FAIR DIVISION OF PIE

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1. Introduction

This project will discuss a problem of fair division, more specifically, the problem of cutting pieces of cake and sectors of pie 'fairly'. Intuitively, when we say fair division, we are talking about dividing cake or pie among *n* people so that every person is "happy" with their piece – we will make precise what we mean by "happy". This project will be based on the article, "Cutting a Pie Is Not a Piece of Cake", by Julius Barbanel, Steven Brams, and Walter Stromquist [BBS09]. We will begin by discussing cake-cutting and the many different properties that can occur when cutting a cake. There are many different combinations of these properties that can occur and we will prove each one. For instance, many cases we will examine will happen when there are two or three players and we must divide the cake so that each player is happy with their piece. Each player must also put a "value" to each piece such that one player's value for the entire cake adds to one. Happiness can be based on the size and the position of the piece. Also, we will look at properties where each player must like his own piece at least as much as the other players, and that there is no other way to divide the cake such that each player is at least as happy with the original division and one player is more happy. Next, we will look at pie-cutting when there are three or more players. For this section, we will look at how the combinations of the properties that could occur in cake cutting can not occur in pie cutting. When cutting pies, the sectors must go through the center of the pie and each player's valuation of the entire pie is no longer required to add to one. Pie-cutting is harder to analyze than cake-cutting.

2. Definitions

To start, we must define some words that will be used in proving the different cases and the properties that we will be using.

Definition 2.1. [BBS09] *A **cake** is the half open interval* $[0,1)$. *A **piece of cake** is represented by the subinterval* $[\alpha,\beta)$ *where* $0 \leq \alpha \leq \beta \leq 1$. *A **pie** is the closed interval* $[0,1]$, *with* 0 *and* 1 *identified. A **sector of pie** is represented by the subinterval* $[\alpha,\beta)$ *where* $0 \leq \alpha \leq 1$ *and* $\alpha \leq \beta \leq \alpha+1$, *where* β *is interpreted mod 1.*

Also, according to the article, $[\alpha,\alpha)$ denotes the empty sector, and $[\alpha,\alpha+1)$ denotes the entire pie. Mathematically, a pie is equivalent to the circle $[0,1]$, with the endpoints identified. [BBS09]

Definition 2.2. [BBS09] $\widetilde{(\alpha,\beta)}$ *is the **complement** to the sector* $[\alpha,\beta)$ *and is equivalent to* $[\beta,\alpha+1)$ *for all* $\beta \leq 1$.

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Now that we have some of the basic definitions that we need to know, we can define some of the definitions that are very important when assessing the values of pieces of cake or sectors of pie. More specifically, we will define what we mean by measures.

Definition 2.3. [Zie95] *A nonempty collection Σ of sets E satisfying the following two conditions is called a* ***σ-algebra:***

(i) if $E \in \Sigma$, then $\widetilde{E} \in \Sigma$, where $\widetilde{E}$ is the complement of E,

(ii) $\bigcup_{i=1}^{\infty} E_i \in \Sigma$ provided each $E_i \in \Sigma$.

Definition 2.4. [Zie95] *Let X be a set and $\mathscr{M}$ a σ-algebra of subsets of X. A* ***measure*** *on $\mathscr{M}$ is a function μ: $\mathscr{M} \to [0,\infty]$ satisfying the properties*

(i) $\mu(\emptyset) = 0$

(ii) if $\{E_1, E_2, \ldots, E_k, \ldots\}$ is any family of disjoint sets in $\mathscr{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Definition 2.5. [Zie95] *Let X be a set and $\mathscr{M}$ a σ-algebra of subsets of X and let $k \in \mathbb{N}$. A* ***finitely additive measure*** *on $\mathscr{M}$ is a function μ: $\mathscr{M} \to [0,\infty]$ satisfying the properties*

(i) $\mu(\emptyset) = 0$

(ii) if $\{E_i\}_{i=1}^{k}$ is a sequence of disjoint sets in $\mathscr{M}$, then

$$\mu\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} \mu(E_i).$$

We will be using the following theorem to prove that measures are continuous on an interval $[0,1]$.

Theorem 2.6. [Zie95] *Let μ be a measure and suppose $\{E_i\}$ is a sequence of sets in $\mathscr{M}$. Then μ is left continuous, that is if $\{E_i\}$ is an increasing sequence of sets with $E_i \subset E_{i+1}$ for each i, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu(\lim_{i\to\infty} E_i) = \lim_{i\to\infty} \mu(E_i).$$

The same holds true for continuity from the right, except $\{E_i\}$ is a decreasing sequence of sets.

Definition 2.7. *An allocation $\langle S_1, S_2, \ldots, S_n\rangle$ is an ordered collection of pieces or sectors.*

Now that we have defined what we mean when we say we will be working with allocations of cake and pie, we can define the different properties of these allocations that we will be working with. Furthermore, we must explain what we mean by ‘happy’ as stated earlier. To define ‘happy’, we must define v_i to denote player i’s measure, S_i to denote player i’s piece, d_i to denote the length of player i’s piece, and $v_i(S_i)$ to denote a player i’s value to his own piece. The following definition will be used throughout the entire paper.

Definition 2.8. [BTZ97] *An allocation of cake $\langle S_1, S_2, \ldots, S_n \rangle$ is called **envy-free** if each player thinks he or she receives a piece of cake that is at least tied for largest. Mathematically, $\langle S_1, S_2, \ldots, S_n \rangle$ is called **envy-free**, if $\forall i, j\ v_i(S_i) \geq v_i(S_j)$.*

Using the following allocation we will show an example of how the envy-free property can be applied.

	$[0, 1/2)$	$[1/2, 1)$
A	$\frac{1}{3}$	$\frac{2}{3}$
B	$\frac{5}{6}$	$\frac{1}{6}$

Thus, looking at picture above, an example of envy-free would be letting player A receive the piece $[1/2, 1)$ and player B receive the piece $[0, 1/2)$. As seen in the picture, we know that player A would value his piece $2/3$ and player B's piece $1/3$; we also know that player B would value his own piece $5/6$ and player B's piece $1/6$. Thus, both players would like their own piece more than they like the other players piece.

An example that violates the envy-freeness definition is assume the opposite as above. Then player A would receive piece $[0, 1/2)$ which he values only $1/3$, but he would value player B's piece with a value of $2/3$. The same result is for player B. Thus, they would both envy the other player's piece.

Definition 2.9. [BJK08] *An allocation $\langle S_1, S_2, \ldots, S_n \rangle$ is **undominated** if there is no $\langle T_1, T_2, \ldots, T_n \rangle$ that is better for some player and no worse for the others. In other terms, an allocation $\langle S_1, S_2, \ldots, S_n \rangle$ is **undominated** if there does not exist an allocation $\langle T_1, T_2, \ldots, T_n \rangle$ such that $\exists i$ with $v_i(T_i) > v_i(S_i)$ and $\forall j \in \{1, \ldots, n\}, v_j(T_j) \geq v_j(S_j)$, where T_i denotes player i's value to a different piece or sector. When an allocation is not undominated, we will say it is dominated.*

Now using the following allocation, we will show an example of undominated and dominated.

	$[0, 1/4)$	$[1/4, 1/2)$	$[1/2, 3/4)$	$[3/4, 1)$
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$

Therefore, using the picture above, an example of undominated is, letting player A receive piece $[0, 3/4)$ and player B receive piece $[3/4, 1)$. Thus, player A's value for his piece is $3/4$ and his value for player B's piece is $1/4$. Likewise, player B's value for his own piece is $1/2$ as well as his value for player A's piece. Thus, looking at the picture, we can see that there is no way to cut the cake such that either player is happier with the new piece compared to the original. Therefore, this is an example of undominated.

