Power Indices, Explaining Paradoxes Dealing With Power and Including Abstention

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1 Introduction

In voting systems, power can be defined in many different ways. Voting power, in general, is the amount of influence a specific voter has over the outcome of the vote. In this paper, our goal is to find the power of the President of the United States in the United States Federal system using a measure of voting power. There are four indices of power that will be discussed. These indices are the Shapley-Shubik, Banzhof, Johnston and Deegan-Packel indices. First we will define these four indices and mention what the power of President of the United States is determined to be with each of the four indices. Next, we will discuss some of the problems, or paradoxes, that arise with these indices. We will define the Donation Paradox, the Bloc Paradox and the Paradox of New Members. Through some of these paradoxes we will go through calculations of the indices in order to show how the Paradoxes are displayed.

In previous calculations of the President present in literature, abstention is ignored. However, abstentions are not ignored in the actual United States Federal System. Therefore, we will go through the area of voting power that includes abstentions. First we will define a new set of definitions that include abstentions. We will go through an example where we calculate the Banzhof index of voting power and the Banzhof measure of voting power. We will calculate the Banzhof measure of voting power of the President when abstentions are ignored, and then we will calculate the Banzhof measure of voting power of the President when abstentions are not ignored.

Through these calculations we will be able to see the difference between including abstentions and ignoring abstentions. This paper is explaining chapters from "The Measurment of Voting Power," by Dan Felsenthal as well as using some information from "Mathematics and Politics" by Alan D. Taylor. The calculations in this paper, including the calculations of the Banzhof measure of voting power, are origninal. The few examples that are taken from a source are cited. Although, the development and expansion of these examples are original.

2 Indices and Voting Power

Power is, informally, the amount of influence a single voter has over the outcome of a vote. In order to define this power of a voter mathematically we must first define a few terms.

Definition 2.1. [FM98] *A* **simple voting game***, or a SVG, is a collection of subsets $\mathcal{W}$ of a finite set N, that satisfy the following conditions:*

1. $N \in \mathcal{W}$
2. $\emptyset \notin \mathcal{W}$
3. *Monotonicity holds: If* $X \subseteq Y \subseteq N$ *and* $X \in \mathcal{W}$*, then* $Y \in \mathcal{W}$.

The set N is called the **assembly** *of the SVG $\mathcal{W}$, where N are the voters of $\mathcal{W}$. A subset of these voters, M, is a coalition. If $M \in \mathcal{W}$, then M is a* **winning coalition**.

An example of a coalition would be the members of the set N that are in the same political party that would be voting the same, either yes or no.

As we can see from Definition 2.1, the SVG $\mathcal{W}$ is a set of winning coalitions. Therefore, a vote of yes from all of the members of a winning coalition will ensure an outcome of yes.

Definition 2.2. [FM98] *If a winning coalition does not include any other winning coalitions then it is called a* **minimal winning coalition***, or MWC.*

For example, let a winning coalition $M = \{a, b, c, d\}$, where a, b, c and d are voters in N. Let another coalition $H = \{a, b, c, d, e\}$, where e is another voter in N. Since voters a, b, c and d are in H, we know that H is also a winning coalition. Since H would still be a winning coalition without voter e, it is not a minimal winning coalition.

Definition 2.3. [FM98] *A* **dummy** *is a voter who is not a member of any MWC.*

This means that a voter that is a dummy does not contribute to any winning coalition. So, their vote does not affect the outcome of the vote in any way.

Definition 2.4. [FM98] *Let $\mathcal{W}$ and $\mathcal{W}'$ be SVG's with assemblies N and N' respectively. An* **isomorphism** *from $\mathcal{W}$ to $\mathcal{W}'$ is a bijection f from N to N' (that is, a 1-1 map from N onto N'), such that for any $X \subseteq N$,*

$$X \in \mathcal{W} \iff f[X] \in \mathcal{W}'$$

If such an f exists, then we say that $\mathcal{W}$ is **isomorphic** *to $\mathcal{W}'$.*

In order to discuss power mathematically, we will formalize the measure of power through the next definition.

Definition 2.5. [FM98] *A mapping $\mathcal{P}$ that assigns a non-negative real number $\mathcal{P}_a[\mathcal{W}]$ to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$, is called a* **measure of voting power** *if the following conditions hold:*

1. *Iso-invariance: If there is an isomorphism of SVGs from $\mathcal{W}$ to $\mathcal{W}'$ that maps voter a to a', then $\mathcal{P}_a[\mathcal{W}] = \mathcal{P}_{a'}[\mathcal{W}']$.*

2. *Ignoring Dummies: If $\mathcal{W}$ and $\mathcal{W}'$ are SVGs with the same MWCs, then $\mathcal{P}_a[\mathcal{W}] = \mathcal{P}_a[\mathcal{W}']$.*

3. *Vanishing just for dummies: Voter $a \in \mathcal{W}$ is a dummy iff $\mathcal{P}_a[\mathcal{W}] = 0$.*

$\mathcal{P}$ is a *index* of voting power if, in addition to the above conditions, $\sum_{a\in N} \mathcal{P}_a[\mathcal{W}] = 1$ for any SVG $\mathcal{W}$.

In order to measure this voting power we can use different indices. Each index calculates the power to be *very* different, sometimes having a 70% difference. These calculations appear in [Tay95]. We will use four different indices of voting power: Shapley-Shubik, Johnston, Banzhof and Deegan-Packel. To show the extreme differences of each index, we will also state the power each index gives to the President in the United States Federal system to pass a bill to law without abstentions.

