, we can write these two down at the same time. Remembering that $v=\omega r$, we can write down the translational kinetic energy as $K.E._{translational}=\frac{1}{2}\mathrm{M}v^2=\frac{1}{2}\mathrm{M}\omega^2 r^2$. The rotational kinetic energy is given by $K.E._{rotational}=\frac{1}{2}I\omega^2=\frac{1}{2}(\frac{1}{2}\mathrm{MR}^2)\omega^2=\frac{1}{4}\mathrm{MR}^2\omega^2$, where we have used the moment of inertia of a disk rotated about its center.

Setting the change in potential energy equal to the sum of the two kinetic energies, we have $\mathrm{M}gh=\frac{1}{4}\mathrm{MR}^2\omega^2+\frac{1}{2}\mathrm{M}\omega^2 r^2$. This gives $\omega=\sqrt{\frac{gh}{\frac{1}{2}r^2+\frac{1}{4}\mathrm{R}^2}}$. Putting in the numbers, this gives $\omega=73.6\frac{\text{radians}}{\text{second}}$.

If the hub's radius is very much less than the disk's radius, note that the vast majority of the energy is in the rotational part of this. We'd be justified in approximating the angular speed, in this case, by $\omega=\sqrt{\frac{4gh}{\mathrm{R}^2}}$. Again inserting the numbers, we get $\omega=73.8\frac{\text{radians}}{\text{second}}$. Clearly the approximation is quite a good one—this answer varies from the one above by only about 0.3%.

6. **An ice skater whose mass is 55 kg is spinning at 11 radians per second. She can be approximated as a cylinder 50 cm in diameter. Someone throws a cat to her and the cat attaches itself to her with its sharp claws (neglect the speed with which the cat is thrown). The mass of the cat is 7 kg. What will her angular speed be with the cat attached?**

It is very important that you recognize the basic form of this problem: This is just another inelastic collision (the cat *sticks*, whenever something sticks, it's inelastic) and so is essentially identical to the other inelastic collision problems that you've done—like the carts in lab. In those cases, we were only concerned with "regular" (i.e., linear) momentum. Here, we're only concerned with angular momentum. Linear momentum and angular momentum are conserved independently, so you could also solve this problem for the skater's new linear momentum if I'd given you the velocity of the cat. You could do this using precisely the same methods as you've done previously. In this case, we just need to worry about angular momentum, however.

Angular momentum is conserved *always*. Just as in the linear case, we have $\vec{L}_{before} = \vec{L}_{after}$. The addition of the cat to the spinning skater changes her moment of inertia. Therefore, her angular speed must change in order for her angular momentum to remain unchanged. (If we hadn't neglected the thrown speed of the cat, this would not necessarily be true. We'd have to allow for the cat's angular momentum as well.)

The size of the initial angular momentum of the skater is $L = I\omega$. If we treat her as a cylinder, here moment of inertia is given by $I = \frac{1}{2}Mr^2$. Thus her total angular momentum is $L = \frac{1}{2}Mr^2\omega = \frac{1}{2}\times 55\,kg \times (.25\,m)^2 \times 11\frac{1}{s} = 18.9\frac{kg\cdot m^2}{s}$. This will be the final angular momentum of the system as well.

We are now left with a bit of a judgment call to make: What shall we use for her moment of inertia once the cat attaches itself to her? The cat is a single mass which is attached at a distance *r* from the skater's axis. The moment of inertia of the cat, therefore, is $I_{cat} = M_{cat}r^2$. I think the best technique would be to add this moment of inertia to that of the skater. However, an alternative method, which is also valid (although I find it a bit less justifiable) would be simply to consider the cat to be part of the skater and just use $I = \frac{1}{2}Mr^2$ but for the mass use the sum of the two masses—skater and cat. I'll go ahead and use the method I prefer, but if you used the other one, it's alright. (This is the sort of thing that simply must be communicated in a problem's solution. We can quibble about how valid an approximation is, but we really should all agree on what approximation has been used.)