Now to show an example of an allocation that is dominated, assume player A receives piece $[0, 1/2)$ and player B receives piece $[1/2, 1)$. Thus player A's value for his own piece and player B's piece is $1/2$. Likewise, player B's value for both his and player A's piece is $1/2$. However, if we extend player A's piece to the same as above, player A would like his piece more than before since $3/4$ is greater than $1/4$. Furthermore, player B's value would not change because he is loosing a part of his piece that he has no value for. Therefore, this allocation is dominated.

Definition 2.10. [BBS09] *The measures $v_1,...,v_n$ are **absolutely continuous** with respect to each other if whenever $v_i(S) = 0$ for some i, then $v_j(S) = 0\ \forall j \in \{1,\dots,n\}$. When assuming the measures are absolutely continuous, we will contract to a point each piece or sector to which all the players assign value 0 and will therefore assume there is no piece or sector of positive length to which any player assigns value 0.*

Definition 2.11. [BBS09] *An allocation is **equitable** if all players assign exactly the same value (in their respective measures) to the pieces or sectors they receive (and so no player envies another's "degree of happiness"), i.e, the allocation $\langle S_1,S_2,\dots,S_n\rangle$ is **equitable** if $v_1(S_1) = \cdots = v_n(S_n)$.*

We will use the same allocation as mentioned after the definition of undominated,

	$[0,1/4)$	$[1/4,1/2)$	$[1/2,3/4)$	$[3/4,1)$
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$

An example of equitable is if player A receives the piece $[0,1/2)$ and player B receives the piece $[1/2,1)$. We can see from the picture above, that both value their own pieces with value $1/2$. Thus, the allocation is equitable. An example of an allocation that is not equitable, is letting player A receive piece $[0,3/4)$ and player B receive $[3/4,1)$. Thus the values would be $3/4$ and $1/2$, respectively. Clearly, these two values are not equal.

Now that we have defined some of the definitions and properties that we will be using, we can start looking at the specific theorems. In both of the following sections, when we are finding the value of an interval, we will be using the notation, $v_A(0,x)$ instead of $v_A([0,x))$ in order to clean up the notation.

3. Cake-Cutting

This section will discuss cake-cutting and the many different combinations of the properties that we can see and use. Note that in this section we require that a person's value to the entire cake must equal 1. Also, note that each cake must have $n-1$ cuts in the cake in order to have n pieces. We will use this fact in Lemma 3.1.

Lemma 3.1. *Let $\langle S_1,S_2,\dots,S_n\rangle$ and $\langle T_1,T_2,\dots,T_n\rangle$ be allocations of cake. Then, $\exists j \in \{1,\dots,n\}, \exists i \in \{1,\dots,n\}$ such that $T_j \subseteq S_i$.*

Proof. Let $\langle S_1,S_2,\dots,S_n\rangle$ and $\langle T_1,T_2,\dots,T_n\rangle$ be allocations of cake. Thus there are $n-1$ cuts in a cake. By way of contradiction, assume $\forall j \in \{1,\dots,n\}, \forall i \in \{1,\dots n\} \quad T_j \nsubseteq S_i$. Since $T_j \nsubseteq S_i$, there can be no cuts within the first or nth allocation of $\langle S_1,S_2,\dots,S_n\rangle$. Let $S_1 = [0,a]$ and $T_1 = [0,b]$ where a and b are the end of the pieces. Thus since $T_1 \nsubseteq S_1, b > a$. Let $S_n = [c,1]$ and $T_n = [d,1]$, where c and d are the beginning of the pieces. Since $T_n \nsubseteq S_n, d < c$. Therefore, there are only $n-2$ allocations left to cut within. By the pigeon hole principle, since we still must make $n-1$ cuts, there must be at least one allocation in $\langle S_1,S_2,\dots,S_n\rangle$ that has more than one cut within it. Thus, there is some allocation T_j, that is strictly contained within some allocation $S_i, T_j \subseteq S_i$ which is a contradiction. □

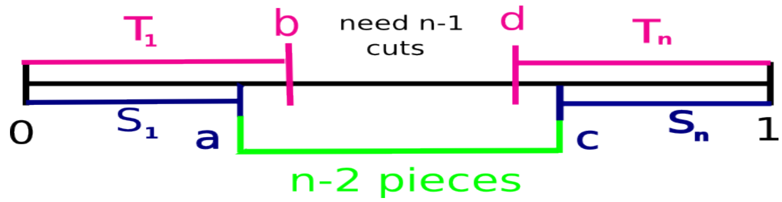

FIGURE 3.1. This picture shows that because there can be no piece of $\langle T_1, T_2, \ldots, T_n \rangle$ cut within the first or nth piece of cake in $\langle S_1, S_2, \ldots, S_n \rangle$, there must be $n-1$ cuts in only $n-2$ pieces.

Theorem 3.2 (2.1a in [BBS09])**.** *Any envy-free allocation of cake among two or more players whose measures are absolutely continuous with respect to one another is also undominated.*

Proof. Let $\langle S_1, S_2, \ldots, S_n \rangle$ be any envy-free allocation of cake. Let v_i denote player i's measure. Since $\langle S_1, S_2, \ldots, S_n \rangle$ is envy-free, we know by Definition 2.8 that each player i prefers his piece to any other piece, i.e. $v_i(S_i) \geq v_i(S_j) \quad \forall i, j$. Furthermore, since the measures are absolutely continuous, we will assume that there is no piece of positive length to which any player assigns value 0, by Definition 2.10. Now, let $\langle T_1, T_2, \ldots, T_n \rangle$ be any other allocation. There are two cases we must consider.
Case 1: Assume $\langle T_1, T_2, \ldots, T_n \rangle$ consists of the same intervals as $\langle S_1, S_2, \ldots, S_n \rangle$ but allocated to different players. Since $\langle S_1, S_2, \ldots, S_n \rangle$ is envy-free, we know $v_i(S_i) \geq v_i(S_j)$. Thus, we also know that $v_i(T_i) \ngtr v_i(S_i)$ for any i. Therefore, $\langle S_1, S_2, \ldots, S_n \rangle$ is not dominated by $\langle T_1, T_2, \ldots, T_n \rangle$.
Case 2: Assume $\langle T_1, T_2, \ldots, T_n \rangle$ consists of at least one different interval from $\langle S_1, S_2, \ldots, S_n \rangle$, i.e., $\exists j$ such that $\forall i$ $T_j \neq S_i$. In this case, some interval of $\langle T_1, T_2, \ldots, T_n \rangle$ must be strictly contained in some interval $\langle S_1, S_2, \ldots, S_n \rangle$ by Lemma 3.1. Thus, $T_j \subseteq S_i$, and $S_i - T_j$ is a positive length. Therefore, we know that $S_i = T_j \cup (S_i - T_j)$ and $v_j(S_i) = v_j(T_j) + v_j(S_i - T_j)$, because the measures are finitely additive. Now, assume $v_j(S_i) = v_j(T_j)$. Thus, $v_j(S_i - T_j) = 0$. However, by our assumption, there is no piece of positive length to which any player assigns value 0. Thus, a contradiction.

Now, assume that $v_j(T_j) > v_j(S_i)$. However, since $T_j \subseteq S_i$, and $S_i = T_j \cup (S_i - T_j)$, thus $v_j(S_i) = v_j(S_i - T_j) + v_j(T_j) \geq v_j(T_j)$. However, $v_j(S_i - T_j) + v_j(T_j) \geq v_j(T_j)$ cannot be true unless $v_j(S_i - T_j) = 0$. Again, this is a contradiction because there is no piece of positive length to which any player assigns value 0.

