An Investigation of Closed Geodesics on Regular Polyhedra

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1 Introduction

This paper was undertaken to examine, in detail, results from Closed Geodesics on Regular Polyhedra by Dmitry and Ekaterina Fuchs [1]. The original paper combined the Gauss-Bonnet Theorem, which is interesting in itself in that it relates geometric and topological quantities so nicely, with not at all obvious proofs using group theory.

Definition 1 *Let M be a smooth surface. Then let γ be at least a C^2 curve on M,*

$$\gamma'' = \kappa_n n + \kappa_g V,$$

*where n is the normal to the surface and V is the tangential component. γ is a **geodesic** if and only if γ'' is normal to the surface everywhere.*

Definition 2 *A **geodesic on a polyhedron** is a piecewise regular curve that is linear in each face and at the edges we simply consider the unfolding of the edge so that adjacent faces of the polyhedron lie in the same plane. The curve on the surface then becomes a straight line.*

Definition 3 *Let $\gamma : [a, b] \to \mathbb{R}^3$ be a geodesic on a polyhedron. We call γ a **closed geodesic** if and only if $\gamma(a) = \gamma(b)$.*

Definition 4 *Let $\gamma : [a, b] \to \mathbb{R}^3$ be a geodesic on a polyhedron. We call γ a **closed non-self-intersecting geodesic** if and only if $\gamma(a) = \gamma(b)$ and $\gamma(x) \neq \gamma(y) \forall x, y \in (a, b)$.*

Definition 5 *Let $\gamma : [a, b] \to \mathbb{R}^3$ be a geodesic on a polyhedron. We call γ a **prime geodesic** if and only if the set $\{x \in [a, b] \mid \exists y \in [a, b]$ such that $x \neq y$ and $\gamma(x) = \gamma(y)\}$ is finite.*

Let P be a polyhedron, not necessarily regular, and let p be a point on the polyhedron. The Gaussian curvature $K(p)$ at a point on a face or edge of a polyhedron is defined to be $K(p) = 0$. We define the Gaussian curvature at a vertex to be

$$K(p) = 2\pi - \Sigma\theta_i.$$

Where $\Sigma\theta_i$ is the sum of the planar angles at the vertex in question. Since the faces of the polyhedron are planar, the Gaussian curvature is concentrated in the vertices.

With this definition of Gaussian curvature $K(p)$, the usual Gauss-Bonnet formula holds:

$$\int_R KdA + \int_\gamma \kappa_g ds + \sum \alpha_i = 2\pi,$$

where R is the simply connected region of the surface bounded by the piecewise regular curve γ, a regular curve is one which is at least once differentiable and its derivative is non-zero at every point on the curve. κ_g is the geodesic curvature and α_i is the jump angle at a point where γ is not differentiable.

In the case where γ is a closed non-self-intersecting geodesic on a polyhedron we can simplify the Gauss-Bonnet formula. Within each face γ is linear and thus continuously differentiable, therefore is has no jump angles. At the edges we simply consider the unfolding of the edge so that adjacent faces of the polyhedron lie in the same plane; γ is still a straight line and again no jump angles. By definition $\kappa_g = 0$, thus in the case of a closed geodesic on a polyhedron, the Gauss-Bonnet formula may be reduced to

$$\int_R KdA = 2\pi.$$

Similar analysis shows that in order for a convex polyhedron to have any closed non-self-intersecting geodesics it must contain some subset of vertices with Gaussian curvature, $K = 2\pi$ [2].

The Gauss-Bonnet Theorem states that if M is a compact, that is, closed and bounded, surface then

$$\int_M KdA = 2\pi\chi,$$

where χ is the Euler characteristic of the polyhedron in question. The Euler characteristic χ is defined to be $F - E + V = \chi$. In the case of regular polyhedra, $\chi = 2$. This means that the Gaussian curvature over a regular polyhedron that is topologically a sphere is equal to 4π.

The implication of the Gauss-Bonnet theorem in our investigation is this: a closed geodesic cuts the surface of the polyhedron into two pieces. The Gauss-Bonnet formula tells us that the region bounded by the geodesic has a sum of Gaussian curvatures equal to 2π. But there are two regions bounded by the geodesic, namely the two regions into which the polyhedron must be divided by a closed non-self-intersecting geodesic, each having a sum of Gaussian curvatures equal to 2π. We will restrict our investigation to regular polyhedra as their symmetry would seem to imply that each contains closed non-self-intersecting geodesics.

We will examine three cases, the tetrahedron, cube and octahedron. We, like Fuchs and Fuchs, gave some thought to the case of the dodecahedron but could not formulate a valid approach using similar tools to the other four cases.

We first develop the notion of the development of a polyhedron along a geodesic in section 2. In section 3 we examine the case of the tetrahedron. This case is relatively straightforward and we find that it contains non-self-intersecting closed geodesics of arbitrary length. We find in section 4 that the case of the cube yields three closed, non-self-intersecting geodesics. Two of which are planar, the third is not. Section 5 examines the case of the octahedron, and it yields two closed, non-self-intersecting geodesics. We have not necessarily given a full description of closed geodesics on the cube and octahedron. Classification will be complete upon showing that any closed geodesic can be parallel translated to obtain other closed geodesics.

2 The Development of a Polyhedron Along a Geodesic

Let P be a polyhedron, not necessarily regular, and let γ be a geodesic on the surface of P. Pick an edge of P which γ intersects at least once, call this edge A_1A_n. Think of the point at which edge A_1A_n and γ intersect, call it X, as the start of γ, with γ continuing into face $A_1A_2A_3A_4...A_n$. γ will then intersect edge A_iA_j, with $1 \leq i < j \leq n$, at some point Y and then continue into an adjacent face, $A_iB_1B_2B_3B_4....B_nA_j$. This process continues along the length of γ.

