WHAT IS A MATROID?

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1. Introduction

Originally studied independently by Hassler Whitney and B. L. van der Waerden in the 1930's, the name matroid comes from studying the independence of the columns of a matrix. Before Whitney introduced matroids, he spent most of his time working on graph theory. It was during this time that Whitney began to see similarities between ideas in graph theory and linear independence and dimension in vector spaces. After recognizing the properties of independence, Whitney decided to introduce the concept of a matroid. Around the same time, B. L. van der Waerden was interested in the concept of independence while trying to formalize the definition of linear and algebraic independence. Eventually, we would see matroids as a link between graph theory, linear algebra, and other fields. [Wil73]

In mathematics we aim to identify patterns that occur in order to connect similar concepts to provide a unified idea. The purpose of studying matroids is, "to provide a unifying abstract treatment of dependence in linear algebra and graph theory". [Oxl03] Throughout this paper we will be focusing on understanding "what is a matroid". We will begin by giving a definition of a matroid and exploring some basic examples. After we have established the definition, we will define a uniform matroid while connecting matroids to linear algebra. Another major aspect we will be concerned with is how matroids can be used in graph theory and how matroids arise from graphs.

Definition 1. [Oxl03] A *matroid* M is a pair (E, I) consisting of a finite set E and a collection I of subsets of E satisfying the following properties.

(1) I is non-empty.
(2) Every subset of every member of I is also in I.
(3) If X and Y are in I and $|X| = |Y| + 1$, then there is an element x in $X - Y$ such that $Y \cup \{x\}$ is in I.

E is called the *ground set* of the matroid.

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Using the definition of a matroid, we will look at this simple example below. It is worth noting that the third rule of the definition of a matroid will typically be our focus, since it is the most complex of the three rules.

Example 2. Let $E = \{1, 2, 3\}$ and $I = \{\emptyset, \{1\}, \{2\}, \{3\}\}$. We will show that (E, I) is a matroid.

(1) I is non-empty since we have some element x in I, for example, $x = \{1\}$.
(2) Since $\emptyset$ is in I, the subset of every other element is in I, namely $\{1\}, \{2\}, \{3\}$. Thus every subset of every member of I is also in I,
(3) Let $X = \{1\}$, then $Y = \emptyset$ since $|X| = |Y| + 1$. That gives us $X - Y = \{1\}$. Then, there is an element x such that $Y \cup \{x\}$ is in I, namely $x = 1$. We could show similiarly with $X = \{2\}$ and $X = \{3\}$. Therefore, this is a matroid since it satisfies all three properties.

Looking at this simple example gives us a basic understanding of the definition of a matroid.

2. Number of Non-Isomorphic Matroids on E

Definition 3. [Oxl03] Two matroids are *isomorphic* if there is a bijection from the ground set of one to the other such that a set is independent in the first matroid if and only if its image is independent in the second matroid.

A classical problem is to classify non-isomorphic objects. In particular, we will be looking at the non-isomorphic matroids on E. We will see that one of the major reasons we are interested in non-isomorphic matroids is to relate matroids to graph theory. Some of these important examples will appear again later in this paper when looking at matroids in graph theory.

Example 4. [Oxl03] If we look at the set $E = \emptyset$, we find there is exactly one matroid $I = \{\emptyset\}$. If $E = \{1\}$, we find the two matroids to be $I = \{\emptyset\}$ and $I = \{\emptyset, \{1\}\}$. If $E = \{1, 2\}$ there are five matroids on E. Namely they are, $I = \{\emptyset\}$, $I = \{\emptyset, \{1\}\}$, $I = \{\emptyset, \{2\}\}$, $I = \{\emptyset, \{1\}, \{2\}\}$, and $I = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. We have the same structure in the matroids $I = \{\emptyset, \{1\}\}$ and $I = \{\emptyset, \{2\}\}$. Even though there are five matroids, we have four non-isomorphic matroids on this set since $\{\emptyset, \{1\}\}$ and $\{\emptyset,\{2\}\}$ are isomorphic.

This example might seem straightforward but it is going to be useful in understanding how to find non-isomorphic matroids. In this next example, the first four non-isomorphic matroids on E are the same from $E = \{1, 2\}$.

Example 5. If $E = \{1, 2, 3\}$ we have eight non-isomorphic matroids. They are:

(1) $I = \{\emptyset\}$

(2) $I = \{\emptyset, \{1\}\}$
(3) $I = \{\emptyset, \{1\}, \{2\}\}$
(4) $I = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
(5) $I = \{\emptyset, \{1\}, \{2\}, \{3\}\}$
(6) $I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$
(7) $I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$
(8) $I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

It is easy to show these are matroids and that none are isomorphic to another, but I want to draw attention to one important detail. Looking at number six in the list above, we want to make sure

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$$

is isomorphic to

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}.$$

If we map $1 \longrightarrow 2$ and $2 \longrightarrow 1$, we have an isomorphism between these matroids. So we could write either one of these as a matroid of E. This would similarly be true if

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$$

by mapping $2 \longrightarrow 3$ and $3 \longrightarrow 2$.