Note that in the United States Federal System a bill can be passed in two different ways. One way is the President votes yes and more than half of the senators and more than half of the House vote yes. The other way is that the president votes no and at least 2/3 of the senators and 2/3 of the House vote yes to override the veto.

2.1 The Shapley-Shubik Index

Definition 2.6. [Tay95] *Let $p_1, p_2, ..., p_n$ be an ordering of n voters. Let $\{p_1, ..., p_{i-1}\}$ not be a winning coalition. If $\{p_1, ..., p_i\}$ is a winning coalition, then we say voter p_i*

is **pivotal**.

Definition 2.7. [Tay95] *Suppose that a is a voter in the assembly of the SVG $\mathcal{W}$, and let N be the set of all voters, where $|N| = n$. Let m be the number of orderings of N for which a is pivotal. Also, note that $n!$ is the number of possible orderings of the set N. The* **Shapley-Shubik index** *of a, denoted by $SSI_a[\mathcal{W}]$, is the number given by $SSI_a[\mathcal{W}] = \frac{m}{n!}$.*

When using the Shapley-Shubik index, the power of the President is calculated to be about 16% [Tay95].

2.2 The Banzhof Index

Definition 2.8. [Tay95] *Let M be a winning coalition in the SVG $\mathcal{W}$. Let a be a member of the coalition M. If $M - \{a\}$ is not a winning coalition then a is* **critical**.

When determining whether or not a voter is critical we do not consider the order of the coalition. We only consider the order of the coalition when we calculate whether or not a voter is pivotal, as defined for the Shapley-Shubik index.

Definition 2.9. [Tay95] *Let a be a voter in an SVG $\mathcal{W}$. The* **total Banzhof Power** *of the voter a, $TBP_a(\mathcal{W})$, is the number of coalitions where a is a critical member of a winning coalition M.*

As we can see from this definition, the $TBP(a)$ will be an integer. We will normalize in order to get a fraction between 0 and 1.

Definition 2.10. [Tay95] *Let the voters in an SVG $\mathcal{W}$ be denoted as $p_1, p_2, ..., p_n$. The* **Banzhof index** *of voter p_i, $BI_i(\mathcal{W})$, is given by*

$$BI_i[\mathcal{W}] = \frac{TBP_i(\mathcal{W})}{TBP_1(\mathcal{W}) + TBP_2(\mathcal{W}) + ... + TBP_n(\mathcal{W})}.$$

When calculating the power of the President with the Banzhof index, we get 4% [Tay95].

2.3 The Johnston Index

Definition 2.11. [Tay95] *Let p be a voter of the SVG $\mathcal{W}$. Let $C_1, ..., C_k$ be the winning coalitions in which the deletion of p is critical. Let n_i be the number of voters whose deletion from C_i is critical. The* **total Johnston Power** *of p, $TJP_p(\mathcal{W})$, is given by*

$$TJP_p(\mathcal{W}) = \frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k}.$$

Let's look at coalition C_1. C_1 is a coalition in which p is critical and n_1 is the number of voters from C_1 that are critical. So, the more voters in C_1 that are critical, the smaller $1/n_1$ becomes. Therefore, the fewer voters that are critical in each of the coalitions gives a higher power for p. If p is one of few critical voters then it makes sense that the power should be greater. Now, we will normalize the total Johnston power to get the Johnston Index.

Definition 2.12. [FM98] *Let $p_1, p_2, ..., p_n$ be the voters of the SVG $\mathcal{W}$. Then the* **Johnston index** *of p_i, $JI_i(\mathcal{W})$, is given by*

$$JI_i(\mathcal{W}) = \frac{TJP_i(\mathcal{W})}{TJP_1(\mathcal{W}) + ... + TJP_n(\mathcal{W})}.$$

The power of the President is calculated to be 77% when using the Johnston index.

Note the differences and similarities between the Banzhof and Johnston indices. Both indices deal with critical players. Although, the Banzhof index counts the number of coalitions in which a voter is critical, and the Johnston index counts the number of voters in a coalition where a voter is critical.

2.4 The Deegan-Packel Index

Definition 2.13. [FM98] *Let p be a voter of the SVG $\mathcal{W}$. Let $C_1, ..., C_k$ be the minimal winning coalitions in which p belongs. Let n_i be the number of voters in C_i. The* **total Deegan-Packel power** *of p, $TDPP_p(\mathcal{W})$, is given by*

$$TDPP_p(\mathcal{W}) = \frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k}.$$

Now we get the final index of voting power, Deegan-Packel index, by normalizing the TDPP.

Definition 2.14. *Let $p_1, p_2, ..., p_n$ be the voters of the SVG $\mathcal{W}$. Then the* **Deegan-Packel index** *of p_i, $DPI_i(\mathcal{W})$, is given by*

$$DPI_i(\mathcal{W}) = \frac{TDPP_i(\mathcal{W})}{TDPP_1(\mathcal{W}) + ... + TDPP_n(\mathcal{W})}.$$

The power of the President is calculated to be 0.4% when using the Deegan-Packel index. As we can see, the power of the President is calculated to be very different values when using the different indices of voting power. Not only are the values different, but they vary widely. The power varies from 77% for the Johnston index and 0.4% for the Deegan-Packel index.