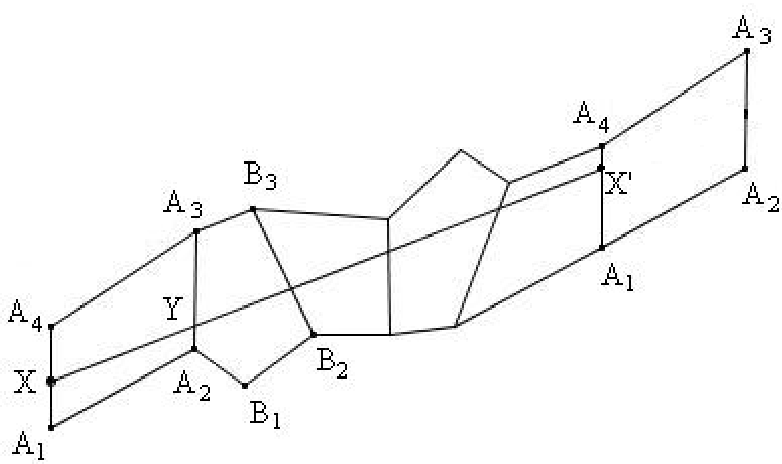


Figure 1. The development of a polyhedron along a geodesic.

If P is convex then this process can be visualized as the rolling of P along the plane in such a way that γ becomes a straight line in the plane. Consecutive faces encountered by γ are then adjacent in the plane, sharing one common edge. If γ is closed then the linear distance from either vertex A_1 or A_n, on the initial edge A_1A_n, to X will be equal to the linear distance from vertex A_1 or A_n to a point $X' \neq X$ in the plane. The edge A_1A_n that γ intersects at X' is an edge of an identically labelled translation of face $A_1A_2A_3A_4...A_n$.

3 The Case of the Regular Tetrahedron

Label the vertices of the tetrahedron A, B, C and D as in Figure 2. Let a geodesic, γ, start at a point on edge AB and continue into face ABC.

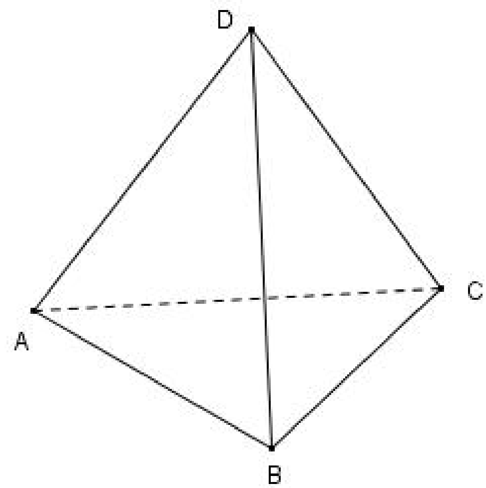


Figure 2

Claim 6 *The labeling of faces in the plane from the development of a tetrahedron along a geodesic will always match the labelling of the standard triangular tiling of the plane as seen in Figure 3.*

Proof. It is clear that the plane can be tiled with equilateral triangles since such a tiling is invariant under reflection in an edge or under rotation by $\frac{\pi}{3}$at a vertex. The only question is how the labels should be assigned to vertices. Let γ be any geodesic on the tetrahedron with vertices labeled A, B, C and D, and without loss of generality assume the geodesic passes through edge AB and into face ABC. The development of the tetrahedron along the geodesic necessitates a labeling of an adjacent face for the tiling, that is, the next face the geodesic will pass through. The tetrahedron has four vertices, three of which are determined necessarily by previous face visited. The adjacent face must contain two of three vertices of the previous face visited as they share an edge and the third vertex must differ from the vertex of the previous face not on the shared edge. Thus the third vertex of an adjacent face can only attain one label, and it is the vertex which is not contained in the previous face. Thus in both the unrolling of the tetrahedron and the labeling of the vertices in the plane the labels of adjacent triangles are necessarily determined by its neighbors. ■

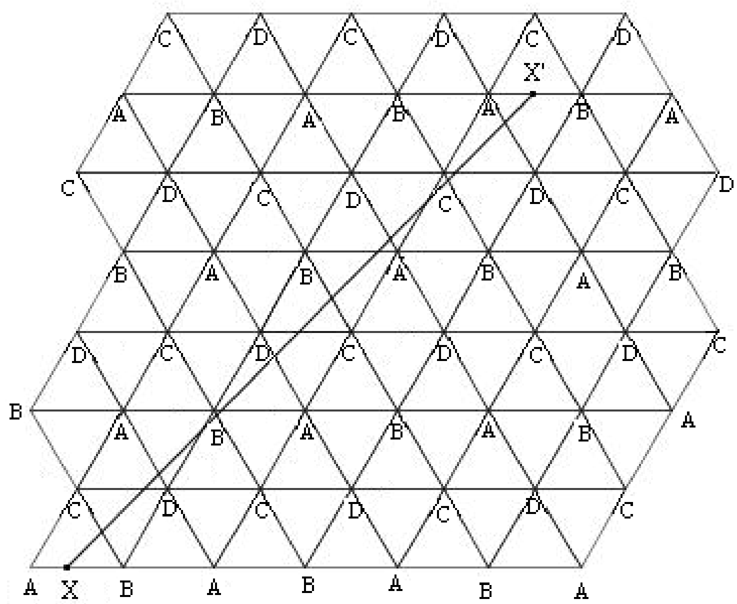


Figure 3. The development of the tetrahedron which is contained in the tiling of the plane by the faces of the tetrahedron shown in Figure 2.

The development of the tetrahedron along γ will be contained in the triangular tiling of the plane by the faces of the tetrahedron (See Figure 3). γ is closed if and only if it joins the point X on edge AB of face $ABCD$ with a point in the plane $X' \neq X$, also on edge AB of face $ABCD$. X and X' must also be the same distance away from the vertex A along the edge AB. Every closed geodesic on the tetrahedron can be found using the standard triangular tiling.