When looking at the previous example, we notice the first four matroids on $E = \{1,2,3\}$ are the same as the four non-isomorphic matroids on $E = \{1,2\}$. We will use this strategy to simplify are next example when dealing with $E = \{1,2,3,4\}$. It is worth noting that from the previous two examples, we can notice a patern if we create a table for non-isomorphic matroids.

E	Number of Non-Isomorphic Matroids
$\{\emptyset\}$	1
$\{1\}$	2
$\{1,2\}$	4
$\{1,2,3\}$	8

Table 1. Number of Non-Isomorphic Matroids on E

We would perhaps believe that $E = \{1,2,3,4\}$ would have sixteen non-isomorphic matroids based off Table 1, but interestingly enough that is not the case. In fact, we have seventeen non-isomorphic matroids. We will be using cases to simplify this proof into different steps.

Theorem 6. *If $E = \{1, 2, 3, 4\}$, then there are seventeen non-isomorphic matroids on E.*

Proof. Eight of the non-isomorphic matroids are the same from Example 5. The next obvious matroid is

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}. \tag{1}$$

In order to check the third rule of a matroid, we will use a simple table to simplify computations. When looking at Table 3, the first row is X, the first column is Y, and the entries are doubles forced by the third rule of a matroid. Remember the third rule states that if X and Y are in I and $|X| = |Y|+1$, then there is an element x in $X - Y$ such that $Y \cup \{x\}$ is in I.

We will define doubles as sets with two elements and triples as sets with three elements. We will refer to Table 3 frequently throughout this proof. Consider the six different cases of having doubles with $I \supseteq \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$.

Up to isomorphism, assume $X = \{1, 2\}$ is in I. Also, let $Y = \{4\}$. From the third rule of the definition of a matroid, since $|X| = |Y| + 1$, we have some x in $X - Y = \{1, 2\}$ such that $Y \cup \{x\}$ is in I. Therefore, we must have $\{1, 4\}$ or $\{2, 4\}$ in I. Now let $Y = \{3\}$. Taking $\{1, 2\} - \{3\}$, we arrive at $\{1, 3\}$ or $\{2, 3\}$ must be in I. This shows we must have $\{1, 4\}$ or $\{2, 4\}$ and $\{1, 3\}$ or $\{2, 3\}$ in I. So the minimum number of doubles in I has to be greater than or equal to three since $\{1, 2\}$, $\{1, 4\}$ or $\{2, 4\}$, and $\{1, 3\}$ or $\{2, 3\}$ must be in I. This rules out the possibility of only having one or two doubles be a matroid in I.

	$\mathbf{\{1,2\}}$	$\mathbf{\{1,3\}}$	$\mathbf{\{1,4\}}$	$\mathbf{\{2,3\}}$	$\mathbf{\{2,4\}}$	$\mathbf{\{3,4\}}$
$\mathbf{\{1\}}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,2\}$ or $\{1,3\}$	$\{1,2\}$ or $\{1,4\}$	$\{1,3\}$ or $\{1,4\}$
$\mathbf{\{2\}}$	$\{1,2\}$	$\{1,2\}$ or $\{2,3\}$	$\{1,2\}$ or $\{2,4\}$	$\{2,3\}$	$\{2,4\}$	$\{2,3\}$ or $\{2,4\}$
$\mathbf{\{3\}}$	$\{1,3\}$ or $\{2,3\}$	$\{1,3\}$	$\{1,3\}$ or $\{3,4\}$	$\{2,3\}$	$\{2,3\}$ or $\{3,4\}$	$\{3,4\}$
$\mathbf{\{4\}}$	$\{1,4\}$ or $\{2,4\}$	$\{1,4\}$ or $\{3,4\}$	$\{1,4\}$	$\{2,4\}$ or $\{3,4\}$	$\{2,4\}$	$\{3,4\}$

Table 3. Doubles that are forced by the third rule of a matroid

Case 1: Suppose you have exactly three doubles in I and all subsets of those doubles. I is non-empty and every subset of every member in I is also in I. We must now look at the third rule of a matroid. Without loss of generality, assume $\{1, 4\}$ is the second double in I. Applying the third rule and using Table 3 with $X = \{1, 4\}$ and $Y = \{3\}$, we arrive at $\{1, 3\}$ or $\{3, 4\}$ must be in I. Looking at the two doubles in I, $\{1, 2\}$ and $\{1, 4\}$, we must have $\{1, 3\}$ or $\{2, 3\}$ and $\{1, 3\}$ or $\{3, 4\}$ be in I. Thus, the third double in I must be $\{1, 3\}$ if we are looking for a matroid with only three doubles. Using Table 3, to check if the third rule holds for $\{1,3\}$, we can see $\{1,3\}$ must have $\{1,3\}$, $\{1,2\}$ or $\{2,3\}$, and $\{1,4\}$ or $\{3,4\}$. Therefore, I is a matroid containing the doubles $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ on E.