3 The Donation Paradox

There are many paradoxes that are exhibited when dealing with voting power. One of these paradoxes is called the Donation Paradox. Before I can describe this paradox

I will define a few terms.

Definition 3.1. [FM98] *Let $N = \{x_1, ..., x_n\}$. A* **weighted voting game***, or WVG, $\mathcal{U}$, is a voting game using a value $q > 0$ as a quota and weights $w_i \geq 0$ for each x_i in N, so that in order for the outcome to pass, the sum of the w_i corresponding to the x_i voting yes must be greater than or equal to q. The weighted voting game can be represented as $\mathcal{U} = [q; u_1, ..., u_n]$. Also, we will call x_i voter i.*

Example 3.2. [Tay95] WVGs are voting systems that are used in the real world. For example the U.N. Security Council is a weighted voting system. The U.N. Security council is made up of fifteen countries where five countries are considered permanent members. These five countries are China, England, France, Russia and the United States. For passage of a resolution, all five permanent members must vote yes, and there must be at least nine members voting yes.

This can be transferred into a weighted voting system by giving all members weights and setting a quota. Let the non-permanent members be given a weight of 1. Let q be the quota and let p be the weight of the permanent members. Then we know that $4p + 10 < q$ and $5p + 4 \geq q$. This implies that $5p + 4 > 4p + 10$, which implies that $p > 6$. We can let $p = 9$. If $p = 9$, then $46 < q$ and $49 \geq q$. So, we can let $q = 49$, and we have a WVG $\mathcal{U} = [49; 9, 9, 9, 9, 9, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$.

Definition 3.3. [FM98] *Let $\mathcal{U} = [q; u_1, ..., u_n]$, and $\mathcal{W} = [q; w_1, ..., w_n]$ where $\mathcal{U}$ and $\mathcal{W}$ are WVGs with the same assembly $N = \{x_1, ..., x_n\}$ where $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} w_i$. Then voter i is a* **donor** *if $u_i > w_i$ and voter i is a* **recipient** *if $u_i < w_i$.*

A donor, say voter i, is a voter that essentially gives some of his weight to another voter and the voter that receives this weight is the recipient. The weight of voter i, the donor, in $\mathcal{U}$, or u_i, is greater than the weight of voter i in $\mathcal{W}$, or w_i.

Now since we have defined the necessary terms, we can discuss the first paradox, the donation paradox.

Definition 3.4. [FM98] *Let* $\mathcal{U} = [q; u_1, u_2, u_3, ..., u_n]$ *and* $\widetilde{\mathcal{U}} = [q; u_1+\delta, u_2-\delta, u_3, ..., u_n]$. *A measure* $\mathcal{P}$ *is said to display the* **donation paradox** *if* $\mathcal{P}_1[\widetilde{\mathcal{U}}] < \mathcal{P}_1[\mathcal{U}]$.

The donation paradox describes when one voter donates weight and another voter receives weight and all other voters' weights and the quota stays the same, but then the recipient's power decreases. This is considered a paradox because of the fact that the voter gains more weight, but then they lose power. Some indices display this paradox while others do not.

Theorem 3.5. *The Shapley Shubik index is immune to the donation paradox.*

Proof. Let p_i and p_j be voters in the WVG $\mathcal{U}$. Let the WVG $\widetilde{\mathcal{U}}$ be the voting game where p_j donates weight to voter p_i. Suppose the number of orderings of X where p_i is pivotal is m. There will be two different positions of p_i and p_j in the orderings, either p_i comes before p_q or p_j comes before p_i. We need to show that $SSI_i(\widetilde{\mathcal{U}}) \geq SSI_i(\mathcal{U})$. In other words, we need to show that the number of orderings of X for which p_i is pivotal either increases or stays the same.

Case 1: [p_i is before p_j]

If p_i was already pivotal in this ordering it will still be pivotal because the amount of points before p_i is the same and p_i has increased in weight. And if p_i was not already pivotal it will either become pivotal or it will not become pivotal. In either situation the number of orderings in which p is pivotal increases or the number of orderings in which p is pivotal stayed the same.

Case 2: [p_i is after p_j]

If p_i was already pivotal, then it will still be pivotal. This is because there will be fewer points before p_i, but when p_i is added there will be the same amount of points as before p_j donated the weight to p_i , so the coalition will become winning. Therefore p_i is still pivotal. If p_i was not pivotal, it will either become pivotal or not become pivotal. It could become pivotal in the case that the voter in the position right before p_i was pivotal in $\mathcal{U}$ but then became not pivotal in $\widetilde{\mathcal{U}}$ because points were taken away from p_j. Then p_i will have the points that were taken away from p_j resulting in a winning coalition and making p_i pivotal. In either situation the number of orderings in which p is pivotal stays the same or the number of orderings in which p is pivotal increases.