Let the planar coordinates of A, B and C be $(0,0), (x_b, 0)$ and $(\frac{x_b}{2}, \frac{x_b\sqrt{3}}{2})$ respectively with $x_b \in \mathbb{R}$. Let X be at $(\alpha, 0)$ $0 < \alpha < 1$ then the coordinates of X' are $(\alpha + x_b(q + 2p), qx_b\sqrt{3})$.

Proposition 7 *Any geodesic on a regular tetrahedron is non-self intersecting.*

Proof. Any two triangles in the tiling which share a labeling are parallel to each other or can be obtained by a rotation of π. Parallel labelling will preserve the orientation of the geodesic within the face as will a rotation by π. Thus, segments of a geodesic in any face are parallel to each other. ■

The regular tetrahedron then contains closed non-self-intersecting geodesics of arbitrary length.

4 The Case of the Cube

The case of the cube is not quite as straightforward as the case of the tetrahedron. Label the vertices of the cube A, B, C, D, A', B', C' and D' as in Figure 4. Consider a geodesic starting at a point on edge AB and proceeding into face $ABCD$. The development of the cube along the geodesic is part of the tiling of the plane by squares. As before, the vertices get labels that coincide with the vertices of the faces visited by the geodesic. However, we can not label the vertices in such a way that the tiling of the plane is guaranteed to match the development of the cube along the geodesic.

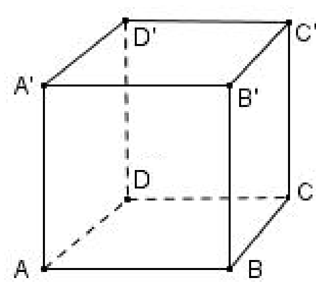


Figure 4

In order to see this, let the coordinates of A, B, C and D be $(0, 0), (x_b, 0), (x_b, x_b)$ and $(0, x_b)$ respectively. Take two parallel geodesic segments the first joining points $(\alpha_1, 0)$ and $(x_b+\alpha_1, 2x_b)$ and the second joining $(\alpha_2, 0)$ and $(x_b+\alpha_2, 2x_b)$. Let $\alpha_1 < \frac{1}{2}$, the first edge encountered is DC thus the next face encountered is $DCC'D'$. The vertices C' and D' take on the coordinates $(x_b, 2x_b)$ and $(0, 2x_b)$ respectively. The cube is then rolled about edge CC' and B and B' take on coordinates $(2x_b, x_b)$ and $(2x_b, 2x_b)$. Now let $\alpha_2 > \frac{1}{2}$, then the first edge encountered is BC rather than DC and the vertices B', C' take on coordinates $(2x_b, 0)$ and $(2x_b, x_b)$. Thus the final result depends on α as seen in Figure 5. Since we can not tile the plane in such a way to guarantee the tiling matches the development of the cube along a geodesic as in the case of the tetrahedron, we must modify our approach to classifying geodesics on the cube.

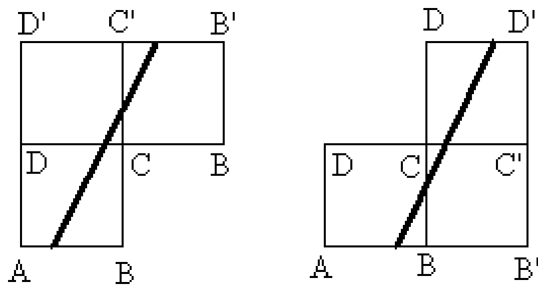


Figure 5. Parallel geodesics with different α produce different tilings of the plane with the faces of the cube.

In order to represent a closed geodesic in the plane we must join the points $X = (\alpha, 0)$ and $X' = (\alpha + x_b p, x_b q)$ with $0 < \alpha < x_b$ and (say something about vertex coordinates here). Then we roll the cube along XX' and label the vertices accordingly. The segment XX' represents a closed geodesic if and only if the vertices immediately on either side of X' are labelled A and B where AB are vertices of the face $ABCD$. The coordinates will be (px_b, qx_b) and $(px_b + 1, qx_b)$. We aim to find (α, p, q) with these properties. From this point forward we will assume without loss of generality that $0 \leq p \leq q, 0 < q$. The general cases can all be reduced to this case by reflections in lines in the plane.

Proposition 8 *Within every face, a geodesic may have at most two directions, the geodesic segments are either parallel or perpendicular to each other.*

Proof. Faces which are labeled with the same vertices may appear in at most eight different orientations in the plane. Each of these orientations can be obtained from the others by rotation by $n\frac{\pi}{2}$ with $n \in \mathbb{Z}$. The rotations either preserve the direction or produce a face in which geodesic segments are perpendicular. ■

Proposition 9 *If (p, q) corresponds to a closed geodesic and $p + q \geq 7$, then the geodesic is self-intersecting.*

Proof. Assume that $p + q \geq 7$ and that the geodesic is non-self-intersecting. The geodesic segment from X to X'(not including the endpoints) crosses p horizontal lines and $q - 1$ vertical lines. The geodesic crosses $p + q$ edges; if $p + q \geq 7$, then it must visit at least one of the six faces twice. If the two geodesic segments in a face are perpendicular to each other then they must intersect in that face or in an adjacent face. If however, they are parallel then we must address the region between the segments. It must be a part of one of the two regions into which the geodesic divides the cube. This region must at some point contain a vertex since the Gaussian curvature of a polyhedron is concentrated in the vertices and the result of Gauss-Bonnet gives that the sum of the Gaussian curvatures of this region must be equal to 2π. If the sum is non-zero there must be more than one vertex in each region the geodesic divides the cube into. We follow the geodesic until one of these vertices is between them.