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}\} \tag{2}$$

Case 2: Suppose you have four doubles in I. Now we either have every number appearing exactly twice throughout the four doubles, such as $\{1,2\}$, $\{1,3\}$, $\{2,4\}$, and $\{3,4\}$, or the case where not every number appears twice in the four doubles, such as $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, and $\{3,4\}$. In both of these cases, I is non-empty and every subset of every member in I is also in I.

Suppose you have the case where not every number appears twice in the four doubles. Assume the first three doubles in I are $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$. Up to isomorphism, assume we pick $\{2,3\}$ as the fourth double. Applying the third rule of the definition of a matroid, let $X = \{2{,}3\}$ and $Y = \{4\}$. Then $\{2,4\}$ or $\{3,4\}$ must be in I. Thus, this is a contradiction since neither $\{2,4\}$ nor $\{3,4\}$ is in I.

Now suppose you have the case where every number appears exactly twice in the four doubles. Up to isomorphism, assume $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, and $\{1,4\}$ are in I. Let us use Table 3 to help us with the third rule of a matroid. Let $X = \{1,2\}$. Then $\{1,2\}$, $\{1,4\}$ or $\{2,4\}$, and $\{1,3\}$ or $\{2,3\}$ must be in I in order for I to be a matroid. From our original assumption, we know $\{1,2\}$, $\{1,4\}$, and $\{2,3\}$ are in I. Now let us apply the third rule to $\{1,4\}$. We will have $\{1,4\}$, $\{1,2\}$ or $\{2,4\}$, and $\{1,3\}$ or $\{3,4\}$ must be in I. Again, this holds true since $\{1,4\}$, $\{1,2\}$, and $\{3,4\}$ are in I. Doing this simple process for $\{2,3\}$ and $\{3,4\}$, we are able to show that $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, and $\{1,4\}$ hold up under the third rule. Therefore, having every number appear exactly twice in the four doubles is in fact a matroid.

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1{,}4\}\} \tag{3}$$

Case 3: Suppose we have five doubles in I. Up to isomorphism, let $\{3,4\}$ be the missing double. I is non-empty and every subset of every member in I is also in I. Now let us look at the third rule of a matroid. Let $X = \{1,3\}$. This means I must contain $\{1,3\}$, $\{1,2\}$ or $\{2,3\}$, and $\{1,4\}$ or $\{3,4\}$. No matter what double is not in I, we will always have a pair to choose from to satisfy the third rule. The only double that does not have two options we know is in I since it is X. Thus, every combination of five doubles is a matroid. This same reasoning holds for when when we have all six doubles in I. Therefore, we have two more non-isomorphic matroids on E.

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\} \tag{4}$$

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\} \tag{5}$$

Now that we have finished looking at the cases where I contains only singles and doubles, we need to look at possible matroids containing triples. Let us use a new table, Table 4, in a similar way we used Table 3.

	$\mathbf{\{1,2,3\}}$	$\mathbf{\{1,2,4\}}$	$\mathbf{\{1,3,4\}}$	$\mathbf{\{2,3,4\}}$
$\mathbf{\{1,2\}}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,2,3\}$ or $\{1,2,4\}$	$\{1,2,3\}$ or $\{1,2,4\}$
$\mathbf{\{1,3\}}$	$\{1,2,3\}$	$\{1,2,3\}$ or $\{1,3,4\}$	$\{1,3,4\}$	$\{1,2,3\}$ or $\{1,3,4\}$
$\mathbf{\{1,4\}}$	$\{1,2,4\}$ or $\{1,3,4\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{1,2,4\}$ or $\{1,3,4\}$
$\mathbf{\{2,3\}}$	$\{1,2,3\}$	$\{1,2,3\}$ or $\{2,3,4\}$	$\{1,2,3\}$ or $\{2,3,4\}$	$\{2,3,4\}$
$\mathbf{\{2,4\}}$	$\{1,2,4\}$ or $\{2,3,4\}$	$\{1,2,4\}$	$\{1,2,4\}$ or $\{2,3,4\}$	$\{2,3,4\}$
$\mathbf{\{3,4\}}$	$\{1,3,4\}$ or $\{2,3,4\}$	$\{1,3,4\}$ or $\{2,3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$

TABLE 4. Triples that are forced by the third rule of a matroid

Looking back at the matroid containing three doubles, let us apend on a triple. Applying the third rule of the definition of a matroid and using Table 4, let $X =$ {1,2,3} and $Y = \{1,4\}$; then $\{1,2,3\}$ or $\{2,3,4\}$ must be in I. This shows that there must be a minimum of two triples in I. If we must have at least two triples in I, we can narrow down a minimum number of doubles that must be in I by the subset rule of a matroid. Without loss of generality, assume we let $\{1,2,3\}$ and $\{1,2,4\}$ be in I. Looking at the subsets of $\{1,2,3\}$, we have the doubles $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. The subsets of $\{1,2,4\}$ are $\{1,2\}$, $\{1,4\}$, and $\{2,4\}$. Looking at these subsets we must have $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,4\}$, and $\{2,4\}$ in I. Notice that having two triples results in a minimum of five doubles, since $\{1,2\}$ appears in both triples. The only double that appears twice in the subset rule are the two numbers the triples share in common. We will always have this case for any two triples we pick, since every triple is only missing one number. Also, $\{3,4\}$ is the only double that does not appear in this case. This happens to always be the two numbers the triples do not share in common. We have just shown that we must have a minimum of five doubles and two triples. Let us now consider the case of five doubles and two triples using the same numbers.