Thus, $SSI_i(\widetilde{\mathcal{U}}) \geq SSI_q(\mathcal{U})$. Therefore, the Shapley-Shubik index is immune to the Donation Paradox. □

Theorem 3.6. [FM98] *The Banzhaf and Deegan-Packel indices are vulnerable to the donation paradox.*

Proof. Let $\mathcal{U} = \{10; 4, 6, 1, 1, 1\}$ and $\widetilde{\mathcal{U}} = \{10; 3, 7, 1, 1, 1\}$. So, u_1 is the donor and u_2 is the recipient. Now, in order to calculate the Banzhof and Johnston indices we must list the winning coalitions for both $\mathcal{U}$ and $\widetilde{\mathcal{U}}$. The winning coalitions for $\mathcal{U}$ are $(1, 2), (1, 2, 3), (1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 3, 4, 5), (1, 2, 4), (1, 2, 4, 5)$,and (1,2,5). By using these winning coalitions and the definition of the Banzhaf index we can calculate the power of u_2 in $\mathcal{U}$.

$$\begin{aligned} BI_2[\mathcal{U}] &= \frac{8}{8+8+0+0+0} \\ &= \frac{1}{2}. \end{aligned}$$

Then the winning coalitions for $\widetilde{\mathcal{U}}$ are (1,2), (1,2,3), (1,2,3,4), (1,2,3,5), (1,2,3,4,5), (1,2,4), (1,2,4,5), (1,2,5) and (2,3,4,5). Now we can calculate the Banzhaf index in $\widetilde{\mathcal{U}}$.

$$\begin{aligned} BI_2[\widetilde{\mathcal{U}}] &= \frac{9}{7+9+1+1+1} \\ &= \frac{9}{19} < \frac{1}{2}. \end{aligned}$$

Therefore, $BI_2[\mathcal{U}] > BI_2[\widetilde{\mathcal{U}}]$.This shows that u_2 gained weight in $\widetilde{\mathcal{U}}$ but then lost power. Therefore, the Banzhaf index is vulnerable to the Donation Paradox.

Now, using the same $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, as above lets show that the Deegan-Packel index is vulnerable to the Donation Paradox. First, we need to list all of the MWCs of $\mathcal{U}$. There is only one: (1,2). Then by using the definition of the Deegan-Packel index we can calculate the power of u_2 in $\mathcal{U}$.

$$\begin{aligned} DPI_2[\mathcal{U}] &= \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}} \\ &= \frac{1}{2}. \end{aligned}$$

Now, we need to list the MWCs of $\widetilde{\mathcal{U}}$. There are two; they are (1,2) and (2,3,4,5). Now, by using the definition of the Deegan-Packel index we can calculate the power of u_2 in $\widetilde{\mathcal{U}}$.

$$\begin{aligned} DPI_2[\widetilde{\mathcal{U}}] &= \frac{\frac{3}{4}}{\frac{1}{2}+\frac{3}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} \\ &= \frac{3}{8} < \frac{1}{2}. \end{aligned}$$

Therefore, $DPI_2[\mathcal{U}] > DPI_2[\widetilde{\mathcal{U}}]$.This shows that u_2 gained weight in $\widetilde{\mathcal{U}}$ but then lost power. Therefore, the Deegan-Packel index is vulnerable to the Donation Paradox.

□

In the example used above, we can see that in $\mathcal{U}$ there were three dummies that were not members of any minimal winning coalition, u_3, u_4 and u_5. But, when u_2 gained more weight in $\widetilde{\mathcal{U}}$, these dummies were able gain power by means of becoming members of a minimal winning coalition (2,3,4,5). Note that the Johnston index is also vulnerable to the Donation Paradox [FM98].

4 The Bloc Paradox

The Bloc Paradox is a form of the Donation Paradox because it deals with the transfer of weights between voters. Only now, the Bloc Paradox deals with the annexation of a voter.

Definition 4.1. *Let $\mathcal{W}$ be a WVG. Let a and b be voters in $\mathcal{W}$. The* **bloc** *$a\&b$ represents wehen voters a and b always vote the same.*

Definition 4.2. [FM98] *Let $\mathcal{W}$ be a WVG. Let voters a and b be in the assembly of $\mathcal{W}$. We can form a new voting game $\mathcal{W}|a\&b$ where voter b gives all their weight to voter a. This means that voter a is replacing the bloc $a\&b$, and voter a* **annexes** *voter b.*

This leaves voter a with more weight and voter b with a weight of zero. Now we can state the Bloc Paradox.

Definition 4.3. [FM98] *Let $\mathcal{U}$ be a WVG. Let a and b be two voters in $\mathcal{U}$. Let $\mathcal{U}|a\&b$ be the WVG where a annexes b. Then a measure displays the* **Bloc Paradox** *if $\mathcal{P}_a[\mathcal{U}|a\&b] < \mathcal{P}_a[\mathcal{U}]$.*

Theorem 4.4. [FM98] *The Shapley-Shubik index is immune to the bloc paradox.*

Proof. Since the Bloc Paradox is a form of the Donation Paradox, the Shapley Shubik index is immune to the Bloc paradox by Theorem 3.5. □

Theorem 4.5. *The Deegan-Packel, Johnston and Banzhof indices are vulnerable to the Bloc Paradox.*

Proof. Let $\mathcal{U} = [8; 6, 1, 1, 1, 1, 1, 1, 1, 1]$ and let $\mathcal{U}|2\&3 = [8; 6, 2, 0, 1, 1, 1, 1, 1, 1]$. Then we have voter 3 as the donor and voter 2 as the recipient. In $\mathcal{U}$ we have the minimal winning coalitions (1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,2,7), (1,2,8), (1,2,9), (1,3,4), (1,3,5), (1,3,6), (1,3,7), (1,3,8), (1,3,9), (1,4,5), (1,4,6), (1,4,7), (1,4,8), (1,4,9), (1,5,6), (1,5,7), (1,5,8), (1,5,9), (1,6,7), (1,6,8), (1,6,9), (1,7,8), (1,7,9), (1,8,9), and (2,3,4,5,6,7,8,9). Now we can find the power of voter 2 in $\mathcal{U}$, by using the definition of the Deegan-Packel index.