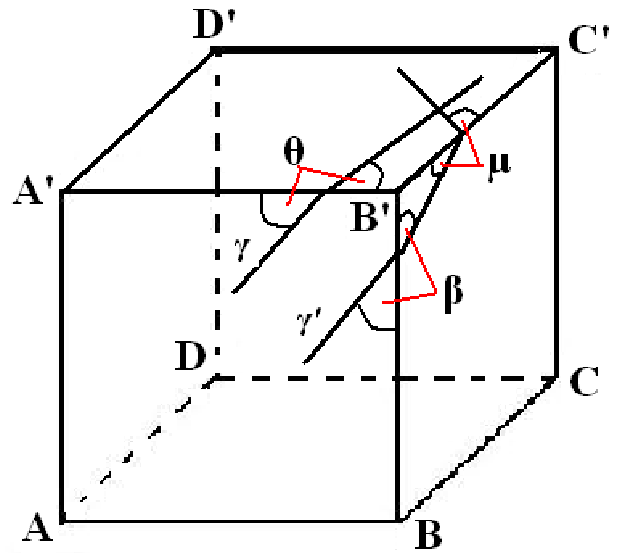


Figure 6. If $p+q \geq 7$ then the geodesic is self-intersecting. The geodesic intersects itself in an adjacent face which shares the vertex that is between the two geodesic segments.

Without loss of generality let γ and γ' be the two geodesic segments intersecting edges $A'B'$ and BB' respectively. The vertex between the two geodesic segments is then B'. Let the angle of incidence that segment γ makes with edge $A'B'$ be $\theta \geq \frac{\pi}{4}$, and the angle that segment γ' makes with edge BB' be β. Since γ and γ' form angles of incidence with perpendicular lines they are complimentary, thus

$$\beta = \frac{\pi}{2} - \theta.$$

The intersection of edge BB' and geodesic segment γ' forms two pair of vertical angles. γ' traverses face $BCC'B'$ and forms one leg of a right triangle. $\angle C'B'B = \frac{\pi}{2}$ therefore β and μ are complimentary and thus

$$\mu = \frac{\pi}{2} - \beta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right) = \theta.$$

The two segments γ and γ' intersect perpendicular edges $A'B'$ and BB'at the same angle of incidence and are thus perpendicular to each other. ■

Definition 10 *We refer to* (p, q) *with* $0 \le p \le q, q > 0$ *as a* ***good pair*** *if* $(\alpha, 0), (p + \alpha, q)$ *corresponds to a closed geodesic on the cube.*

Theorem 11 (1)*For any pair of integers* (p, q) *such that* $0 \le p \le q, q > 0$*, and* p, q *are relatively prime, there exists a unique positive integer* k *such that* (kp, kq) *is a good pair.*

(2) *If* p, q *are both odd, then* $k = 3$*.*
(3) *If one of* p, q *is even then* $k = 2$ *or* 4*.*

Proof. Let T_1, T_2 be the automorphisms of the cube from the rolling of the cube over edges BC and DC, see Figure 7.

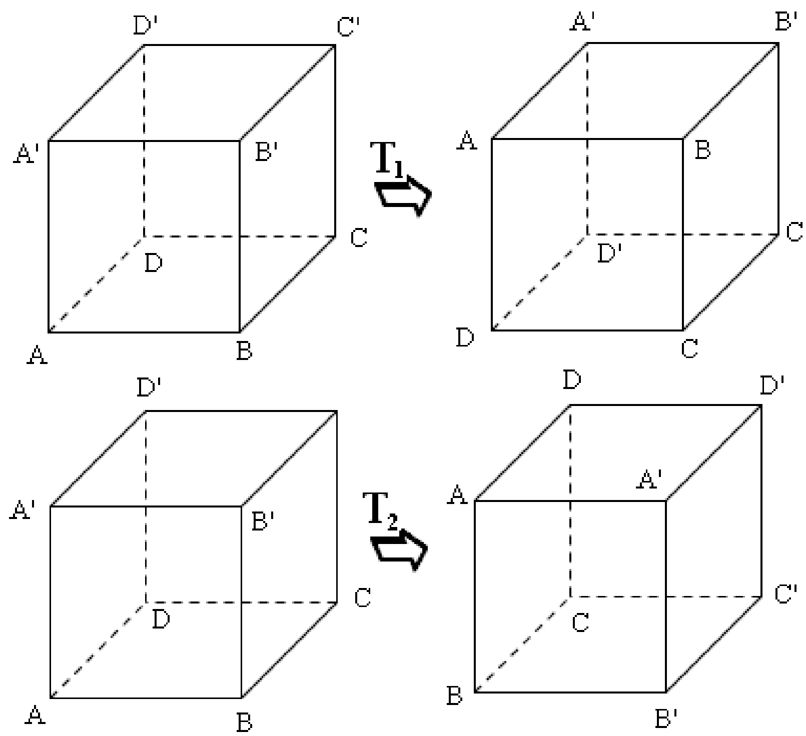
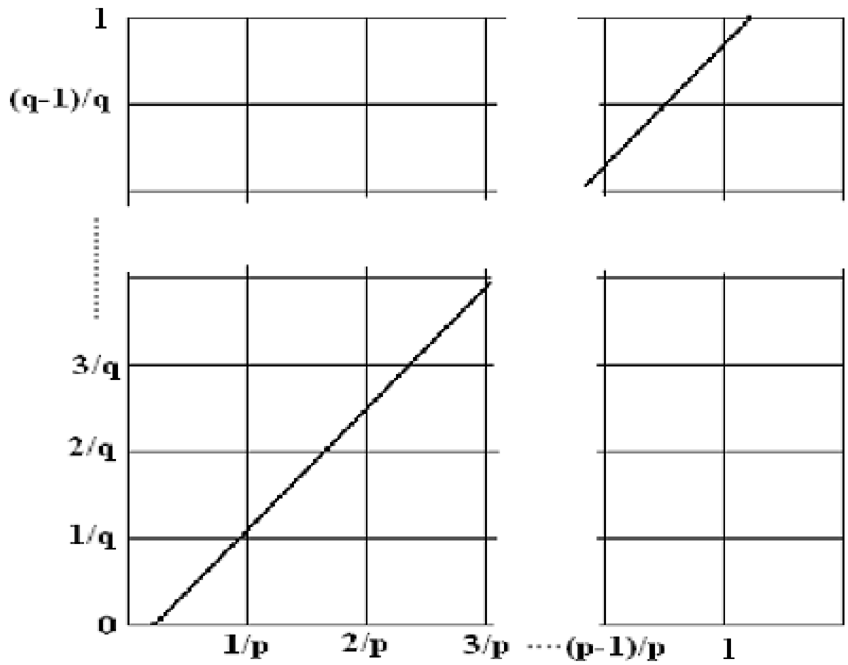


Figure 7. The automorphisms of the cube corresponding to rollings across edges BC and DC.