Case 4: Suppose we have five doubles and two triples. Up to isomorphism, assume we pick $\{1,2,3\}$ and $\{1,2,4\}$ to be the triples in I. I is non-empty and every subset of every member in I is also in I. Also, we just showed in the previous paragraph that the missing double must be $\{3,4\}$. Using Table 4, we need to check the third rule of a matroid. We know the five doubles hold up under the rule from case 3, but we need to check the triples. Looking at Table 4, we can very quickly verify that only $\{1,2,3\}$ and $\{1,2,4\}$ need to be in I. Therefore, we have a matroid with five doubles and two triples.

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}\} \quad (6)$$

Also, we can easily show that five doubles and three or more triples is not a matroid because the subset rule for the three triples would result in all six doubles needing to be in I.

Now let us consider the case if we have six doubles and two triples. Assume we have $\{1,2,3\}$ in I. Now applying the third rule and using Table 4, we must have $\{1,2,3\}$, $\{1,2,4\}$ or $\{1,3,4\}$, $\{1,2,4\}$ or $\{2,3,4\}$, and $\{1,3,4\}$ or $\{2,3,4\}$. Looking closely, we can narrow it down to a minimum of three triples in I. This means, we must have at least three triples when dealing with six doubles. Lets look at the case with six doubles and three triples.

Case 5: Up to isomorphism, assume we have $\{2,3,4\}$ be the triple missing in I. I is non-empty and every subset of every member in I is also in I. Looking at the third rule of a matroid, let $X = \{1,2,3\}$. This means I must contain $\{1,2,3\}$, $\{1,2,4\}$ or $\{1,3,4\}$, $\{1,2,4\}$ or $\{2,3,4\}$, and $\{1,3,4\}$ or $\{2,3,4\}$. Notice we always have two triples to pick from besides X. Since we are only missing one triple, the other triple must be in I. Therefore, six doubles and three triples is a matroid.

$$\begin{aligned} I = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \\ &\{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\} \end{aligned} \quad (7)$$

This same reasoning proves that six doubles and four triples is a matroid.

$$\begin{aligned} I = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \\ &\{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\} \end{aligned} \quad (8)$$

Finally, we have one more matroid.

$$\begin{aligned} I = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \\ &\{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\} \end{aligned} \quad (9)$$

$\square$

Next we are going to define some important terms so we can begin to relate matroids to linear algebra and graph theory.

3. Uniform matroids and Fields

Definition 7. [Oxl03] Let r, n be natural numbers with $0 \leq r \leq n$. A *uniform matroid* $U_{r,n}$ is a matroid such that E is an n-element set and I is the collection of all subsets of E with at most r elements.

For example, we can take the uniform matroid $U_{2,4}$. If $E = \{a, b, c, d\}$, then

$$U_{2,4} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}.$$

We will mainly be looking at uniform matroids and their connections with matroids. In order to look at uniform matroids and fields, we must define several terms including a group, ring, and field.

Definition 8. [Fra03] A *group* is a set G, closed under a binary operation $*$, satisfying the following axioms:

(1) Associativity: For all $a, b, c \in G$, we have $(a * b) * c = a * (b * c)$.
(2) Identity element: There is an element e in G such that for all $x \in G$, $e * x = x * e = x$.
(3) Inverse: Corresponding to each $a \in G$, there is an element a' in G such that $a * a' = a' * a = e$.

Definition 9. [Fra03] A *ring* is a set R together with two binary operations $+$ and $\cdot$, which we call addition and multiplication, defined on R such that the following axioms are satisfied:

(1) $R, +$ is a group that is commutative.
(2) $R, \cdot$ is associative.
(3) For all $a, b, c \in R$, the left distributive law $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and the right distributive law $(a + b) \cdot c = (a \cdot b) + (a \cdot c)$ hold.

Now that we have defined a group and ring, we can define a field.

Definition 10. [Fra03] A *field* is a ring with unity such that all non-zero elements have a multiplicative inverse and multiplication is commutative.

Next, we want to discuss a certain type of field. This is described in [Oxl03]. Let p be a prime and k be a positive integer. A finite field $GF(p^k)$ has exactly p^k elements. When $k = 1$, we can view the finite field as $\{0, 1, 2, .., p - 1\}$ with the operations of addition and multiplication modulo p. When $k > 1$ the structure of

$GF(p^k)$ is more complex, and we will only be looking at one particular case in this format, the structure of $GF(2^2) = GF(4)$. We will discuss the properties of this field later when looking at Exercise 17.