$$\begin{aligned} DPI_2[\mathcal{U}] &= \frac{\frac{59}{24}}{\frac{28}{3} + \frac{59}{24} + \frac{59}{24} + \frac{59}{24} + \frac{59}{24} + \frac{59}{24} + \frac{59}{24} + \frac{59}{24} + \frac{59}{24}} \\ &= \frac{59}{696}. \end{aligned}$$

Now, the minimal winning coalitions for $\mathcal{U}|2\&3$ are (1,2), (1,4,5), (1,4,6), (1,4,7), (1,4,8), (1,4,9), (1,5,6), (1,5,7), (1,5,8), (1,5,9), (1,6,7), (1,6,8), (1,6,9), (1,7,8), (1,7,9), (1,8,9), and (2,4,5,6,7,8,9). Now we can find the power of voter 2 in $\mathcal{U}|2\&3$ by using the definition of the Deegan-Packel index.

$$\begin{aligned} DPI_2[\mathcal{U}|2\&3] &= \frac{\frac{9}{14}}{\frac{11}{2} + \frac{9}{14} + 0 + \frac{38}{21} + \frac{38}{21} + \frac{38}{21} + \frac{38}{21} + \frac{38}{21} + \frac{38}{21}} \\ &= \frac{27}{638} < \frac{59}{696}. \end{aligned}$$

Therefore, since the power of voter 2 decreases when it gains all the weight of voter 3 this displays the Bloc Paradox. □

Examples of the Banzhof and Johnston indices displaying the bloc paradox are found in Felsenthal's "The Measurement of Voting Power."

Example 4.6. [FM98] Let $\mathcal{W} = [11; 6, 5, 1, 1, 1, 1, 1]$. When looking at the winning coalitions we get $TBP_1[\mathcal{W}] = 33$, $TBP_2[\mathcal{W}] = 31$, $TBP_3[\mathcal{W}] = TBP_4[\mathcal{W}] = TBP_5[\mathcal{W}] = TBP_6[\mathcal{W}] = TBP_7[\mathcal{W}] = 1$. Now we can calculate the Banzhof index.

$$\begin{aligned} BI_1[\mathcal{W}] &= \frac{33}{33 + 31 + 1 + 1 + 1 + 1 + 1} \\ &= \frac{11}{23}. \end{aligned}$$

Now, we let voter 1 annex voter 3 . Then we get the new WVG $\mathcal{W}|1\&3$, where $\mathcal{W}|1\&3 = [11; 7, 5, 0, 1, 1, 1, 1]$. If we look at these winning coalitions we get $TBP_1[\mathcal{W}|1\&3] = 34$, $TBP_2[\mathcal{W}|1\&3] = 30$, $TBP_3[\mathcal{W}|1\&3] = 0$, $TBP_4[\mathcal{W}|1\&3] = TBP_5[\mathcal{W}|1\&3] = TBP_6[\mathcal{W}|1\&3] = TBP_7[\mathcal{W}|1\&3] = 2$. Now we can calculate the Banzhof index.

$$\begin{aligned} BI_1[\mathcal{W}|1\&3] &= \frac{34}{34 + 30 + 0 + 1 + 1 + 1 + 1} \\ &= \frac{17}{36} < \frac{11}{23}. \end{aligned}$$

So we have $BI_1[\mathcal{W}|1\&3] < BI_1[\mathcal{W}]$. Therefore, the Banzhof index dispays the Bloc Paradox.

5 The Paradox of New Members

The Paradox of New Members concerns a weighted voting game, $\mathcal{U}$, as well. Although now we let a new voter join, $n + 1$, where n was the original number of voters in the assemby of $\mathcal{U}$. This new voting game will be called $\mathcal{V}$. The quota, q, is not changed, and the proportions of the old weights are the same. When dealing with the Paradox of New Members, we will use weights between 0 and 1 instead of positive integers. Now, the weights of the voters in $\mathcal{V}$, v_i, are equal to $u_i(1 - v_{n+1})$ for $i = 1, ..., n$ and where $u_i \in [0, 1]$ and $v_i \in [0, 1]$. In this voting game, $\mathcal{V}$, v_i are the new weights of the original n voters and v_{n+1} is the weight of the new voter. Now we can state the paradox.

Definition 5.1. [FM98] *In the WVGs stated above, a measure $\mathcal{P}$ displays the* **Paradox of New Members** *if $\mathcal{P}_a[\mathcal{V}] > \mathcal{P}_a[\mathcal{U}]$ for some a in the assembly of $\mathcal{U}$.*

This means that if any of the original voters in the original voting game, $\mathcal{U}$, have an increase in power in $\mathcal{V}$, it displays the paradox of new members. When adding a completely new voter to a game, other voters who were originally dummies now may become 'empowered.' By the definition of a dummy, we know that a dummy is not a member of any MWC and it has a measure of power of 0. This implies that a voter who is not a dummy has a measure greater than zero. Therefore, in a situation where a voter is added to an assembly and a voter who was a dummy becomes a non-dummy, the paradox of new members is displayed. Therefore, this should apply to any measure of voting power. We will show this through the following example.