This gives $I = \frac{1}{2}M_{skater}r^2 + M_{cat}r^2$ so, setting the two angular momenta (before the cat and after the cat) equal to each other, we have $\frac{1}{2}M_{skater}r^2\omega_{initial} = \left(\frac{1}{2}M_{skater}r^2 + M_{cat}r^2\right)\omega_{final}$. Solving for the final angular speed, we have

$$\omega_{final} = \frac{\frac{1}{2}M_{skater}r^2\omega_{initial}}{\left(\frac{1}{2}M_{skater}r^2 + M_{cat}r^2\right)} = \frac{\frac{1}{2}M_{skater}\omega_{initial}}{\left(\frac{1}{2}M_{skater} + M_{cat}\right)} = 0.797\omega_{initial} = 8.77\frac{\text{radians}}{\text{second}}.$$

7. Two men are standing on a merry-go-round which is turning at a constant angular speed, ω. One man is standing very near the center of the merry-go-round, the other is standing very near the edge. The man at the center is trying to throw a baseball directly to his friend at the edge but for some reason the ball always winds up very far from its intended target. Please explain the mistake the man at the center is making. (This is an example of the "coriolis" effect.)

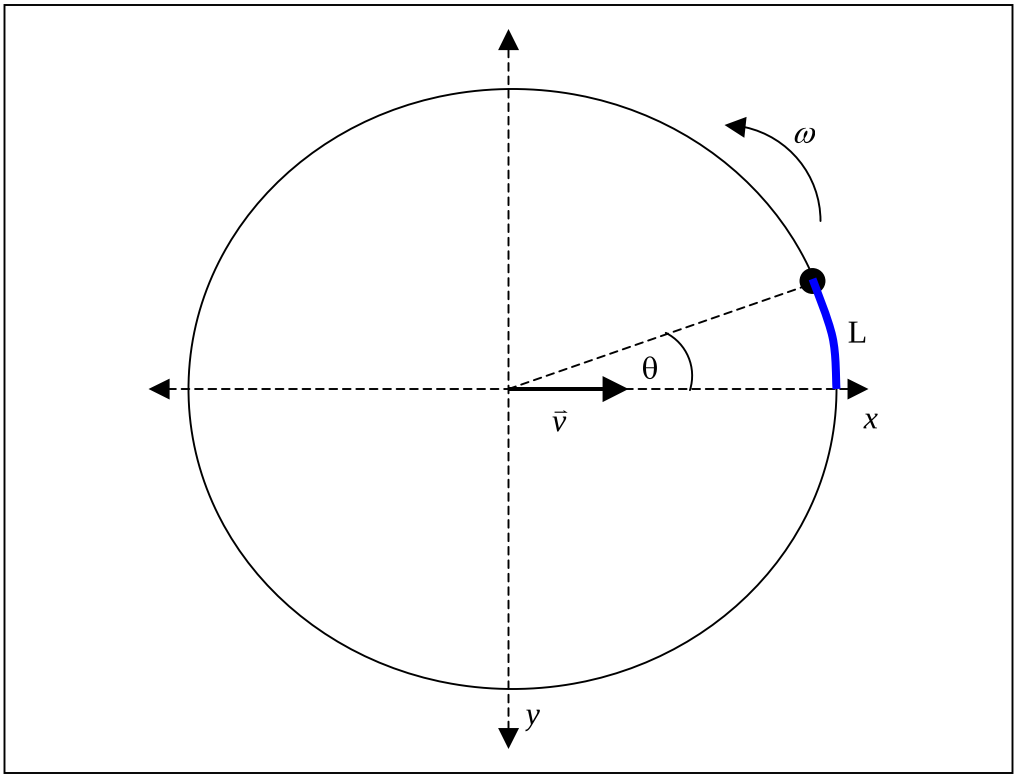
Both men are standing on a rotating object. They very naturally (it's really the way humans are put together) perceive the merry-go-round to be a good frame of reference. That is, since they are not moving relative to the merry-go-round, they think of the merry-go-round as being motionless, even though they know it's spinning around. Further, even when their intellect overcomes their intuition sufficiently to convince them that they're moving, they still don't recognize that they are accelerating. *Any* object forced to move in anything other than a straight path must accelerate! Thus, their reference frame is non-inertial.

When the man at the center throws the ball to his friend, he doesn't allow for the fact that his friend has a different velocity than he does. The man at the edge is moving faster than the man at the center—even though they have the same *angular* speed. The ball, when thrown, will have whatever velocity the man at the center gives it. As seen by an observer in an inertial reference frame, the ball will have that velocity plus whatever velocity the man at the center has (relative to the inertial frame). But the man at the edge will have a different velocity. If the men presume their reference frame to be inertial, it will appear that the ball has some extra velocity not given to it by the thrower. Since the only thing that can change a velocity is a force, the men will come to the conclusion that there is some force that acts on the ball after it is thrown. In reality, no such force exists. It is fictitious.