The general case can be reduced to the case where $0 \le p \le q, 0 < q$. In this case if the geodesic starts on edge AB and continues into face $ABCD$. Since $0 \le p \le q, 0 < q$ the geodesic, represented by a straight line in plane has positive slope thus the only two possible edges the geodesic can cross after edge AB are BC and DC and this corresponds to a rolling of the cube over these edges. T_1 and T_2 are elements of the group S_4, the symmetries of the cube. $T_1 = (1324)$ and $T_2 = (1342)$. For given (p, q) consider the fractions $\frac{1}{p}, ..., \frac{p-1}{p}$ and $\frac{1}{q}, ..., \frac{q-1}{q}$; these fractions correspond to distinct rollings of the cube over edges. In order to visualize this, take the geodesic in the plane and scale the horizontal and vertical axes such that $p = 1$ and $q = 1$ respectively. The vertical and horizontal lines now take on the equations $y = \frac{n}{p}$ and $x = \frac{m}{q}$ with $0 \leq n \leq p$ and $0 \leq m \leq q$, see Figure 8. Each of the horizontal and vertical lines represent edges of the cube and thus the fractions $\frac{1}{p}, ..., \frac{p-1}{p}$ and $\frac{1}{q}, ..., \frac{q-1}{q}$ correspond to edges.



Arrange these fractions in increasing order

$$0 < f_1 < f_2 < ...... < f_{p+q-2} < 1.$$

For $i = 1, 2, ...p + q - 2$ set

$$c(i) = \begin{cases} 1, & \text{if } qf_i \in \mathbb{Z}, \\ 2, & \text{if } pf_i \in \mathbb{Z}. \end{cases}$$

Then we define

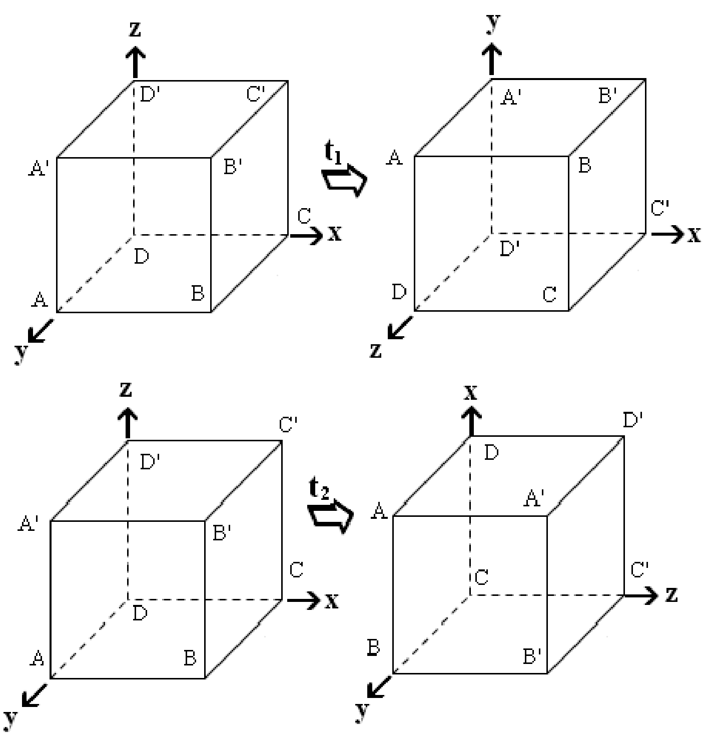
$$T_{p,q} = T_1 T_2 T_{c(p+q-2)} .....T_{c(2)} T_{c(1)}.$$

Note that $c(i)$ is well defined only if p and q are relatively prime. This definition for $T_{p,q}$ does not work in the trivial case where $p = 0$, $q = 1$ which corresponds to a geodesic segment which is represented by a vertical line in the plane. This geodesic is planar and cuts a square section of the cube. In this case we define

$T_{0,1} = T_1$. $T_{p,q}$ corresponds to the total rolling of the cube along the geodesic joining $(\alpha, 0), (p + \alpha, q)$. This is not the general case but works for small $\alpha' s$. Only if α is small can we guarantee that f_1 corresponds to the automorphism T_1, and thus the ordering of $f_{i's}$ corresponds to the ordering of the intersections of the geodesic segment with horizontal and vertical lines in the tiling. The pair (kp, kq) is good if and only if k is the order of $T_{p,q}$ in S_4. This proves part (1) of the theorem. Any element of S_4 has an order of either $1, 2, 3$ or 4.

If one of p, q is even then $p + q$ is odd. Since T_1 and T_2 are both even length cycles they are odd permutations. $T_{p,q}$ is then a composition of an odd number of odd permutations, thus the order of $T_{p,q}$ is even and in S_4 this must be 2 or 4. This proves part (3).

To prove part (2) consider the homomorphism of S_4 onto S_3 (in the terms of the cube this assigns the automorphism of the cube with respect to diagonals to the permutation of the x, y and z directions). This homomorphism takes T_1 and T_2 to $t_1 = (23)$ and $t_2 = (13)$ respectively.