Example 5.2. Let $\mathcal{U} = [\frac{41}{50}; \frac{30}{50}, \frac{30}{50}, \frac{20}{50}, \frac{10}{50}]$. Then let a new voter join the assembly and create the WVG $\mathcal{V} = [\frac{41}{50}; \frac{15}{50}, \frac{15}{50}, \frac{10}{50}, \frac{5}{50}, \frac{25}{50}]$. In $\mathcal{U}$, the MWCs are (1,2),(1,3) and (2,3). We can see that voter four is not contained in any MWCs in $\mathcal{U}$. So, voter 4 is a dummy and has a measure of voting power of zero. But, then when we look at the MWCs of $\mathcal{V}$, we see that we have MWCs (1,2,3,4),(1,4,5),(2,4,5),(1,2,5),(1,3,5) and (2,3,5). Here we see that voter 4 is now a member of some MWCs. This means that voter 4 is no longer a dummy and has a measure of voting power greater than zero. The power of voter 4 has increased and therefore, it displays the paradox of new members.

There are also cases where adding a new member to an assembly can add power one of the most powerful voters. This can be shown through an example. But first we must discuss a few new definitions.

Definition 5.3. [FM98] *A voter i is said to be a* **blocker** *if its membership is required in every winning coalition.*

All blockers in a SVG have the same power. We can see that all blockers have the same power if we look at the definitions of power and iso-invariance. Let a and b be blockers in an SVG $\mathcal{W}$. Let $\mathcal{W}'$ be the SVG where there is an isomorphism f from $\mathcal{W}$ to $\mathcal{W}'$ where $f(a) = b$, $f(b) = a$ and $f(x) = x$ for all other x in the assembly. $\mathcal{P}_a[\mathcal{W}] = \mathcal{P}_{f(b)}[\mathcal{W}']$. Since the only difference in $\mathcal{W}'$ from $\mathcal{W}$ is that a and b switch places, the power of a is equal to the the power of b. So, from the definition of iso-invariance all blockers have the same power.

A blocker in a voting game is a very powerful voter that is required in order to pass a vote. A blocker in a real world situation is a vetoer. The measure of voting power with relation to blockers and non-blockers is addressed in the following definition.

Definition 5.4. [FM98] *A measure $\mathcal{P}$* **prefers blockers** *if whenever b is a blocker and c is a non-blocker in the SVG $\mathcal{W}$, then $\mathcal{P}_b[\mathcal{W}] > \mathcal{P}_c[\mathcal{W}]$.*

Now, we can show an example where the most powerful voter in an WVG gains power when another voter is added to the assembly.

Example 5.5. Let $\mathcal{U} = [\frac{84}{100}; \frac{60}{100}, \frac{20}{100}, \frac{5}{100}, \frac{5}{100}]$. Then let a new voter join the assembly and create the WVG $\mathcal{V} = [\frac{84}{100}; \frac{24}{100}, \frac{8}{100}, \frac{2}{100}, \frac{2}{100}, \frac{60}{100}]$. Looking at $\mathcal{U}$ we can see that voter 1 and voter 2 are blockers since they are required in a winning coalition, therefore they have the same power. Then we also see that voter 3 and voter 4 are non-dummies with a measure greater than zero. Now, let's assume that $\mathcal{P}_1[\mathcal{U}] \geq 1/2$. Then we also have $\mathcal{P}_2[\mathcal{U}] \geq 1/2$. So, $\mathcal{P}_1[\mathcal{U}] + \mathcal{P}_2[\mathcal{U}] + \mathcal{P}_3[\mathcal{U}] + \mathcal{P}_4[\mathcal{W}] > 1$, which is a contradiction by the definition of an index. Therefore, $\mathcal{P}_1[\mathcal{U}] < 1/2$.

In $\mathcal{V}$ we have voter 2, voter 3 and voter 4 as dummies since they are no longer a part of any MWCs. We have their measure of power as zero. We have voter 1 and voter 5 as blockers since both of them are required to have a winning coalition. So,they have the same power. We have $\mathcal{P}_1[\mathcal{V}] = 1/2 > \mathcal{P}_1[\mathcal{U}]$. The power of voter 1, the most powerful voter in $\mathcal{U}$, has increased when another voter is added to the assembly.

In these examples we have seen how the paradox of new members is displayed in different simple voting games. We have seen that by adding a voter we can change a voter from dummy into a non-dummy, resulting in increasing their power. We have also seen the most powerful voter in a WVG increase their power by changing a non-dummy into a dummy.

6 Abstention

Most literature on voting power ignores the idea of voters having the choice of a third vote — abstention. According to Felsenthal in "The Measurement of Voting Power," abstention is mainly ignored because it is viewed as illogical for someone to not vote [FM98]. As we will see here, when absention is taken into account, the power changes. When dealing with absentions, we have to deal with a new kind of set, a tripartition.

Definition 6.1. [FM98] *Let N be a finite set of voters where abstention is not ignored. Then a* **Tripartition** *T is a map from N to {-1,0,1}. The inverse images of {-1}, {0} and {1} are denoted by* T^-, T^0, T^+.

This is how we will be dealing with abstention mathematically. We assign each voter in the assembly a value that represents the vote of yes, no or abstention. A vote of no is denoted as the {-1}, a vote of yes is denoted as the {1} and an abstention is denoted as the {0}.