Therefore, the value for T_j is strictly less than the value S_i, i.e. $v_j(T_j) < v_j(S_i)$, and since $\langle S_1, S_2, \ldots, S_n \rangle$ is envy-free, player j prefers his own piece over player i's piece, $v_j(S_i) \leq v_j(S_j)$. Also, since $v_j(T_j) < v_j(S_i)$ and $v_j(S_i) \leq v_j(S_j)$ we know $v_j(T_j) < v_j(S_j)$. Thus, the allocation $\langle T_1, T_2, \ldots, T_n \rangle$ does not dominate the allocation $\langle S_1, S_2, \ldots, S_n \rangle$.
Therefore, any envy-free allocation of a cake among two or more players whose measures are absolutely continuous with respect to one another is also undominated. ☐

As a result found in "How to Cut a Cake Fairly", Walter Stromquist, we know that for two or more players, there is an allocation x and a way to assign the pieces to the players such that all players prefer their assigned pieces [Str80]. The next theorem that we will look at, we need to know the definition for uniform. We define the word uniform to mean that a player's value for the piece is spread evenly over the entire piece. Mathematically, let I be a given interval for a piece. We say that v is uniform on a set A if for every $I \subseteq A$,

$$v(I) = v(A)\frac{\text{length}(I)}{\text{length}(A)}.$$

Theorem 3.3 (2.2 in [BBS09]). *For three or more players, there are measures that are not absolutely continuous with respect to one another, such that there does not exist an allocation that is envy-free and undominated.*

Proof. Suppose that there are three players, and let the cake be the interval $[0,1)$. Define the measures v_A, v_B, v_C by the following table with measures for players A, B, and C as follows:

	$[0,1/6)$	$[1/6,1/3)$	$[1/3,1)$
Player A	$\frac{1}{3}$	0	$\frac{2}{3}$
Player B	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
Player C	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

The measures defined above, were ones that we created in order to show a counter example, to prove that there may be no allocation that is envy-free and undominated. Assume that each player's measure is uniform on each of the three segments shown in the chart. As an example, let us look at $v_A(0,1/12)$, $v_B(0,1/12)$, and $v_C(0,1/12)$. We know the measures are respectively, 1/6, 1/12, and 1/12, because $(0,1/12)$ is half the length of the first interval. Thus, since the measures are uniform, we can divide each player's value by 2, and we will get each of their values for the suggested pieces. Also, we know that these measures are not absolutely continuous because player A has value 0 for the middle piece and players B and C do not. Also, note that player B's and player C's measures are uniform on the entire cake, and that all of the player's measures are the same on the interval $[1/3,1)$.

Consider an allocation $\langle S_A, S_B, S_C\rangle$. We will show that this allocation, $\langle S_A, S_B, S_C\rangle$ is not both envy-free and undominated. First note that in order for $\langle S_A, S_B, S_C\rangle$ to be envy-free, we must have $v_i(S_i) \geq 1/3$ for each $i = A,B,C$. This is because they all value each piece with value of $1/3$. For example, the interval $[0,1/3)$ we must add the first two columns. Thus, player A values this piece $1/3+0$. This is clearly $1/3$. For players B and C, since they have the same measures over the entire cake, we know that they value this piece $1/6+1/6$ which equals $1/3$. Then for the next piece, we look at the interval $[1/3,2/3)$, we must divide the last column by two, since $[1/3,2/3)$ is half of $[1/3,1)$. Therefore, players A, B and C all value these last two pieces with value $1/3$. Thus, each player must receive at least $1/3$ of the cake according to their own measure.

Case 1: Suppose that player A receives the leftmost piece: $S_A = [0,x)$ for some $x < 1$. Thus, there are three cases. Case one, assume $x > 1/3$, then, there is not enough cake remaining for B and C to have at least 1/3 each. This is because the each value the piece $[0,1/3)$ with a value of $1/3$. Also, since each player's value of the entire cake must add to 1, there is only $2/3$ value remaining. However, players B and C have the same measure for the entire cake, so dividing the piece by two will result in a value of less than $1/3$, so they would each envy player A's piece. Case two, assume $x < 1/3$, then $v_A(S_A) \leq 1/3$.

Therefore, in order for player A's measures to add to one, there is more than $2/3$ value remaining. Thus, when split among players B and C, we know in order to maintain the envy-free property, players B and C must divide the remainder of the cake equally based on the measures. So, $v_A(S_B) = v_A(S_C) = \frac{(1-x)}{2} > 1/3$. This violates the envy-freeness property.

Case three, assume that $x = 1/3$, then the pieces are $[0,1/3)$, $[1/3,2/3)$, and $[2/3,1)$ or $[0,1/3)$, $[2/3,1)$, and $[1/3,2/3)$, depending if player B or player C receives the right most piece. This allocation is envy-free, but it is dominated by the allocation $\langle T_A, T_B, T_C\rangle$ where $T_A = [0,1/6)$, $T_B = [1/6,7/12)$, and $T_C = [7/12,1)$, since $v_A(S_A) = 1/3$, $v_B(S_B) = 1/3$, and $v_C(S_C) = 1/3$, but $v_A(T_A) = 1/3$, $v_B(T_B) = 5/12$, and $v_C(T_C) = 5/12$.

Case 2: Suppose that player A does not receive the leftmost piece. That means that B or C receives the leftmost piece. Without loss of generality, since players B and C each have the same measures throughout the cake, we can assume that B receives the leftmost piece: $S_B = [0,x)$, for some x. Again, there are three cases. Case 1, assume $x > 1/3$, then, there is not enough cake for both A and C to have at least $1/3$ each. Since they each value the remaining part of the cake with less than $2/3$ value, we know that if we cut the remaining piece into two pieces for players A and C, then they can not both receive a piece that they value $1/3$. Therefore, it is not envy-free. Case two, assume $x < 1/3$, then A and C must divide the remainder of the cake. Since $v_B(S_A) + v_B(S_B) + v_B(S_C) = 1$, we know that that $v_B(S_A) + v_B(S_C) = 1 - v_B(S_B) \Rightarrow v_B(S_A) + v_B(S_C) > 2/3$. Thus, $v_B(S_A) > 1/3$ or $v_B(S_C) > 1/3$. Thus, player B will envy player A's piece or player B's piece, thus this allocation is not envy-free. Case three, assume that $x = 1/3$. In order to have an allocation that is envy-free, the pieces must be $[0,1/3)$, $[1/3,2/3)$, $[2/3,1)$. However the allocation $\langle S_A, S_B, S_C\rangle$ is dominated by the allocation $\langle T_A, T_B, T_C\rangle$ where $T_A = [0,7/12)$, $T_B = [1/6,7/12)$, and $T_C = [7/12,1)$, since $v_A(S_A) = 1/3$, $v_B(S_B) = 1/3$, and $v_C(S_C) = 1/3$, but $v_A(T_A) = 1/3$, $v_B(T_B) = 5/12$, and $v_C(T_C) = 5/12$. Since, $v_A(S_A) = 1/3$, $v_B(S_B) = 1/3$, and $v_C(S_C) = 1/3$, but $v_A(T_A) = 1/3$, $v_B(T_B) = 5/12$ and $v_C(T_C) = 5/12$. Since, $v_A(T_A) = 1/3 \geq v_A(S_A) = 1/3$, $v_B(T_B) = 5/12 > v_B(S_B) = 1/3$, and $v_C(T_C) = 5/12 > v_C(S_C) = 1/3$. Therefore, since the allocation $\langle T_A, T_B, T_C\rangle$ dominates $\langle S_A, T_B, T_C\rangle$, we know for three or more players with measures that are not absolutely continuous with respect to one another, there need not exist an allocation that is envy-free and undominated. □

Theorem 3.4 (2.3 in [BBS09]). *For two players and any cake and corresponding measures,*

a. *there exists an allocation that is both envy-free and undominated.*
b. *there exists an allocation that is both envy-free and equitable; and*
c. *if the measures are absolutely continuous with respect to one another, then there exists an allocation that is envy-free, undominated, and equitable.*

Proof. For part b, assume there are two players. By Theorem 2.6, if we let $g(x) = v_A(0,x)$,

$$\lim_{x\to c^-} g(x) = \lim_{x\to c^-} v_A(0,x) = \lim_{i\to\infty} v_A(0,x_i) = v_A(0,c) = f(c),$$

where (x_i) is an increasing sequence approaching c. The same argument holds true for continuity from the right, and for proving $v_B(0,x)$ is continuous. Define $f(x) = v_A(0,x) - v_B(x,1)$. Note that $f(x)$ is continuous on the interval $[0,1]$. Note that $f(0) = v_A(0,0) -$

$v_B(0,1) < 0$, and $f(1) = v_A(0,1) - v_B(1,1) > 0$, so by the Intermediate Value Theorem there is some $x \in [0,1]$ so that $f(x) = 0$. Let player A's piece be denoted by $S_A = [0,x)$ and player B's piece by $S_B = [x,1)$. We know that $v_A(S_A) + v_A(S_B) = 1$ and that $v_B(S_A) + v_B(S_B) = 1$. Thus, we can assume that $v_A(S_A) = v_B(S_B) \geq 1/2$, because if not we can simply exchange pieces to make this true. This shows that there exists an allocation that is both envy-free and equitable.