Consider the fact that the sequence $f_1, ..., f_{p+q-2}$ is symmetric, that is

$$1 - f_i = f_{p+q-1-i},$$

and thus the sequence $c(i)$ is symmetric. That is

$$c(i) = c(p + q - 1 - i).$$

To see this consider that $1 - \frac{n}{p} = \frac{p-n}{p}$. If p and q are both odd then $p+q$ is even, and thus

$$t_{c(p+q-2)} \ldots\ldots t_{c(1)} = (t_{c(p+q-2)} \ldots t_{c(\frac{p+q-2}{2})})(t_{c(\frac{p+q}{2})} \ldots t_{c(1)}).$$

Since the sequence $c(i)$ is symmetric, the products in parentheses are inverses. Therefore the homomorphism $S4 \to S3$ annihilates $T_{c(p+q-2)} \ldots\ldots T_{c(2)} T_{c(1)}$ and thus takes $T_{p,q}$ into $t_1 t_2 = (123)$. $t_1 t_2$ has order 3 in $S3$ thus the order of $T_{p,q}$ must be divisible by 3 therefore it is equal to 3 in S_4. This proves part (2). ■

Corollary 12 *Up to parallelism and cube symmetries there are precisely three non-self-intersecting closed geodesics on a cube. They correspond to the good pairs* $(0,4)$, $(3,3)$ *which are planar, and* $(2,4)$.

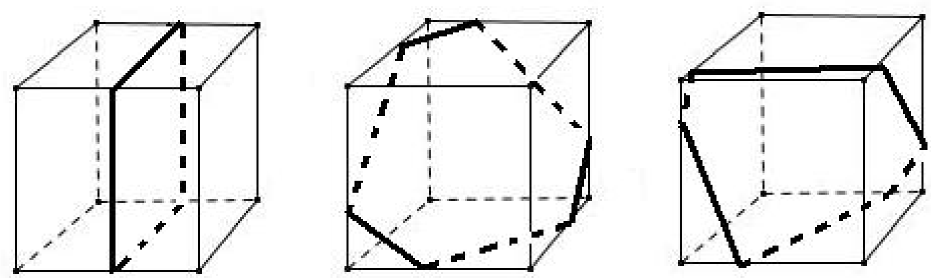


Figure 10. Three non-self-intersecting geodesics on the cube.

5 The Case of the Octahedron

The faces of the octahedron can be painted black and white in such a way that any two adjacent faces will always have different colors. We must also label the vertices in some manner, we choose to label them with unordered pairs $12, 13, 14, 23, 24$ and 34, see Figure 11. Two vertices are joined by an edge if they share a digit and three vertices belong to the same face if they all share a digit (white faces) or they all avoid some digit (black faces).

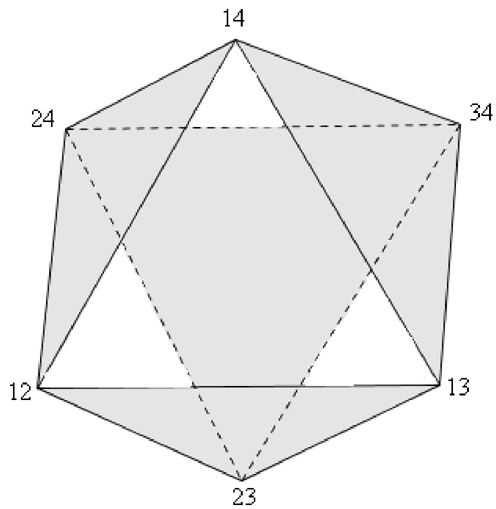


Figure 11

We can not tile the plane in such a way that the labelings in the development of a geodesic and the labeling of the standard triangular tiling of the plane are guaranteed to match however, we can use the standard triangular tiling and color it black and white so no two faces which share an edge are the same color. The triangular bicolored tiling of the plane can then be transformed

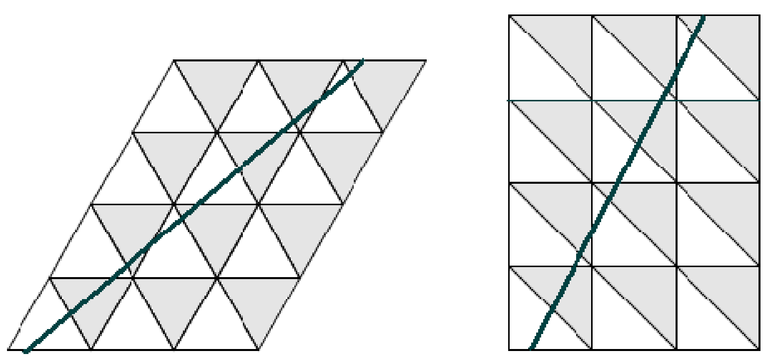


Figure 12. The tiling of the plane with the bi-colored faces of the octahedron (left) and the result of that tiling after application of the shear transform in (12).

into a square tiling by application of the shear transformation,

$$\begin{bmatrix} 1 & \frac{-x_b}{2} \\ 0 & 1 \end{bmatrix},$$

see Figure 12. Similarly to the case of the cube, a closed geodesic joins the points $X = (\alpha, 0)$ and $X' = (\alpha + x_b p, x_b q)$. Again we assume $0 \leq p \leq q, 0 < q$.

Proposition 13 *Within every face, a geodesic may have at most three directions and these three directions form angles of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with each other.*

Proof. Faces labelled with the same vertices may appear in at most three different orientations in the tiling of the plane. These orientations can be obtained from each other by rotation through $n(\frac{2\pi}{3})$ with $n \in \mathbb{Z}$. The rotations either preserve the direction of the geodesic or produce a face in which geodesic segments are at angles of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ relative to each other. ■

Proposition 14 *If $p + q \geq 5$ then the geodesic is self-intersecting.*

Proof. The geodesic crosses $2(p + q) - 1$ edges thus if $p + q > 4$ the geodesic must visit at least one face at least twice. We can show in a fashion similar to the case of the cube that if we follow parallel geodesic segments until there is a vertex between them, again there must be by Gauss-Bonnet, the two segments intersect in the white face that shares the vertex which passes between the segments. ■

As in the case of the cube, a good pair (p, q) is one which corresponds to a closed prime geodesic.