Example 6.2. Let the set $N = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$, where $p_1, ..., p_7$ are the voters. Then we have the tripartition T_1 so that $T_1(p_2) = T_1(p_3) = T_1(p_4) = -1$, $T_1(p_1) = T_1(p_6) = 0$ and $T_1(p_5) = T_1(p_7) = 1$. This means that voters p_2, p_3 and p_4 voted no, voters p_1 and p_6 abstained, and voters p_5 and p_7 voted yes.

The above example is one of the possible tripartitions of the set N. When we were dealing with SVGs we only took into account two choices of voting, yes and no. So, now while we are allowing abstention, we need to define a new structure other than SVGs. This new structure will be called a ternary voting rule.

Definition 6.3. [FM98] *Let N be a nonempty finite set. A* **ternary voting rule***, TVR, is any map* $\mathbf{W}$ *from the set {-1,0,1} of all possible tripartitions to {-1,1} where the follwing conditions hold.*

1. $T^+ = N \implies \mathbf{W}T = 1.$

2. $T^- = N \implies \mathbf{W}T = -1.$

3. $Monotonicity : T_1 \leq T_2 \implies \mathbf{W}T_1 \leq \mathbf{W}T_2.$

Note that $T_1 \leq T_2$ means that for all $x \in N$, $T_1(x) \leq T_2(x)$.

In a TVR, N are the voters of $\mathbf{W}$. As before, a set of these voters is a coalition. Now, $\mathbf{W}T$ is the outcome of T, the tripartition, under $\mathbf{W}$. The purpose of using the TVR is so that we can obtain an outcome of the vote that is either yes or no, while still allowing voters to abstain. Condition 1 is saying that if all of the voters are voting yes, then the outcome of the vote will be a yes. Condition 2 is saying that if all the voters are voting no, the outcome of the vote will be a no.

Definition 6.4. [FM98] *Let N be a finite set. Then the* **sum**, **S**, *and the* **margin**, **M**, *of a tripartition T are defined as follows.*

$$\mathbf{S}T = \sum_{x \in N} T(x)$$

and

$$\mathbf{M}T = |\mathbf{S}T|.$$

The sum and margin defined above are used to calculate the outcome of a vote.

Definition 6.5. [FM98] *A TVR* **W** *is a* **majority TVR** *if* $\mathbf{W}T = 1 \iff \mathbf{S}T > 0$ *for every tripartition of the assembly.*

Therefore, under a majority TVR, a bill is passed if and only if more people vote for it than against it. We can also talk about the relationship between what the voter votes and the outcome of the vote.

Definition 6.6. *Let N be a finite set, and let T be the tripartition of the assembly N. Let a be a voter of the TVR* **W**. *We say that a* **agrees** *with the outcome T under* **W** *if $T(a) = \mathbf{W}T$. We say that a* **agrees positively** *or a* **agrees negatively** *with the outcome of T under* **W** *with the common values of 1 and -1 respectively. Similarly, we can say that a* **disagrees positively or negatively** *with the outcome of T under* **W,** *if $T(a) = -\mathbf{W}T$.*

Definition 6.7. [FM98] *If$T(a) \leq 0$, then we denote by $T_{a\uparrow}$, the tripartition of N so that $T_{a\uparrow}(a) = T(a) + 1$ and for all other $x \in N$, $T_{a\uparrow}(x) = T(x)$. Then we say that a is* **negatively W-critical** *for T if $\mathbf{W}T = -1$ and $\mathbf{W}T_{a\uparrow} = 1$.*

If $T(a) \geq 0$, then we denote by $T_{a\downarrow}$, the tripartition of N so that $T_{a\downarrow}(a) = T(a) - 1$ and for all other $x \in N$, $T_{a\downarrow}(x) = T(x)$. Then we say that a is **positively W-critical** *for T if $\mathbf{W}T = 1$ and $\mathbf{W}T_{a\downarrow} = -1$.*

If a voter is either positively or negatively **W**-critical we can say that they are **W**-critical. This means that when a bill is not passing and one voter, who is either voting no or abstaining, improves their vote by one, the bill will then pass. Also, when a bill is passing and one voter, who is either voting yes or abstaining, decreases their vote by one, the bill will then not be passing.

Example 6.8. Let W be a majority TVR. Let the assembly be $N = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. Let T be the tripartition so that $T^{-1} = \{p_1, p_3\}$, $T^0 = \{p_2, p_6\}$ and $T^1 = \{p_4, p_5\}$. Now we can calculate the sum of $\mathbf{W}T$.

$$\mathbf{S}T = T(p_1) + T(p_2) + T(p_3) + T(p_4) + T(p_5) + T(p_6)$$

$$= -1 + 0 + -1 + 1 + 1 + 0$$

$$= 0$$

Therefore, since **S**T is not greater than 0, the bill will not pass. But now let p_2 improve their vote from abstention to yes. So, now we have $T^{-1}_{p_2\uparrow} = \{p_1, p_3\}$, $T^0_{p_2\uparrow} = \{p_6\}$ and $T^1_{p_2\uparrow} = \{p_2, p_4, p_5\}$. Now we can calculate the sum of $\mathbf{W}T_{p_2\uparrow}$ and see whether the sum changes or not.

$$\mathbf{S}T_{p_2\uparrow} = T_{p_2\uparrow}(p_1) + T_{p_2\uparrow}(p_2) + T_{p_2\uparrow}(p_3) + T_{p_2\uparrow}(p_4) + T_{p_2\uparrow}(p_5) + T_{p_2\uparrow}(p_6)$$

$$= -1 + 1 + -1 + 1 + 1 + 0$$

$$= 1$$

As we can see, $\mathbf{S}T_{p_2\uparrow} > 0$ and therefore the bill is now passing. So, by the improvement of p_2's vote alone the bill went from not passing to passing. So, voter p_2 is negatively **W**-critical for T.