For part c, assume that the measures are absolutely continuous with respect to one another. Then, by Theorem 3.2, any envy-free allocation of cake among two or more players whose measures are absolutely continuous with respect to one another is also undominated. Therefore, we have an allocation that is absolutely continuous, envy-free and undominated. Then, by part b, we know that there does exist an allocation that is both envy-free and equitable. Therefore, we know that there exists an allocation that is envy-free, undominated, and equitable.

For part a, we note that if the allocation $\langle S_A, S_B \rangle$ from part b is dominated, then, by Theorem 3.2, we know that the players' measures are not absolutely continuous with respect to one another. Assume $\langle S_A, S_B \rangle$ is dominated which means, without loss of generality, there exists an allocation $\langle T_A, T_B \rangle$ such that $v_A(T_A) > v_A(S_A)$ and $v_B(T_B) \geq v_B(S_B)$ for player A and player B.

We will show that regardless of which piece each player gets and the values for $v_A(S_A)$ and $V_B(S_B)$ are we can move the x from part b from the original position and still find a piece where it is dominated.

Case 1: Assume that $v_A(S_A) > 1/2$. As a consequence, since $v_B(S_B) = v_A(S_A)$, we know $v_B(S_B) > 1/2$. Therefore, if $v_A(S_A) > 1/2$, then $v_A(S_B) < 1/2$. This is because player A's value for the entire cake must equal 1. Therefore, if player A values his own piece greater than $1/2$, then his value for the other piece must be less than $1/2$. Also, for the same reason as player A, player B's value for the entire cake must also equal 1. Thus, since $v_B(S_B) > 1/2$, then $v_B(S_A) < 1/2$. There are five different allocations that we must look at. The allocations could be as follows:

Case (i), assume player A's piece is the left piece and $T_A \subset S_A$. Since $\langle T_A, T_B \rangle$ dominates $\langle S_A, S_B \rangle$, we know that player A's value remains the same, $v_A(T_A) = v_A(S_A)$. Then, since $S_B \subset T_B$ and $\langle T_1, T_2, \ldots, T_n \rangle$ dominates $\langle S_1, S_2, \ldots, S_n \rangle$, we know that player B's value increases, $v_B(T_B) > v_B(S_B)$. The same argument works for case (ii), when player A receives the right piece. Case (iii), assume player A's piece is the left piece and $S_A \subset T_A$. Since $T_B \subset S_B$ and $\langle T_A, T_B \rangle$ dominates $\langle S_A, S_B \rangle$, we know that player B's value remains the same, $v_B(T_B) = v_B(S_B)$. Thus, we know that player A's value must increase to follow the dominated requirement, $v_A(T_A) > v_A(S_A)$. The same argument holds for case (iv), when player A receives the right piece. Hence, regardless of which piece player A receives as long as he receives the same piece on the same side in both allocations, x can be moved to the left or to the right some positive distance, and the assumption that S_A is dominated by T_A still holds true. Case (v), without loss of generality, assume player A's piece in allocation S is on the left side and in allocation T is on the right side. Also, since $v_A(S_A) > 1/2$, we know that $v_A(S_B) < 1/2$. However, since $\langle T_A, T_B \rangle$ dominates $\langle S_A, S_B \rangle$, we know that $v_A(T_A) > v_A(S_A)$ or $v_B(T_B) > v_B(S_B)$, therefore, $v_A(T_A) > v_A(S_B)$

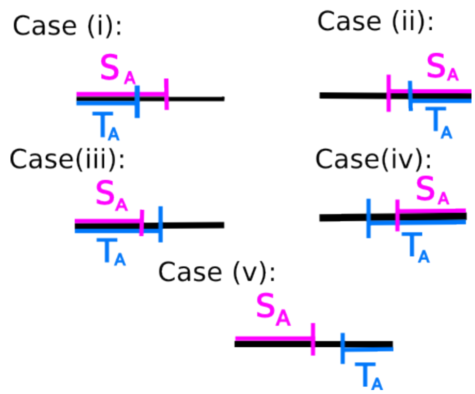

FIGURE 3.2. This picture shows the four different allocations that could happen when Player A's piece is on the same side for both allocation S and allocation T. It also shows in case (v) when allocation S and allocation T are on different sides.

or $v_B(T_B) > v_B(S_B)$. However, since $T_A \subset S_B$, and we know that $v_A(T_A) > v_A(S_B)$ we know this is a contradiction. Hence, when player A receives the right piece, x can be moved to the left or to the right some positive distance and the assumption still holds true. Therefore, when $v_A(S_A) = v_B(S_B) > 1/2$, it is possible to move x to the left or to the right some positive distance and have one player's value for his own piece increase while the other player's value of his own piece remains the same.

Case 2: Assume that $v_A(S_A) = 1/2$. Without loss of generality for case (i) and (ii), assume player A's piece is the left piece in both allocation S and allocation T, and in case (iii) and case (iv), player A receives the left piece in allocation S and the right in allocation T. As a consequence, since $v_B(S_B) = v_A(S_A)$, we know that $v_B(S_B) = 1/2$. Therefore, if $v_A(S_A) = 1/2$, then $v_A(S_B) = 1/2$ and if $v_B(S_B) = 1/2$, then $v_B(S_A) = 1/2$. There are four different allocations that we must look at. The allocations could be as follows: Case

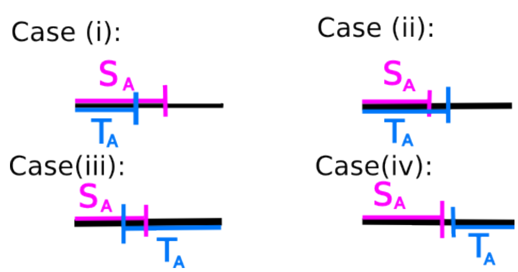

FIGURE 3.3. This picture shows the four different allocations that could happen. Case (i) and case (ii) show the results when player A's piece is on the same side for both allocation S and allocation T. Then, it shows case (iii) and case (iv) where Player A's piece for allocation S is on a different side than his piece for allocation T.

(i), assume $T_A \subset S_A$. Since $\langle T_A, T_B \rangle$ dominates $\langle S_A, S_B \rangle$, we know that player A's value remains the same, $v_A(T_A) = v_A(S_A)$. Then, since $S_B \subset T_B$, we know that player B's value increases, $v_B(T_B) > v_B(S_B)$. Case (ii), assume $S_A \subset T_A$. Thus, we know that player A's value increases, $v_A(T_A) > v_A(S_A)$. Then, since $T_B \subset S_B$ and $\langle T_A, T_B\rangle$ dominates $\langle S_A, S_B\rangle$, we know that player B's value remains the same, $v_B(T_B) = v_B(S_B)$. Case (iii), swap S_A and S_B. We can do this because the values for both S_A and S_B are equal to $1/2$, $v_A(S_A) = 1/2$, $v_A(S_B) = 1/2$, $v_B(S_B) = 1/2$, $v_B(S_B) = 1/2$. Thus, after the swap, S_A and T_A are on the same side as each other, and $S_A \subset T_A$. Thus, player A's value increases, $v_A(T_A) > v_A(S_A)$. Then, since $T_B \subset S_B$ and $\langle T_A, T_B\rangle$ dominates $\langle S_A, S_B\rangle$, we know that player B's value remains the same $v_B(T_B) = v_B(S_B)$. Case (iv), swap S_A and S_B again. Also, note, that the values are the same for both players again. After the swap, S_A and T_A are on the same side as each other, and $T_A \subset S_A$. Since $\langle T_A, T_B\rangle$ dominates $\langle S_A, S_B\rangle$, we know that player A's value remains the same, $v_A(T_A) = v_A(S_A)$. Then, since $S_B \subset T_B$, we know that player B's value increases, $v_B(T_B) > v_B(S_B)$. Therefore, when $v_A(S_A) = v_B(S_B) = 1/2$, it is possible to move x to the left or to the right some positive distance and have one player's value for his own piece increase while the other player's value of his own piece remains the same.