Notice that the coriolis force acts "tangentially"—i.e., in a direction tangent to the rotation. This is perpendicular to another fictitious force that we've studied, the "centrifugal" force, which acts "radially"—i.e., along the radius of the circle. Don't confuse the two!

The coriolis force is *very* important in weather: As air travels from one latitude to another, the ground has a different speed. This results in air moving relative to the ground until it gets accelerated to the same speed as the ground. This is a major cause of wind and is the reason that things like hurricanes rotate.

8. Consider again the situation in problem #7. The angular speed of the merry-go-round is 0$.1\frac{\text{radian}}{\text{second}}$ **and it is 12 meters in diameter. The baseball is thrown with a velocity in the** $\hat{x}$ **direction of** $13\frac{\text{meters}}{\text{second}}$ **(i.e., ignore the** $\hat{y}$ **component of the velocity). How far (measured along an arc) from the catcher does the ball wind up?**



Let's assume the merry-go-round is spinning counter-clockwise as seen from above. (The answer will be the same which ever way you take the rotation since I only asked for the distance from the catcher the ball winds up, not the direction.) This is shown in the figure. Now, take the "god's eye view"—look down on the rotating merry-go-round from the perspective of a fixed observer in an inertial reference frame directly above the center of the merry-go-round. This isn't essential, but it makes life easier. From this perspective, the ball will be thrown with a velocity $\vec{v}$ and will keep this velocity through the entire problem—it will not change since there is no real force acting on it after it is thrown. However, in the time that it takes for the ball to make it from the center of the merry-go-round to its edge, the catcher will have moved some angle θ from his starting point.

To figure out what θ is, we first realize that we know how θ changes with time. The angular speed of the merry-go-round is just $\omega = \frac{\Delta\theta}{t}$. This is just like our "regular" speed except with "angle" replacing "distance". We read this as "the change in angle divided by the time it takes to make the change." So, we can conclude that $\theta = \omega t$ (taking our initial angle to be zero). Now all we need to do is find the time that it takes for the ball to reach the edge of the merry-go-round.

Since the time that it takes for the ball to reach the edge of the merry-go-round doesn't depend on the angular speed (note that this is because of the high level of symmetry in a circle; imagine an elliptical merry-go-round, or some other shape, then the problem would require more work!), all we need to do is realize that the speed (that is, the "regular" speed, not the angular speed) is just the distance traveled divided by the time it takes to travel that distance. In this case, the distance traveled is the radius of the circle, *r*. We have $t = \frac{r}{v}$. Substituting numbers, this gives $t = \frac{6 \text{ meters}}{13 \frac{\text{meters}}{\text{second}}} = 0.46 \text{ seconds}$.

Now we use this with our angular speed to find the angle $\theta = \omega t = 0.1 \frac{\text{radian}}{\text{second}} \times 0.46 \text{ seconds} = 0.046 \text{ radians}$. But wait, we're not done yet. The problem asked how *far* from the catcher the ball winds up. That is, it asks for the distance. This is indicated on the figure as "L." (I am sorry that we are using the same letter, L, to indicate both the arc length and the angular momentum. This sort of thing is maddeningly confusing, but inevitable. There are just too many concepts to reserve a letter for each of them!) Here's where using radians to measure angles really comes in handy: By *definition*, if we measure our angle in radians, we have $\theta \equiv \frac{\text{L}}{r}$. Thus $\text{L} = \theta r = 0.046 \text{ rad} \times 6 \text{ meters} = 0.276 \text{ meters}$.

(Some folks who are new to radians will feel uncomfortable with the fact that this unit seems to have magically vanished from the answer above. In some sense, this is because it was never really there. Recall the definition of an angle measured in radians $\theta \equiv \frac{\text{L}}{r}$. Note that the numerator and the denominator both have the same units. So θ never really had a unit. By saying the angle is measured in "radians" we keep track of how the number was created, but there's not really a unit there. I know this is confusing. This is one place where I'll just say to memorize the fact that this is so without wasting too much time on understanding why it is. There won't be too many other cases like this.)

9. Yet again, consider the merry-go-round of problem #7. The catcher perceives that a force is acting on the ball to skew its path. This is a "fictitious force." What size force would be needed to cause the motion observed by the catcher if it were real?