Theorem 15 (1) *For any pair of integers (p, q) such that $0 \leq p \leq q, q > 0$, and p, q are relatively prime,there exists a unique positive integer k such that (kp, kq) is a good pair.*

(2)*If* $p \equiv q \bmod 3$,*then*$k = 2$
(3) *If* $p \not\equiv q \bmod 3$*, then* $k = 3$.

Proof. A combination of two consecutive rollings of the octahedron along a geodesic segment from $X = (\alpha, 0)$ to $X' = (\alpha + x_b p, x_b q)$ corresponds to a transition from a square to a square in the plane. This combination along with a parallel translation is an automorphism of the octahedron preserving orientation and colors of faces. There are 12 automorphisms of this type and they form a group which is isomorphic to A_4. ϕ takes the vertex ij to the vertex $\phi(i)\phi(j)$. For $0 \leq p \leq q$ we will use U_1 and U_2 because we can reduce the general case to this case.

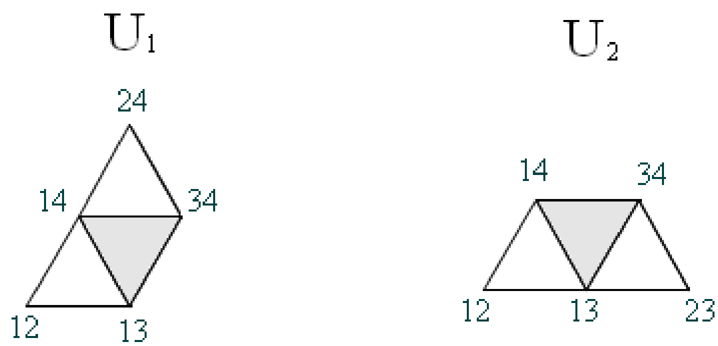


Figure 13. Permutations of the octahedron U_1 and U_2.

U_1 takes the vertices $12, 13$ and 14 to $14, 34$ and 24 respectively. The vertices were labeled as unordered pairs thus $14, 34$ and 24 can also be written as $41, 43$ and 42. Writing them this way we see that σ takes $1 \to 4, 4 \to 2$ and $2 \to 1$. Thus $U_1 = (142)$. Likewise U_2 takes the vertices $12, 13$ and 14 to $13, 23$ and 34 respectively. The vertices can again be rewritten as $31, 32$ and 34. Writing them this way we see that σ takes $1 \to 3, 3 \to 2$ and $2 \to 1$. Thus $U_2 = (132)$. If p, q are relatively prime then again the total rolling of the octahedron along the geodesic segment is

$$U_{p,q} = U_1 U_2 U_{c(p+q-2)} \ldots . U_{c(1)},$$

where $c(i)$ are defined in the same manner as the proof of the cube. Then (kp, kq) is a good pair if and only if k is the order of $U_{p,q}$ in A_4. This proves part (1), that $k = 1, 2$ or 3.

The projections of $U_1$and U_2 into $A_4 \cong \mathbb{Z}_3$ are $u_1 = (312)$ and $u_2 = u_1^{-1} = (213)$. Label the vertices of the initial face as shown in figure 14. Transition from the initial face to the black face is a reflection across edge 23. This sends vertex 1 to the remaining vertex in the black face. A reflection across edge 31 of the black face results in fixing 1 and 3 and sending 2 to the remaining vertex of the second white face. A parallel translation of this face into the position of the initial face is an automorphism sending vertex 1 to 2, 2 to 3 and 3 to 1.

We can find u_2 using the same method.

Figure 14. Automorphisms of the cube $u_1$and u_2.

Since $\mathbb{Z}_3$ is commutative, $0 \leq p \leq q, q > 0$, and p and q are relatively prime, and $u_2 = u_1^{-1}$, p of the element u_2 composed with q of the element $u_1$annihilates all but $q - p$ of the element u_1. Thus the image of $U_{p,q}$ in A_3 is $u_{p,q} = u_1^{q-p}$. If $p \not\equiv q \bmod 3$, then $u_{p,q}$ has order 3. If $u_{p,q}$ has order 3 then $U_{p,q}$ must have an order of 3. This proves part (3).

All that remains is to prove part (2). In order to do so we observe the following with regard to $U_1$and U_2:

$$U_1^3 = (142)(142)(142) = e$$

$$U_2^3 = (132)(132)(132) = e$$

$$U_1U_2U_1 = (142)(132)(142) = (123) = U_2^{-1} \tag{1}$$

$$U_2U_1U_2 = (132)(142)(132) = (124) = U_1^{-1} \tag{2}$$

and thus

$$(U_1U_2)^2 = U_1U_2U_1U_2 = U_2^{-1}U_2 = e = U_1^{-1}U_1 = U_2U_1U_2U_1 = (U_2U_1)^2. \tag{3}$$

■

Lemma 16 *If $p \equiv q \bmod 3$, then $U_{p,q} = U_1U_2$ or U_2U_1.*

Proof. $U_{p,q} = U_1U_2V_{p,q}$, where $V_{p,q} = U_{c(p+q-2)}....U_{c(1)}$ is symmetric in $U_1 = (142)$ and $U_2 = (132)$. Firstly we reduce $V_{p,q} = U_{c(p+q-2)}....U_{c(1)}$ to a symmetric word (palindrome) of the form

$$...U_1^{\pm 1}U_2^{\pm 1}U_1^{\pm 1}U_2^{\pm 1}U_1^{\pm 1}...,$$

using the relations

$$U_i^3 = e \text{ and } U_i^2 = U_i^{-1}. \tag{4}$$

The total degrees in U_1 and $U_2 \bmod 3$ stay the same and the number of letters after this reduction is 0 or odd. If the total number of letters is more than 1 then the middle three letter section must be one of the following:

$$\begin{array}{l} (U_1U_2U_1)^{\pm 1} \\ (U_2U_1U_2)^{\pm 1} \\ (U_1^{-1}U_2U_1^{-1})^{\pm 1} \\ (U_2^{-1}U_1U_2^{-1})^{\pm 1}. \end{array}$$