Therefore, in any of the cases, if we move x from its original position to obtain a piece where one player's value increases as much as possible and has no effect on the other player's value of his piece, then the resulting allocation will be envy-free and undominated. Therefore, there exists an allocation that is both envy-free and undominated. □

Corollary 3.5. *For two players there exists a measure and an allocation that is envy-free, undominated and not equitable.*

Proof. Assume there are two players, player A and player B. Also, let the measures v_A and v_B for the allocation $\langle S_A, S_B\rangle$ be as follows:

	$[0,1/2)$	$[1/2,1)$
A	$\frac{1}{3}$	$\frac{2}{3}$
B	$\frac{5}{6}$	$\frac{1}{6}$

Let $S_A = [1/2,1)$ and $S_B = [0,1/2)$. Thus, $v_A(S_A) = 2/3$, $v_A(S_B) = 1/3$, $v_B(S_A) = 1/6$, and $v_B = 5/6$. Therefore, $v_A(S_A) \geq v_A(S_B)$ and $v_B(S_B) \geq v_B(S_A)$ for player A and player B, where v_A denotes player A's measure and S_A denotes player A's value to a piece. Thus, $\langle S_A, S_B\rangle$ is envy-free. Now, consider moving the 'cut' to the left, as player A's piece gets larger player A's value gets larger and player B's value gets smaller. Now, consider moving the 'cut' to the right. Player A's piece gets smaller along with his value, whereas player B's piece is increasing and along with his value. Thus, changing the 'cut' for the allocations cannot be moved to the left or right. Thus, $\langle S_1, S_2, \ldots, S_n\rangle$ is undominated. We have shown that $\langle S_A, S_B\rangle$ is envy-free and undominated. We also know that this allocation is not equitable since each player's values are not equal, $v_A(S_A) = 2/3$ and $v_B(S_B) = 5/6$. Also, we know that we can not swap the pieces because of the measures they gave the opposite pieces. Hence, for two players with allocations that are both envy-free and undominated, we showed that the allocation may not be equitable. □

Corollary 3.6. *For two players with measures that are not absolutely continuous with respect to one another, there exists an allocation that is envy-free, equitable and dominated.*

Proof. Assume there are two players, player A and player B. Also, let the measures v_A and v_B for the allocation $\langle S_A, S_B\rangle$ be as follows:

	$[0,1/4)$	$[1/4,1/2)$	$[1/2,3/4)$	$[3/4,1)$
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$

Also, note that their measures, v_A and v_B, are not absolutely continuous with respect to one another. Let $\langle S_A, S_B\rangle = \langle [0,1/2), [1/2,1)\rangle$. Let v_A denote player A's measure and S_A denote player A's piece. Therefore, $v_A(S_A) = 1/2$, $v_A(S_B) = 1/2$, $v_B(S_B) = 1/2$, $v_B(S_A) = 1/2$. Since $v_A(S_A) = v_A(S_B)$, and $v_B(S_A) = v_B(S_B)$, this given allocation is envy-free. Furthermore, since both player A and player B's values for their own pieces have the same value, $v_A(S_A) = v_B(S_B)$, this allocation is also equitable. This allocation is dominated by the allocation $\langle T_A, T_B\rangle = \langle [0,3/4), [3/4,1)\rangle$, since, $v_A(T_A) = 3/4$, $v_A(T_B) = 1/4$, $v_B(T_A) = 1/2$, and $v_B(T_B) = 1/2$. Therefore, $v_A(T_A) > v_A(S_A)$ because $3/4 > 1/2$, and $v_B(T_B) = v_B(S_B) = 1/2$. Therefore, player A likes T_A better than S_A and player B's value remains the same for both allocations. Thus, $\langle S_A, S_B\rangle$ is dominated by $\langle T_A, T_B\rangle$. Therefore, for two players with measures that are not absolutely continuous with respect to one another and allocations that are both envy-free and equitable, we know that the allocation may not be undominated. □

Corollary 3.7. *For two players, there are some measures that are not absolutely continuous with respect to one another, such that for all allocations $\langle S_A, S_B\rangle$, either $\langle S_A, S_B\rangle$ is dominated or $\langle S_A, S_B\rangle$ is not equitable.*

Proof. Assume there are two players, player A and player B. Also, let the measures v_A and v_B for the allocation $\langle S_A, S_B\rangle$ be as follows:

	$[0,1/4)$	$[1/4,1/2)$	$[1/2,3/4)$	$[3/4,1)$
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$

We will show that in this allocation S, when the values for player A and B are equal, we will be able to find another allocation that will dominate $\langle S_1, S_2, \ldots, S_n\rangle$. Thus, consider the allocation $S_A = [0,1/2)$ and $S_B = [1/2,1)$. Thus, $v_A(S_A) = 1/2 = v_B(S_B)$. However, we see that this piece is dominated by all of the allocations created by sliding the cut to the right up until $S_A = [0,3/4)$ and $S_B = [3/4,1)$. Now, consider the allocation where player A receives the right most piece and player B receives the left. Thus, $S_A = [1/2,1)$ and $S_B = [0,1/2)$. Again, these values are equitable but dominated if the two players were to switch pieces. Therefore, when the values of player A and B are equal, the allocation is dominated. □

In the following section, we will discuss that some of the combinations of properties used together and proved in the cake cutting theorems can not be properties that are combined in pie cutting.

4. Pie-Cutting

For this section, we will be discussing dividing pie among three or more players. We will show that "there exists a pie and corresponding measures for which no allocation is envy-free and undominated" [BBS09]. In order to do this, we must prove nine different lemmas, using the following properties. We must first define some new definitions for this section. In the picture below, a player's window is the white spaces. Player 1's window is $[0,2)$. According to the article, we define a player's window as the following; for $i = 2,\dots,n$, define the ith player's window to be the interval $[i-\frac{7}{6}, i+\frac{10}{6})$. Note that the endpoints are defined mod n, so the last two windows actually end at $\frac{4}{6}$ and $\frac{10}{6}$. We will talk about a player's window when defining what a player's piece will look like. We are told that with certain exceptions, the value of a piece to a player is 1 per unit length inside the player's window, and $1-n^{-4}$ unit length outside the player's window. In some of the player's windows there is an extra value of $C = n^{-8}$, and $\varepsilon = n^{-16}$. The cell including the C is called a player's bonus cell. Player 1 has many positive and negative adjustments:

$+C-\varepsilon$, uniformly over $[0,1/6)$
$-C$, uniformly over $[2/6,3/6)$
$+C$, uniformly over $[3/6,4/6)$
$-C+\varepsilon$, uniformly over $[4/6,5/6)$
$-C+\varepsilon$, uniformly over $[7/6,8/6)$
$+C$, uniformly over $[8/6,9/6)$
$-C$, uniformly over $[9/6,10/6)$
$+C-\varepsilon$, uniformly over $[11/6,2)$.

We will use d_i to represent the length of S_i.