Theorem 11. [Oxl03] *Let A be a matrix over a field F. Let E be the set of column labels of A, and I be the collection of subsets I of E for which the multiset of columns labelled by I is linearly independent over F. Then (E, I) is a matroid.*

Proof. I is non-empty and every subset of every member of I is also in I. We need to show that there is an element x in $X - Y$ such that $Y \cup \{x\}$ is linearly independent. Let X and Y be linearly independent subsets of E such that $|X| = |Y| + 1$. Let W be the vector space spanned by $X \cup Y$. The dimension of $X \cup Y$ is at least the dimension of $|X|$, so $\dim W \geq |X|$. Suppose that $Y \cup \{x\}$ is linearly dependent for all x in $X - Y$. We know that W is contained in the span of Y, since $Y \cup \{x\}$ are linearly dependent. So W has dimension at most $|Y|$. Thus, we have $|X| \leq \dim W \leq |Y|$ which is a contradiction, since we stated $|X| = |Y| + 1$. Therefore, there is an element x in $X - Y$ such that $Y \cup \{x\}$ is linearly independent. □

From the previous theorem, the matroid obtained from the matrix A can be written as $M[A]$ and is called the vector matroid of A.

Definition 12. [Oxl03] A matroid M that is isomorphic to $M[A]$ for some matrix A over a field F is called *F-representable*, and A is called an *F-representation* of M. We say M is *binary* if it is representable over $GF(2)$ and M is *ternary* if it is representable over $GF(3)$.

In the next proposition, we are going to discover whether every matroid is representable over every field. We will be using the idea of uniform matroids and binary and ternary fields.

Proposition 13. *The matroid $U_{2,4}$ is not representable over $GF(2)$.*

Proof. We need to show that no matrix A represents $U_{2,4}$ over $GF(2)$. Suppose that $U_{2,4}$ is represented over $GF(2)$ by a matrix A. From definition 7, the definition of a uniform matroid, we know that the largest independent set in $U_{2,4}$ has two elements. The column space of A has dimension exactly two, since the vector space spanned by its columns has dimension two. By performing row operations and arriving at the identity matrix in A, the row space has dimension two since the column space of A is dimension two. The field $GF(2)$ has at most four vectors when the column space is two, namely $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}$, and $\begin{bmatrix}0\\0\end{bmatrix}$. In order to have a A over $GF(2)$, we must have any two of the four columns be linearly independent.

However, no set containing $\begin{bmatrix}0\\0\end{bmatrix}$ is linearly independent. Thus, we simply run out of linearly independent columns for A since we only have three independent sets to fill four columns. Therefore, matrix A does not represent $U_{2,4}$ over $GF(2)$. $\square$

From the proceeding proposition, we see that not every matroid is representable over every field. However, the matroid is ternary even though it is not binary.

Proposition 14. *The matroid $U_{2,4}$ is represented over $GF(3)$.*

Proof. We need to show that a matrix A is linearly independent for any two vectors in A that are represented in $U_{2,4}$ over $GF(3)$. There are a total of nine possible vectors that represent $U_{2,4}$ over $GF(3)$. We can eliminate $\begin{bmatrix}0\\0\end{bmatrix}$ since no set containing $\begin{bmatrix}0\\0\end{bmatrix}$ is linearly independent. With the eight remaining vectors, we can group these vectors into pairs so that these group of vectors are linearly dependent.

(1) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$

(2) $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$

(3) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$

(4) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}$

Now any pair of these vectors could not both possibly be in A, since it would make any two vectors in A linearly dependent. Eliminating one vector from every pair for the purpose of finding any two linearly independent vectors in A leaves us with only four possible vectors. Without loss of generality, let matrix $A = \begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 2\end{bmatrix}$. Next, we need to show that any two vectors are linearly independent. It is obvious that every pair of vectors besides $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ are linearly independent. Taking the determinant of $B = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$, we get $\det B = 1$. Therefore, $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ is linearly independent. We have now found one matrix that is linearly independent for any two vectors represented in $U_{2,4}$ over A. Thus, the matroid $U_{2,4}$ is represented over $GF(3)$. $\square$

We have shown that not every matroid is representable over every field. In fact, there are matroids which are representable over no fields. There is also a theorem to show when a matroid is representable over every field. Its proof is beyond the scope of this paper.

Theorem 15. [Oxl03] *A matroid is representable over every field if and only if it is both binary and ternary.*

Looking from the previous examples, we can easily arrive at the conclusion that $U_{2,5}$ is not binary or ternary since we would simply not have enough linearly independent vectors in A. Let us take a look at $U_{3,5}$ over $GF(3)$.