As is so often the case, there's a hard way to do this and an easy way. I knew that when I wrote the problem. What I lost track of, however, was just *how* hard the hard way is if you don't know calculus—far too hard for this course. (Try to remember what it was like before you could read. It's difficult to remember, isn't it? That's the way it gets with Math also: Sometimes it's hard to keep hold of the fact that I wasn't born knowing calculus.) So don't feel too bad if you missed it. I'm perfectly happy with the easy answer. But the hard way *is* instructional, however, so I'm not sorry I threw it at you. I just should have given you adequate warning first. The hard way is *much* more informative than the easy way, so I'll do it that way first. Then, I'll make you slap your head in frustration by showing you how easily it can really be done. I'll also include the method using calculus, for those of you who have that skill.

Hard way: You must recognize that the ball moves in a straight line at a constant speed (as required by Newton's first law) for its entire trip as seen by an observer in an inertial reference frame. This line is along a radius of the merry-go-round, so let's call this speed v_{radial}. Also, the merry-go-round moves at a constant angular speed for that entire trip. But, as the ball moves out from the center of the merry-go-round, the speed (not angular) of the merry-go-round increases steadily. Since regions of the merry-go-round that are different distances from its center sweep out larger distances in a given amount of time. This can be seen from the definition of angle $\theta \equiv \frac{\mathrm{L}}{r}$. If we take θ to be the angle swept out in some small interval of time, as r increases, L must also increase to keep θ constant (as it must be since the angular speed is constant). Note that this speed is in a direction perpendicular to v_{radial}.

Now, let's come up with a strategy for solving this problem: We want an acceleration. (We actually want a force, but since we know Newton's second law, we realize that if we can find the acceleration, we'll have also found the force.) We know that acceleration is the change in velocity divided by the time over which that change occurs. We also know that velocity is the change in the displacement vector divided by the time over which *that* change occurs. So, if we can find the displacement as a function of time, we can divide by time to find velocity and then divide that by time to find acceleration. Let's do it.

Now, we can define an angle in terms of the radius and the arc length it subtends: $\theta \equiv \frac{\mathrm{L}}{r}$. Also, the angular speed is the angle swept out in some interval of time: $\omega = \frac{\Delta\theta}{t}$.

Putting these together, we can write $\omega = \frac{\Delta\theta}{t} = \frac{\frac{\mathrm{L}}{r}}{t} = \frac{\mathrm{L}}{rt}$. (You may be troubled that I blithely go between θ and $\Delta\theta$. I can do this by simply taking the angle at the beginning to be zero. Recall that the change in some quantity is its value at the end minus its value at the beginning. So, $\Delta\theta = \theta$ if $\theta_{beginning} = 0$. This is convenient, but be careful to use it only when it is true!) Now, figure out what L is as a function of time: $\text{L} = \omega r t$.

We're almost there. Now, to find the speed (we're now working in one dimension, so we can drop the vector language), we divide this by *t* to get $v = \frac{\text{L}}{t} = \omega r$. Please be careful: Here *r* is not the total radius of the merry-go-round but simply the distance from the center at which the ball happens to be at some time. If that time happens to be the time at which the ball is at the edge of the merry-go-round, *then r* is the full radius.

Now, we find the acceleration of the ball. We make the assumption (if you know a bit of calculus you can prove that this is true; if you don't, you'll just have to take my word for it) that the acceleration is constant in this problem. Thus we can use the definition of acceleration (in one dimension) $a = \frac{\Delta v}{t} = \frac{v_{end} - v_{begin}}{t}$. Remember: We're *not* talking about the speed of the ball as it goes from the center of the merry-go-round to its edge. All observers agree that the ball is not accelerating in that direction. We're talking about the acceleration perpendicular to this. In that direction, v_{begin}=0, so we can just write $a = \frac{\omega r}{t}$. We're almost there. Notice that we can rewrite the above equation as $a = \frac{\omega r}{t} = \omega \frac{r}{t}$. But $\frac{r}{t}$ is *precisely* the initial speed of the ball—v_{radial}! So we can write $a = \omega v_{radial}$. Inserting some numbers, this gives $a = .1\,\frac{\text{radian}}{\text{second}} \times 13\,\frac{\text{meters}}{\text{second}} = 1.3\,\frac{\text{meters}}{\text{second}^2}$. Multiply this by the mass of the ball to find the fictitious force.

But wait, we're not done yet! I pulled a sleight of hand on you. I only included the change in the ball's speed (in the non-inertial frame) in calculating the acceleration. Remember: Acceleration can be a change in the direction of the velocity in addition to (or instead of) a change in its size! So far, we've only calculated the change in the size of the velocity as seen by the catcher. He *also* sees a change in the direction of the velocity: The ball begins headed directly toward him, but at the end, it is shooting off in some other direction. The math here gets way beyond this course, but the result is a factor of 2 applied to the expression above. So the final answer is $a = 2.6\,\frac{\text{meters}}{\text{second}^2}$, but I wouldn't expect anyone in the class to get the extra factor of 2.