In the first two cases we apply the relations from (1) and (2),

$$U_1U_2U_1 = U_2^{-1} \tag{5}$$

and

$$U_2U_1U_2 = U_1^{-1}$$

to simplify the middle three letter section. In the second two cases, if the total number of letters is more than three we can reduce the middle section of letters using relations already shown.

$$U_1(U_1U_2^{-1}U_1)U_1 = U_1^2U_2^{-1}U_1^2 = U_1^{-1}U_2^{-1}U_1^{-1} = (U_1U_2U_1)^{-1} = U_2$$

$$U_1^{-1}(U_1U_2^{-1}U_1)U_1^{-1} = U_2^{-1}$$

$$\begin{aligned} U_2(U_1U_2^{-1}U_1)U_2 &= U_2U_1U_2^{-1}U_1U_2 = U_2U_1U_2U_2U_1U_2 = \\ (U_2U_1U_2)(U_2U_1U_2) &= U_1^{-2} = U_1 \end{aligned}$$

$$\begin{aligned} U_2^{-1}(U_1U_2^{-1}U_1)U_2^{-1} &= U_2U_2(U_1U_2U_2U_1)U_2U_2 = \\ U_2(U_2U_1U_2)(U_2U_1U_2)U_2 &= U_2U_1^{-1}U_1^{-1}U_2 = \\ U_2U_1U_1U_1U_1U_2 &= U_2U_1^3U_1U_2 = U_2U_1U_2 = U_1^{-1}. \end{aligned}$$

We have only shown four reductions dealing with a middle section of $(U_1U_2^{-1}U_1)$ but the rest follow similarly and all reduce to one of $U_1^{\pm 1}$ or $U_2^{\pm 1}$. What is important is that all of these reductions keep $(\deg_{U_1} - \deg_{U_2}) \bmod 3$ the same. But any reduction must keep the degree of $U_1 \bmod 3$ the same as the degree of $U_2 \bmod 3$ so any reduction that results in $U_1^{\pm 1}$ or $U_2^{\pm 1}$ is excluded. We have now reduced $U_{p,q}$ to one of the following cases

$$(U_1U_2)e = U_1U_2,$$

$$U_1U_2(U_1U_2^{-1}U_1) = U_2^{-1}U_2^{-1}U_1 = U_2^2U_2^2U_1 = eU_2U_1 = U_2U_1,$$

$$U_1U_2(U_1^{-1}U_2U_1^{-1}) = U_1U_2U_1U_1U_2U_1U_1 = U_2^{-1}U_2^{-1}U_1 = eU_2U_1 = U_2U_1,$$

$$U_1U_2(U_2U_1^{-1}U_2) = U_1U_2^{-1}U_1^{-1}U_2$$

using (1) and right multiplying by U_1^{-1} we get,

$$U_1U_2 = U_2^{-1}U_1^{-1}.$$

Thus

$$U_1U_2^{-1}U_1^{-1}U_2 = U_1U_1U_2U_2 = U_1^{-1}U_2^{-1} = (U_2U_1)^{-1},$$

by (4)

$$(U_2U_1)^{-1} = U_2U_1.$$

Lastly,

$$U_1U_2(U_2^{-1}U_1U_2^{-1}) = U_1^2U_2^{-1} = U_1^{-1}U_2^{-1} = U_2U_1.$$

We have shown in (3) that the order of U_1U_2 and U_2U_1 in A_4 are both 2. Thus $k = 2$. ■

This completes the proof of Theorem 15. We find that there are two closed, non-self-intersecting geodesics on the octahedron and they correspond to good pairs $(0, 3)$ and $(2, 2)$.

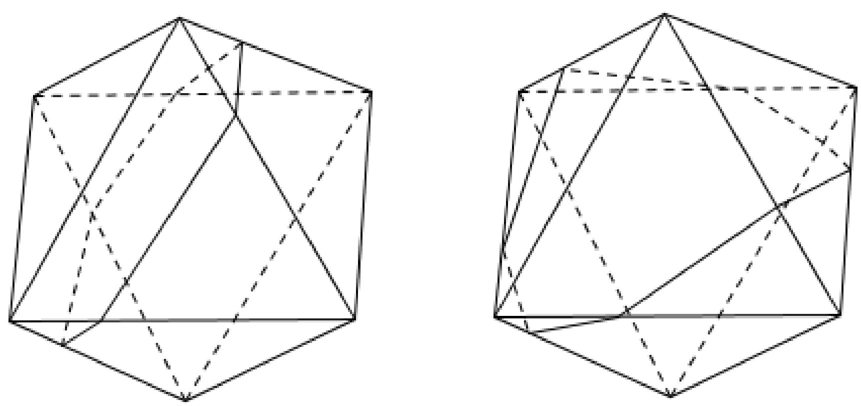


Figure 15. Two non-self-intersecting geodesics on the tetrahedron.

It seems that indeed the symmetry of a regular polyhedron is enough to guarantee it contains closed non-self-intersecting geodesics. The case of the icosahedron is similar to the case of the octahedron, using A_5 to find good pairs from relatively prime p, q. There are three closed non-self-intersecting geodesics on the icosahedron corresponding to good pairs $(0, 5), (3, 3)$ and $(2, 4)$ [1].

We considered the case of the dodecahedron however the pentagonal faces presented a problem. We can not develop the dodecahedron along a geodesic in such a way that we can use the p, q approach to classify geodesics. It may yet yield interesting results, however it will necessitate different methods to solve the problem. We know of no classification of geodesics on the dodecahedron.

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