Proposition 16. *The matroid $U_{3,5}$ is ternary.*

Proof. We need to show that a matrix A is linearly independent for any three vectors in A that are represented in $U_{3,5}$. Looking at the total number of vectors represented in $U_{3,5}$ over $GF(3)$, we would have 27 possibilities. We can eliminate $\begin{bmatrix}0\\0\\0\end{bmatrix}$ since no set containing $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is linearly independent. Similarly from Proposition 14, we can group the remaining 26 vectors into pairs of linearly dependent vectors. Eliminating half of these vectors, we now have 13 vectors remaining. Let us assume that the identity matrix is the first three vectors in A since we can perform row operations to get the identity matrix for any set of linear independent vectors. Next, we can eliminate any remaining vectors with at least one 0 in the vector, since we could simply choose two of the vectors from the identity matrix to make the third vector linearly dependent. For example: $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\0\\2\end{bmatrix}$ are not linearly independent. Eliminating these vectors from the thirteen remaining vectors, we are left with the identity matrix vectors, $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, and 4 other vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\2\end{bmatrix}$, $\begin{bmatrix}2\\1\\2\end{bmatrix}$, and $\begin{bmatrix}2\\2\\1\end{bmatrix}$. Let the first four vectors in A be $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ since for any three of these vectors we pick they are linearly independent with each other. We now have $\begin{bmatrix}1\\2\\2\end{bmatrix}$, $\begin{bmatrix}2\\1\\2\end{bmatrix}$, and $\begin{bmatrix}2\\2\\1\end{bmatrix}$ that could be the last vector in A. Let us assume $\begin{bmatrix}2\\2\\1\end{bmatrix}$

is in A. If $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ is linearly independent, then $U_{3,5}$ over $GF(3)$ is ternary. We can see that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Similarly with the other two possible vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ we can see A is linearly dependent. Therefore, $U_{3,5}$ is not ternary since there are not enough linearly independent vectors in A. □

Next, we will look at an Example 17 dealing with the field $GF(4)$. The behavior of this field is going to be much different than a binary or ternary field. The elements of $GF(4)$ are 0, 1, w, $w+1$ and, in this field, $w^2 = w+1$ and $2 = 0$. We will use Table 5 to help us understand the field described in Example 17.

When looking at Table 5, we are multiplying the first row by the first column, where the entries give us the elements over $GF(4)$.

$\bullet$	0	1	w	$w+1$
0	0	0	0	0
1	0	1	w	$w+1$
w	0	w	$w+1$	1
$w+1$	0	$w+1$	1	w

Table 5. Multiple Table for $GF(4)$

Example 17. $U_{3,6}$ is representable over $GF(4)$.

Proof. We must define a matrix A such that any three columns are linearly independent. Now there would be 64 possible vectors from $U_{3,6}$ over $GF(4)$ since we have four possible elements and the column space of A has dimension three. Narrowing down columns, we can eliminate the zero vector. We can also group the remaining 63 vectors in groups of three that are linearly dependent with each other, leaving us with 21 possible linearly independent vectors. Let the first three columns of the matrix A be the identity matrix $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, since we know they are linearly independent. Now, we only need to find one possible solution that works to make $U_{3,6}$ representable over $GF(4)$. We can continue to narrow down possible linearly independent vectors by eliminating any vectors that contain a 0, since it would be linearly dependent if it was grouped with two of the identity vectors. This leaves us now with nine possible linear independent vectors to go with the identity matrix. In the remaining nine vectors, we have two groups of three that are linearly dependent with each other. Looking at the three remaining vectors with the identity matrix, we can let $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & w \\ 0 & 1 & 0 & w & 1 & 1 \\ 0 & 0 & 1 & 1 & w+1 & 1 \end{bmatrix}$. We can easily see that any three vectors in A are linearly independent besides the last three vectors. Let us check if the last three columns are linear independent by row reduction.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & w \\ 0 & 1 & 0 & w & 1 & 1 \\ 0 & 0 & 1 & 1 & w+1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} w & 0 & w+1 & 1 & 0 & 0 \\ w+1 & w+1 & w & 0 & 1 & 0 \\ 0 & w & w+1 & 0 & 0 & 1 \end{bmatrix}$$

When we row reduce we notice it is indeed linearly independent. Therefore, $U_{3,6}$ is representable over $GF(4)$. $\square$

Example 18. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ and let E be the set $\{1, 2, 3, 4, 5, 6, 7\}$ of column labels of A. Let I be the collection of subsets I of E for which the multiset of columns labelled by I is linearly independent over the real numbers. Now, we are going to look for I consisting of all column labels with at most three elements that make (E, I) a matroid. That would consist of all subsets of $E - \{7\}$ besides $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{2, 3, 6\}$, and any subset containing $\{5, 6\}$.

4. Matroids in Graph Theory

In this next section we will be looking at connecting matroids with graph theory. Before looking at any examples, we must first define a cycle.

Definition 19. [GP02] A *walk* in a graph is an alternating sequence of vertices and edges, beginning and ending with a vertex, in which each edge is incident with the vertex immediately preceding it and the vertex immediately following it. A *path* is a walk in which all vertices are distinct.

Definition 20. [Fra03] A *cycle* of graph G, is a subset of the edge set of G that forms a path such that the first node of the path is the last.

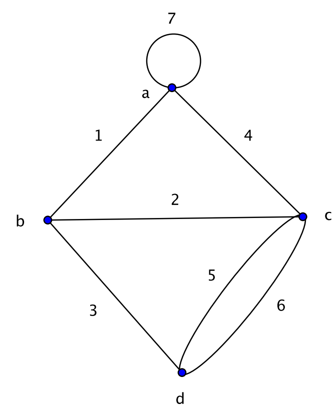

FIGURE 4.1. The Graph G

Next, we are going to be comparing the graph G to the results in Example 18 to help connect matroids with graph theory.