(**Anyone who hasn't had calculus, don't read this paragraph!** For those of you who *have* had calculus, it goes like this: Start with $r(t) = v_{radial} t$ and $\theta(t) = \omega t$. These are the sizes of the components of the position vector in a circular coordinate system as a function of time. The deviation of the ball's position as a function of time will lie in the $\hat{\theta}$ direction and will be $L(t) = r(t)\theta(t) = v_{radial}\omega t^2$. We take the derivative of this with respect to time to find the velocity in the $\hat{\theta}$ direction. This is $v_\theta = \frac{dL(t)}{dt} = 2v_{radial}\omega t$. Now, we take the derivative with respect to time of the velocity to find the acceleration. This is $a = \frac{dv_\theta}{dt} = 2v_{radial}\omega$. Man, does calculus make life easier!)

Easy way: That was pretty tough, huh? Especially that "magical" factor of 2! Let's do it the easy way: Again recognizing that the acceleration is in the direction perpendicular to v_{radial}, we note the solution found in the previous problem: The distance traveled due to this force (in the non-inertial frame) is 0.276 meters. We also know how long it took for this distance to be traveled: 0.46 seconds. Thus, we can write $\mathrm{L} = \frac{1}{2}at^2$ and solve this for the acceleration. This gives $a = \frac{2\mathrm{L}}{t^2}$. Plugging in the numbers above, we have $a = \frac{2\mathrm{L}}{t^2} = \frac{2 \times .276 \text{ meters}}{(.46 \text{ seconds})^2} = 2.6 \frac{\text{meters}}{\text{second}^2}$. Yup, it's that easy!

10. What is the "orbital angular momentum" of the Earth? That is, the angular momentum due to the Earth revolving around the sun.

There are two ways to do this (isn't that a shock?). They're actually identical, but they don't look that way unless you recall where the equations you're using came from. The definition of angular momentum is $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the position vector of an object and $\vec{p}$ is its (regular) momentum. When we work this out and allow for the fact that objects frequently don't come as isolated blobs of matter but are often extended, with some structure, we get a variation which is $\vec{L} = I\vec{\omega}$. (Reread that last sentence a couple of times until you are clear on what it is saying: The two equations are the *same*. One simply takes into account all the component masses that make up an object while the other is the angular momentum of just one of those component masses.) If we treat the Earth going around the sun as an isolated blob of matter moving with some angular speed ω, we get (considering only the size of the vectors—we can neglect the vector cross product because the velocity vector and the position vector are perpendicular to each other to a very good approximation) $L = rp = rmv = mr^2\omega$, where I've used the fact that $v = \omega r$. (If this final relation isn't like second nature to you by now, concentrate on it! It is a very important piece of geometry.) Well, the moment of inertia of an isolated mass a distance *r* from the axis of rotation is $I = mr^2$, so we've really just written $L = I\omega$. I'll use this form since it's more convenient in this case.

The Earth goes around the sun one time per year. Assuming that the distance from the Earth to the sun and the angular speed both remain constant (and Kepler's first and second laws say that this is not true, although it *is* a very good approximation), we can find the angular speed easily. The Earth travels an angle of 2π radians in one year, which is 3.156×10^7 seconds. This gives an angular speed of $\omega = \frac{\Delta\theta}{t} = \frac{2\pi}{3.156 \times 10^7 \text{ seconds}} = 1.99 \times 10^{-7} \frac{1}{s}$.

The mean distance between the Earth and the sun is 1.521×10^{11} meters and the Earth's mass is $5.98 \times 10^{24}\ kg$, so the Earth's moment of inertia about a rotational axis at the sun is

$$I = mr^2 = 5.98 \times 10^{24}\ kg \times \left(1.521 \times 10^{11}\ \text{meters}\right)^2 = 1.38 \times 10^{47}\ \text{kilogram} \cdot \text{meters}^2 .$$

Putting these two results together, we get

$$L = I\omega = 1.38 \times 10^{47}\ kg \cdot m^2 \times 1.99 \times 10^{-7}\ \frac{1}{s} = 2.75 \times 10^{40}\ \frac{kg \cdot m^2}{s}.$$ Now *that's* a big number!