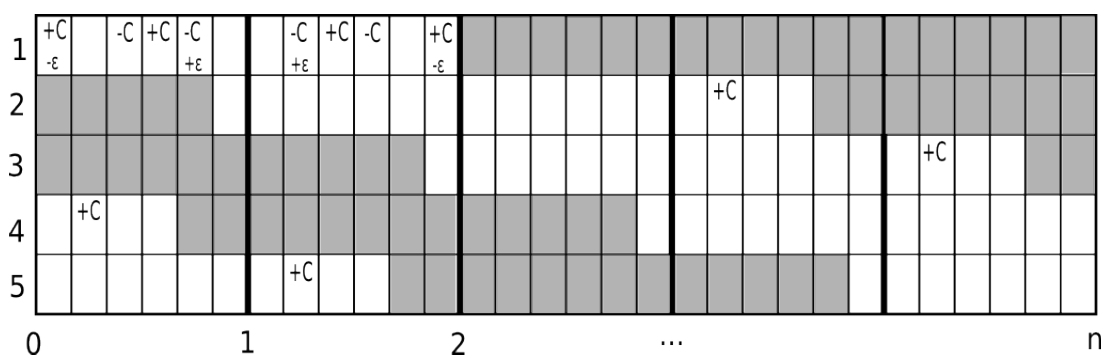

FIGURE 4.1. This picture is from the article that I am using. [BBS09]. The white cells in this picture represent a player's window, and the shaded cells represent pieces outside the window. The C values shown in the picture represent a bonus cell for the players. The $C = n^{-8}$ and the $\varepsilon = n^{-16}$, in the players window.

Let $\langle S_1, S_2, \dots, S_n\rangle$ be envy-free and undominated.

Lemma 4.1. *Each piece has both value and length greater than $1-n^{-4}$. That is, for each i, we have $d_i > 1-n^{-4}$ and $v_i(S_i) > 1-n^{-4}$. Furthermore, the total length $L(k)$ of any k consecutive pieces must satisfy $k-n^{-3} < L(k) < k+n^{-3}$.*

Proof. We will begin this proof by looking only at player 1's values for the entire pie, $[0,n]$. Player 1's value is $2+(n-2)(1-n^{-4})$, where 2 is the value from player 1's window, and

$(n-2)(1-n^{-4})$, is the value outside player 1's window. Thus, simplifying player 1's value for the entire pie we obtain the equation

$$\begin{aligned} v_1(0,n) &= n-(n-2)(n^{-4}) \\ &= n-n^{-3}+2n^{-4}. \end{aligned}$$

Now, since these allocations are envy-free, we know that player 1's value of his own piece must be greater than the average of his value for the entire pie. Therefore, we know

$$v_1(S_1) \geq \frac{v_1(0,n)}{n} = 1-n^{-4}+2n^{-5};$$

otherwise, according to player 1's measure, another player's piece would have value greater than $v_1(S_1)$. Thus, $v_1(S_1) > 1-n^{-4}$. Also, by the measures shown in Figure 4.1 above, we see that there can be no piece that has value more than $C = n^{-8}$ greater than its length, therefore, $v_1(S_1) \leq n^{-8}+d_1$. To show that the length must be greater than $1-n^{-4}$, we know that $1-n^{-4}+2n^{-5} \leq v_1(S_1)$. Also, we have already shown $v_1(S_1) \leq n^{-8}+d_1$. Therefore, we know $1-n^{-4}+2n^{-5} \leq v_1(S_1) \leq n^{-8}+d_1$. This results in $1-n^{-4}+2n^{-5}-n^{-8} \leq d_1$. Since n^{-8} is a smaller number than $2n^{-5}$, we know that the length will be strictly greater than $1-n^{-4}$. The same reasoning applies for players i, for $i = 2,\ldots,n$. To show this, we must find player i's value of his own piece, for $i = 2,\ldots,n$. Therefore, for $i = 2,\ldots,n$,

$$\begin{aligned} v_i([0,n)) &= \frac{17}{6}(1)+C+\left(n-\frac{17}{6}\right)\left(1-n^{-4}\right) \\ &= \frac{17}{6}+C+n-\frac{17}{6}-n^{-3}+\frac{17}{6}n^{-4} \\ &= n-n^{-3}+\frac{17}{6}n^{-4}+C. \end{aligned}$$

In the equation above, the $(n-\frac{17}{6})$ is the length of the pieces outside of players $i = 2,\ldots,n$ window. Multiplying that by $1-n^{-4}$ represents player i's values to those pieces. Each player other than player 1 has 17 cells inside their window. Therefore, for each player i, for $i = 2,\ldots,n$, their value must be greater than their average of the entire cake. Thus,

$$v_i(S_i) \geq 1-n^{-4}+\frac{17}{6}n^{-5}+n^{-7}.$$

By the same reasons as above, the $v_i(S_i) > 1-n^{-4}$, and $d_i > 1-n^{-4}$.

Let $L(k)$ be the length of k consecutive pieces, for $\forall k \in \{1,\ldots,n\}$. Now to show that $L(k) > k-n^{-3}$, we must use the fact that $d_i > 1-n^{-4}$. Multiplying this fact by k, results in $L(k) > k(1-n^{-4}) = k-kn^{-4}$. This shows us that $k-kn^{-4}$ is the value for the k pieces outside of the player's window. Since $k \leq n$, we know that $k-kn^{-4} \geq k-n\cdot n^{-4} = k-n^{-3}$. Thus, the lower bound for our length, $L(k)$ is $k-n^{-3}$.

Now we want to show that our upper bound for $L(k)$, is $L(k) < k+n^{-3}$. To do this, we suppose by way of contradiction that $L \geq k+n^{-3}$. Also, we know that $L(n) = n$. Therefore, for the $n-k$ remaining pieces,

$$\begin{aligned} L(n-k) &\leq n-(k+n^{-3}) \\ &= n-k-n^{-3}. \end{aligned}$$

However, our lower bound states $L(k) > k-n^{-3}$, and we have just shown, $L(n-k) \leq n-kn^{-3}$. Thus, a contradiction. Therefore, $L(k) < k+n^{-3}$. It follows that the total length $L(k)$ of any k consecutive pieces must satisfy $k-n^{-3} < L(k) < k+n^{-3}$.

□

Again, using the measure above, any undominated allocation has the property of the following lemma.

Lemma 4.2. *The total value satisfies* $v_1(S_1)+\cdots+v_n(S_n) \geq n$.

Proof. Suppose by way of contradiction that $v_1(S_1)+\cdots+v_n(S_n) < n$. Now, define a new allocation $\langle T_1,T_2,\ldots,T_n\rangle$ by starting at 0 and letting each piece from the new allocation T_i have length $v_i(S_i)$. Thus, $T_1 = [0,v_1(S_1))$, and $T_2 = [v_1(S_1),v_1(S_1)+v_2(S_2))$. Thus, continuing with this, we get for each i,

$$T_i = [v_1(S_1)+\cdots+v_{i-1}(S_{i-1}),v_1(S_1)+\cdots+v_i(S_i)).$$

Now, to avoid wasting any of the pie, we extend T_n to

$$T_n = [v_1(S_1)+\cdots+v_{n-1}(S_{n-1}),n).$$

By Lemma 4.1, we know that $v_1(S_1)$ has value greater than $1-n^{-4}$, but less than $1+n^{-3}+C$. Also, we know the $d_1 < 1+n^{-3}$. Thus, $T_1 = [0,x)$ where $5/6 < x < 1+n^{-3}+C$. Therefore, in $\langle T_1,T_2,\ldots,T_n\rangle$, T_1 must end within player 1's and player 2's window. Also, according to Lemma 4.1 we know player 1's bonus cells, in $\langle T_1,T_2,\ldots,T_n\rangle$ must add to 0 in order to not violate the upper bound. This is true because looking at the bounds we put on x, player 1's piece ends somewhere between the interval $[5/6,7/6)$. Looking at the picture, we know that this interval has no bonus cells, and the bonus cells in the interval $[0,5/6)$ sum to 0.