Example 21. Looking at the graph G, we have 4 vertices and 7 edges. Let E be the edge set of G, namely $E = \{1, 2, 3, 4, 5, 6, 7\}$, and let I be the collection of subsets of E that do not contain all of the edges of any simple closed path or cycle of G. The cycles of G have edge sets $\{1, 2, 4\}$, $\{1, 3, 5, 4\}$, $\{1, 3, 6, 4\}$, $\{2, 3, 5\}$, $\{2, 3, 6\}$, $\{5, 6\}$, and $\{7\}$. Thus, I contains all of the edge sets that do not contain $\{1, 2, 4\}$, $\{1, 3, 5, 4\}$, $\{1, 3, 6, 4\}$, $\{2, 3, 5\}$, $\{2, 3, 6\}$, $\{5, 6\}$, and $\{7\}$.

Relaying this graph back to matrix A in Example 18, we can see that every cycle in G is related to the columns being linearly dependent in matrix A. Also, we have a matroid for all collection of subsets where every edge set does not contain a cycle in G. We are able to arrive that the pair (E, I) is a matroid. This next definition will help us define a cycle matroid.

Definition 22. [Oxl03] Let G be a graph. The *cycle matroid*, $M(G)$, is the matroid obtained by defining I as the collection of subsets of E that do not contain all of the edges of any simple closed path or cycle of G.

Definition 23. [GP02] A graph is *connected* if and only if there exists a walk between any two vertices.

Definition 24. [GP02] A *component* of a graph is a maximal connected subgraph, that is, a connected subgraph that is properly contained in no other connected subgraph that has more vertices or more edges.

Now that we have defined a cycle matroid, we will be looking at connected components in Lemma 25.

Lemma 25. [Cam98] *If a graph with n vertices, m edges, c connected components contains no circuits, then $n = m + c$.*

Proof. Suppose we have a graph G and we have c connected components where the graph G has no circuits. Let us have n vertices and m edges. Also, we say every edge must begin and end with a vertex. Since we will always begin and end with a vertex in every component and G contains no circuits, we will always have one less edge than the number of vertices in every component. Thus, the number of components and edges will always be equal to the number of vertices. Therefore, we have $n = m + c$. □

We will use Definition 22 and Lemma 25 to help us prove Theorem 26. In this next proof, we will use a graph-theoretic argument to show that the cycle matroid is a matroid.

Theorem 26. [Oxl03] *If G is a graph, then $M(G)$ is a matroid.*

Proof. We need to show that a cycle matroid is indeed a matroid. Let E be the edge set of a graph G and I be the collection of subsets of E. A set is linearly independent if it contains no circuits. From Definition 22, a cycle matroid must not contain all of the edges of any simple closed path or cycle of G. I is non-empty and every subset of every member in I is also in I. From Lemma 25, we know if a graph with n vertices, m edges, c connected components contains no circuits, then $n = m + c$. If X and Y are linearly independent, $|X| = |Y| + 1$, and $n = m + c$, then the subgraph with edge set X (and all vertices) has fewer components then the graph with edge set Y since we include all vertices even if not used by an edge. There is some edge e of X that must join vertices in different components of Y since Y has one more component then X. Adding an edge e to Y creates no circuit since we would need two edges to create a circuit from a different component. Therefore, the cycle matroid $M(G)$ is indeed a matroid. □

Definition 27. [Oxl03] Let A be a matrix and G be a graph. The *vertex-edge incidence matrix*, A_G, is the matrix where the rows of A_G are indexed by the vertices of G and the columns are indexed by the edges of G such that the entry is 1 if the edge meets the vertex and is 0 if the edge does not meet the vertex.

We say a matroid that is isomorphic to the cycle matroid of some graph is called graphic. In this next theorem, we will be looking at which matroids are graphic.

Theorem 28. [Oxl03] *Let G be a graph and A_G be its vertex-edge incidence matrix. When A_G is viewed over $GF(2)$, its vector matroid $M[A_G]$ has as its independent sets all subsets of $E(G)$ that do not contain the edges of a cycle. Thus $M[A_G] = M(G)$ and every graphic matroid is binary.*

Proof. We need to show that a set X of columns of A_G is linearly dependent if and only if X contains the set of edges of a cycle of G. Assume X contains the edge set of some cycle C. Now, C either contains a loop or not. If C contains a loop, then X is linearly dependent and the column is the zero vector. If C does not contain a loop, every vertex in C is met by exactly two edges of C. We see that when adding each row, modulo 2, we get the zero vector. This is always the case since every edge that is not a loop meets exactly two vertices. Therefore, X is linearly dependent.

On the other hand, suppose that X is a linearly dependent set of columns. We need to show that X contains the set of edges of a cycle of G. Take a subset D of X that is minimal and linearly dependent. We say D is minimal if D is linearly dependent but all of its proper subsets are linearly independent. The two possible cases are if D contains a zero column or if D does not contain a zero column. Assume D contains a zero column. We know if D contains a zero column, then it must contain the edge set of a loop. In this case, D contains the edge set of a cycle of G. Assume that D does not contain a zero column. We know every entry is either zero or one since we are in $GF(2)$. Since D is a minimal linearly dependent set, we know that every vertex that meets an edge of D is met by at least two such edges. Taking the sum, modulo 2, of the columns in D is the zero vector, since D is a minimal linear dependent set. Thus, D contains the edges of a cycle of G. $\square$

In Example 29, we will be looking at 8 graphs and how these graphs relate to previous theorems and propositons already shown. This example will be able to connect major concepts as well as show how matroids relate to linear algebra and graph theory.

Example 29. [Oxl03] There are 8 graphs each with 3 edges such that the associated cycle matroids are non-isomorphic.

Connecting these graphs back to Example 5, we can see this is the same as matroids on $E = \{1, 2, 3\}$.

4.2a: $I = \{\emptyset\}$
4.2b: $I = \{\emptyset, \{1\}\}$
4.2c: $I = \{\emptyset, \{1\}, \{2\}\}$
4.2d: $I = \{\emptyset, \{1\}, \{2\}, \{3\}\}$
4.2e: $I = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
4.2f: $I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$
4.2g: $I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$
4.2h: $I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Note that non-isomorphic graphs can have isomorphic cycle matroids. For example, if we look at figure 4.2a, we could also have the graph containing three vertices

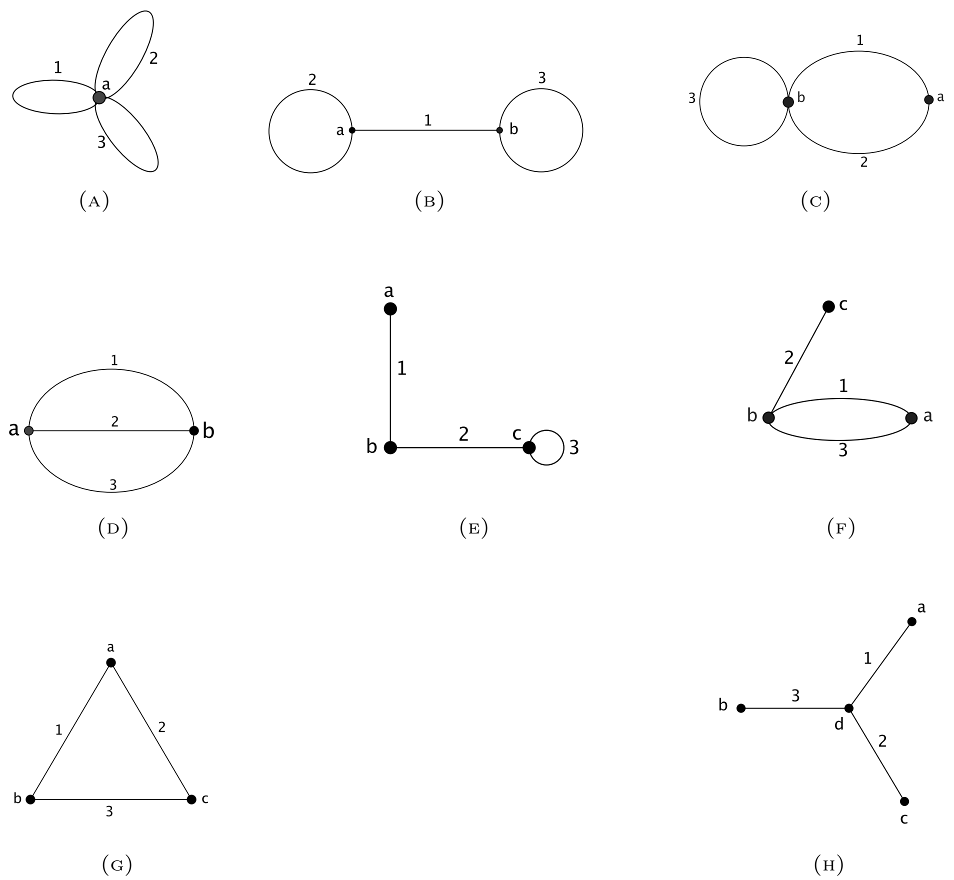

FIGURE 4.2. Eight non-isomorphic graphs

where each vertex contains a single loop. We can do this in a cycle matroid of any graph by adding a collection of vertices that meet no edges. Looking further ahead, if we would try to find 17 graphs each with 4 edges such that the associated cycle matroids are non-isomorphic, we would only be able to graph 16. Even though we were able to find 17 non-isomorphic matroids in Theorem 6, where $E = \{1,2,3,4\}$, there is no graph corresponding to the matroid

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$

We can also connect this to uniform matroids in Proposition 13, where $U_{2,4}$ is not representable over $GF(2)$. This is indeed the one non-binary matroid on a 4-element set that we are unable to graph. In fact, the number of non-isomorphic matroids and non-isomorphic binary matroids are only equal up to $E = \{1,2,3\}$.

Throughout this paper, we were able to establish a basic understanding of a matroid and how it connects with multiple fields such as linear algebra and graph theory. Being able to link Example 29 with several other examples in different aspects shows how much matroids can be used in more than one field.

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