Now, we must show that player i's piece, for $i = 2,\ldots,n$ must also fall within player i's window. Thus, looking at player 2, $v_1(S_1)$ must be greater than or equal to $1-1/6$, since player 2's piece starts where player 1's piece ends, and player 2's piece will fall into his window. This argument is trivial by the lower bound we gave in Lemma 4.1. Also, note in Lemma 4.1, we proved that there can be no piece that has value more than $C = n^{-8}$ greater than its length. Therefore, for T_i no player can have bonus cells included in his piece because of the upper bound, and the same argument as player 1. Thus, we know that $v_2(S_2) < 14/6$. Each player i, for $i = 2,\ldots,n$ can not include bonus cells, so the length of T_i must equal the value $v_i(T_i)$ for $i = 2,\ldots,n-1$. Since T_i has length $v_i(S_i)$, we know that $v_i(S_i) = v_i(T_i)$ for each case, except for player n, because T_n is extended to n. Since it is extended to n, we know that player n's piece in the allocation $\langle T_1,T_2,\ldots,T_n\rangle$ has value greater than S_n in $\langle S_1,S_2,\ldots,S_n\rangle$, so $\langle T_1,T_2,\ldots,T_n\rangle$ dominates $\langle S_1,S_2,\ldots,S_n\rangle$. Thus, we have found a contradiction.

□

Lemma 4.3. *If any players' pieces include parts outside of the players' windows, then the total length of these parts cannot exceed n^{-3}.*

Proof. Assume a player's piece includes parts outside the player's window. We define the excess value of each piece S_i to be $\text{exc}(S_i) = v_i(S_i) - d_i$. By Lemma 4.2, $v_1(S_1) + \cdots + v_n(S_n) \geq n$. Also, we know that $d_1 + \cdots + d_n = n$. Therefore, by definition of excess value, $\text{exc}(S_1) + \cdots + \text{exc}(S_n) = v_1(S_1) - d_1 + v_2(S_2) - d_2 + \cdots + v_n(S_n) - d_n \geq 0$. Therefore, the sum of the excess value is greater than 0. So the sum of every player's value outside their respective windows is $-pn^{-4}$, where p is the number of players with pieces outside their windows. This value would be added to the excess value inside each players' respective window. By looking at Figure 4.1 above, we see that no piece can have value greater than $C = n^{-8}$. Therefore, adding up every players excess value inside their windows, we get $nC = n(n^{-8}) = n^{-7}$. Thus, the excess value must be less than $-pn^{-4} + n^{-7}$. If $p > n^{-3}$, thus, we get $-n^{-6} + n^{-7} < 0$, which can not happen by what we have just shown. However, if $p \leq n^{-3}$, our value would be greater than or equal to 0 which is what we want. Thus, the total lengths of these parts cannot exceed n^{-3}. □

Lemma 4.4. *The players' pieces are in order, with S_{i+1} immediately to the right of S_i for each i.*

Proof. We know that the length of each player's piece is approximately 1, and by Lemma 4.3, nearly all of a player's piece must be within the player's window. Also, looking at Figure 4.1, each player's window, except for player 1, is less than 2 unit length. Therefore, if player 2's piece came before player 1's piece, player 1 would have nearly all of his piece outside of his window, thus violating what we proved in Lemma 4.3. This argument holds for every player. Thus, the order of the windows must be where S_{i+1} immediately follows to the right of S_i for each i. □

Lemma 4.5. *For each $i = 2, \ldots, n$, player i's piece (i.e., S_i) cannot include any part of player i's bonus cell.*

Proof. By Lemma 1, we know each piece has length about 1. Assume player 2's piece includes a bonus cell. So, player 2's piece would fall within the interval $(13/6, 20/6)$. Therefore, player 1's piece would end somewhere between the interval $(13/6, 14/6)$. If this is the case, player 1's piece would extend too far outside of his window, which would contradict Lemma 4.3. Now, to show that this occurs if any player's piece includes a bonus cell, let i be the smallest number such that player i has a bonus cell. Therefore, if we let player 3's piece include his bonus cell, we know by looking at Figure 4.1, that player 2 will also include his bonus cell in his piece. Thus, the same argument holds for all players. □

Now, we must limit the possibilities for $\langle S_1, S_2, \ldots, S_n \rangle$ based on what we have proven so far. Then, once we have limited the possibilities, we will eliminate them to show that these possibilities can not really happen.

Lemma 4.6. *S_1 must have one of these forms:*

- $[0, x)$, *or*
- $[y, 2)$, *or*

- $[\frac{1}{2}+x, \frac{3}{2}+y)$, *where* $|x|+|y| \leq \frac{1}{3}n^{-8}$.

Proof. First, we must note that from Lemma 4.1, there can be no piece that has value more than $1+C$. Also, note from Lemma 4.5, no other player's piece has value greater than its length, $d_i \geq v_i(S_i)$ for $i \neq 1$. Also, note that by Lemma 4.2, the values must sum to at least n. Therefore, we know that $v_1(S_1)$ must be at least equal to the length of S_1. Therefore, these cases listed above are the only possibilities.

Assume that S_1 does not take one of the first two forms. Therefore, we will look at the third form, $[\frac{1}{2}+x, \frac{3}{2}+y)$. Note that we defined excess value of each piece S_i to be $v_i(S_i)$ minus d_i, the length of S_i. Now, consider the third case, the piece $[1/2, 3/2)$ has excess value of $1+2\varepsilon-1$ where the $1+2\varepsilon$ represents the value and 1 is representing the length. Thus, the excess value of $[1/2, 3/2)$ is $+2\varepsilon$. Now consider moving the 1/2 boundary by an amount x. The excess value of this piece, $[1/2+x, 3/2)$ if $x > 0$, would be

$$\begin{aligned} &1+2\varepsilon - \frac{x}{\frac{1}{6}}\left(\frac{1}{6}+C\right)-(1-x) \\ &= 1+2\varepsilon - x - 6xC - 1 + x \\ &= 2\varepsilon - 6xC. \end{aligned}$$

Therefore, moving the left boundary by an amount x to the right reduces the excess by $-6xC = -6xn^{-8}$. Therefore, in order to satisfy the condition that the pieces must have excess of at least 0, we are required to have $6xn^{-8} \leq 2\varepsilon = 2n^{-16}$. Solving for x results in

$$x \leq \frac{2n^{-16}}{6xn^{-8}} = \frac{1}{3}n^{-8}.$$

Now, if $x < 0$, the excess value would be

$$\begin{aligned} &1+2\varepsilon + \frac{|x|}{\frac{1}{6}}\left(\frac{1}{6}-C\right)-(1+|x|) \\ &= 1+2\varepsilon - \frac{x}{\frac{1}{6}}\left(\frac{1}{6}+C\right)-(1-x) \\ &= 1+2\varepsilon - x - 6xC - 1 + x \\ &= 2\varepsilon - 6xC. \end{aligned}$$

Note that this result is the same as above, regardless of which way the left bound is moved. The same argument works when moving the right bound, $[1/2, 3/2+y)$. □

Lemma 4.7. $S_1 \neq [0,1)$ *and* $S_1 \neq [1,2)$.

Proof. Assume $S_1 = [0,1)$. Since $\langle S_1, S_2, \ldots, S_n\rangle$ is envy-free, we know that results in each piece having length 1. Thus, the values vector, $\langle v_1(S_1), \ldots, v_n(S_n)\rangle = \langle 1, 1, \ldots, 1\rangle$. However, this allocation is dominated by the allocation $T_1 = [1/2, 3/2)$, and each piece having length 1. The values for players $i = 2, \ldots, n$ all remain the same with value 1. We know this by looking at the picture above. However, player 1's value becomes $1+2\varepsilon$ which is greater than his previous values. The value vector for allocation $\langle T_1, T_2, \ldots, T_n\rangle$ looks as follows: $\langle v_1(T_1), \ldots, v_n(T_n)\rangle = \langle 1+2\varepsilon, 1, \ldots, 1\rangle$. However, we assumed our $\langle S_1, S_2, \ldots, S_n\rangle$ was undominated. Thus, a contradiction. The same argument holds for $S_1 = [1,2)$. □

Lemma 4.8. *$S_1 \neq [0,x)$ for any x and $S_1 \neq [y,2)$ for any y.*

Proof. Let $S_1 = [0,x)$. Again, let d_i denote the length of piece S_i. By Lemma 4.1, we know $d_i > 1 - n^{-4}$. Also, we know that for each $i = 2,\dots,n-1$, both S_{i+1} and S_i are within player i's window. We know this because each window for $i = 2,\dots,n$, is of length $17/6$. Thus, there can be two players' pieces within one player's window.

Furthermore, the values of each player i's piece, where $i = 2,\dots,n$ must also be about 1 since the values are 1 per unit length when inside the player's window. Therefore, because of the envy-free requirement, we know $v_i(S_{i+1}) \leq v_i(S_i)$. Thus, we know $d_{i+1} \leq d_i$. Also, since player 1's piece is starting at 0, we know that player 1's piece is within player n's window. Therefore using the same argument as above, $d_1 \leq d_n$. Now we have $d_1 \leq d_n \leq d_{n-1} \leq \cdots \leq d_2$. Also, from Lemma 4.7, $x \neq 1$, we know $d_1 < 1 < d_2$. Since player n's window includes $[0,10/6)$, we know $d_1 < 1$ in order to not break the envy-freeness requirement. Also, by Lemma 4.7, we know $S_1 \neq [0,1)$, and by Lemma 4.2, $v_1(S_1)$ must be at least equal to the length of S_1. Also, S_1 must be at least $5/6$ length because we said for each $i = 2,\dots,n-1$, both S_{i+1} and S_i are within player i's window. Therefore, we know $v_1(S_1) = d_1$. To show a contradiction, we will show $v_1(S_2) > v_1(S_1)$ violates the envy-freeness requirement. Player 2's piece has three different possibilities. The first one is that player 2's piece extends outside player 1's window. If this is the case, $v_1(S_2) > v_1(S_1)$. The second possibility is that player 2's piece ends at exactly where player 1's window ends. Since $S \neq [0,1)$ and $d_1 < 1$, player 2's piece begins a little before 1. Therefore, if S_2 ends at 2, $v_1(S_2) > v_1(S_1)$. The last possibility is player 2's piece ends within player 1's window. If this is the case, player 2's piece must end between $11/6$ and 2. To prove this third possibility, we must use the fact that player 1's value just to the left of 2 is $1+6(C-\varepsilon)$ per unit. Thus $v_1(S_2) = v_1(d_1, d_1+d_2) > v_1(d_1, d_1+1)$, for the reasons mentioned above. This equals $v_1(0,2) - v_1(0,d_1) - v_1(d_1+1,2)$. The $v_1(0,2)$ represents player 1's window and we are subtracting $v_1(0,d_1)$ representing player 1's piece, and $v_1(d_1+1,2)$ represents the remaining piece of player 1's window that player 2's piece cannot include because it would violate the envy-free requirement. We know $v_1(0,2) = 2$, $v_1(0,d_1) = d_1$, and $v_1(d_1+1,2) = (2-(d_1+1))(1+6(C-\varepsilon))$. Therefore,

$$\begin{aligned}
v_1(S_2) &> v_1(0,2) - v_1(0,d_1) - v_1(d_1+1,2) \\
&= 2 - d_1 - (2-(d_1+1))(1+6(C-\varepsilon)) \\
&= 2 - d_1 - (2-(d_1+1) + 12(C-\varepsilon) - 6(C-\varepsilon)(d_1+1)) \\
&= 1 + 12C - 12\varepsilon - 6(Cd_1 + C - \varepsilon d_1 - \varepsilon) \\
&= 1 + 6C - 6\varepsilon - 6Cd_1 + 6\varepsilon d_1 \\
&= 1 + 6(C-\varepsilon)(1-d_1).
\end{aligned}$$

Clearly, $v_1(S_2) > 1$ and $v_1(S_1) < 1$. Then $v_1(S_2) > v_1(S_1)$. Therefore, $S_1 \neq [0,x)$ for any x. The argument for $S_1 \neq [y,2)$ for any y is symmetrical to this argument based on the positions of the bonus cells in player 1's window. □

Lemma 4.9. *$S_1 \neq [\frac{1}{2}+x, \frac{3}{2}+y)$ for any x and y with $|x|+|y| \leq \frac{1}{3}n^{-8}$.*

Proof. Suppose $S_1 = [\frac{1}{2}+x, \frac{3}{2}+y)$ where $|x|+|y| \leq \frac{1}{3}n^{-8}$. Again, we suppose for each $i = 2,\dots,n-1$ both S_{i+1} and S_i are within player i's window and S_{i+1} include player i's bonus cell. Since we are using the interval $[\frac{1}{2}+x, \frac{3}{2}+y)$, and even moving a little either way does not shift the pieces enough such that player $i+1$'s piece would not include player i's bonus cell. Therefore, we know $d_{i+1}+C \leq v_i(S_i)$. Therefore, $d_{i+1}+C \leq d_i$, for each i. The same reason applies for player n. Thus, $d_1+C \leq d_n$. We calculate this by the following:

$$\begin{aligned} d_1 &\geq d_1 \\ d_n &\geq d_1 + C \\ d_{n-1} &\geq d_n + C = d_1 + C + C = d_1 + 2C \\ &\vdots \\ d_2 &\leq d_1 + (n-1)C. \end{aligned}$$

The sum of the lengths is n. Thus adding the above inequalities results in

$$\begin{aligned} n &\geq nd_1 + C + 2C + (n-1)C \\ &= nd_1 + \left[\frac{n(n-1)}{2}\right]C, \end{aligned}$$

or when we divide by n, we obtain the equation,

$$1 \geq d_1 + \frac{n-1}{2}C.$$

Now, solving for d_1, we see that

$$\begin{aligned} d_1 &\leq 1 - \frac{n-1}{2}C \\ &\leq 1 - C \\ &= 1 - n^{-8}. \end{aligned}$$

However, $1-n^{-8} < 1-1/3n^{-8} \leq d_1$, which implies $d_1 < d_1$. However, that is a contradiction. ☐

Now that we have proven the different lemmas, the following theorem summarizes the conclusions we have made.

Theorem 4.10. *For three or more players, there exists a pie and corresponding measures for which no allocation is envy-free and undominated.*

Proof. Assume there are three or more players, labeled $1,2,\dots,n$. Also, let us represent the pie as an interval $[0,n]$, where the endpoints are identified. Note, that for this proof, we ignore the requirement that each player's valuation of the entire pie be 1. As mentioned in the above lemmas, we will use two constants when talking about pie, $C = n^{-8}$ and $\varepsilon = n^{-16}$. We will use the same information for this proof as given in the introduction to the pie section. In the lemmas that I have just proven, I have shown that each piece is restricted by an upper and lower bound, and that each piece has both value and length greater than $1-n^{-4}$, as shown in Lemma 4.1. Also, I have shown that the sum of every player's value for their own piece must add to equal n, as shown in Lemma 4.2. Furthermore, in Lemma 4.3, I have shown that the player's piece must have the majority of the piece within the player's window, and if it includes parts outside of the player's window, the total length cannot exceed n^{-3}. Also, proven in Lemma 4.4, I have shown that they player's pieces must be in order S_{i+1} following S_i. Then, for players $i = 2, \ldots, n$ the player's piece cannot include any part of the bonus cell. This is proven in Lemma 4.5. Now that we have proven the requirements and restrictions, we know that the pieces must have one of the three forms as shown in Lemma 4.6. After showing these forms, we then eliminate one form at a time as shown in Lemmas 4.7 through Lemma 4.9. Therefore, we have proven that there exists a pie and corresponding measures for which no allocation is envy-free and undominated. □

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