Definition 6.9. [FM98] *Let* $\mathbf{W}$ *be a TVR with the assembly N. Let* $a \in N$. *Let* $\eta_a[\mathbf{W}]$, *the* **Banzhof score** *of a, be the number of tripartitions of N where a is positively* $\mathbf{W}$-*critical.*

This means that after finding all of the possible tripartitions of N under $\mathbf{W}$, we find the number of tripartitions in which if we decrease voter $a's$ vote by one the bill goes from passing to not passing.

Definition 6.10. [FM98] *Let* $\mathbf{W}$ *be a TVR with the assembly N. Let* $a \in N$. *We define the* **Banzhof index of voting power** *of voter a,* $\beta_a[\mathbf{W}]$, *by*

$$\beta_a[\mathbf{W}] = \frac{\eta_a[\mathbf{W}]}{\sum_{x \in N} \eta_x[\mathbf{W}]}.$$

Example 6.11. Let $\mathbf{W}$ be a TVR with the assembly $\{a, b, c, d\}$, where a bill is passed if and only if a votes for it and at least 2 of b, c and d vote for it. Therefore, $\mathbf{W}T = 1$ iff $T(a) = 1$ and at least two of $T(b), T(c)$ and $T(d)$ are equal to 1. Now we can determine the tripartitions in which the bill passes. This is organized in the table below.

Table 1: Tripartions of $\mathbf{W}$

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
$T(a)$	1	1	1	1	1	1	1
$T(b)$	1	1	1	1	1	0	-1
$T(c)$	1	1	1	0	-1	1	1
$T(d)$	1	0	-1	1	1	1	1

Now we can determine the Banzhof score of each voter, the number of tripartitions in which a voter is positively $\mathbf{W}$-critical. By using the table above, we can see that voter a is positively $\mathbf{W}$-critical in $T_1, T_2, T_3, T_4, T_5, T_6$ and T_7. So, $\eta_a[\mathbf{W}] = 7$. Voter

b is positively **W**-critical in T_2, T_3, T_4 and T_5. So, $\eta_b[\mathbf{W}] = 4$. Voter c is positively **W**-critical in T_2, T_3, T_6 and T_7. So, $\eta_c[\mathbf{W}] = 4$. Voter d is positively **W**-critical in T_4, T_5, T_6 and T_7. So, $\eta_d[\mathbf{W}] = 4$. We can now calculate the Banzhof index of voting power of each of the voters.

$$\begin{aligned}
\beta_a[\mathbf{W}] &= \frac{\eta_a[\mathbf{W}]}{\eta_a[\mathbf{W}] + \eta_b[\mathbf{W}] + \eta_c[\mathbf{W}] + \eta_d[\mathbf{W}]} \\
&= \frac{7}{7+4+4+4} \\
&= \frac{7}{19}. \\
\beta_b[\mathbf{W}] &= \frac{\eta_b[\mathbf{W}]}{\eta_a[\mathbf{W}] + \eta_b[\mathbf{W}] + \eta_c[\mathbf{W}] + \eta_d[\mathbf{W}]} \\
&= \frac{4}{7+4+4+4} \\
&= \frac{4}{19}. \\
\beta_c[\mathbf{W}] &= \frac{\eta_c[\mathbf{W}]}{\eta_a[\mathbf{W}] + \eta_b[\mathbf{W}] + \eta_c[\mathbf{W}] + \eta_d[\mathbf{W}]} \\
&= \frac{4}{7+4+4+4} \\
&= \frac{4}{19}, \\
\beta_d[\mathbf{W}] &= \frac{\eta_d[\mathbf{W}]}{\eta_a[\mathbf{W}] + \eta_b[\mathbf{W}] + \eta_c[\mathbf{W}] + \eta_d[\mathbf{W}]} \\
&= \frac{4}{7+4+4+4} \\
&= \frac{4}{19}.
\end{aligned}$$

Therefore, we have the indices of voting power for each of the four voters. As we can see, voter a has a higher measure of voting power, $\frac{7}{19}$ as compared to $\frac{4}{19}$. This can be explained by how **W** was defined. In order for the bill to be passed, it is required for voter a to vote yes while it is not required for any of b, c and d to vote yes. So, it is makes sense that voter a should have a higher index of voting power than the others.

Definition 6.12. [FM98] *Let* $\mathbf{W}$ *be an SVG with the assembly* N*, where* $|N| = n$*. Let* $a \in N$*. Define the* **Banzhof measure of voting power** *of voter* a*,* $\beta'_a[\mathbf{W}]$ *by*

$$\beta'_a[\mathbf{W}] = \frac{\eta_a[\mathbf{W}]}{2^{n-1}}.$$

Definition 6.13. *Let* $\mathbf{W}$ *be a TVR, with assembly* N*, where* $|N| = n$*. Then the* **Banzhof measure of voting power** *for TVRs is defined as follows.*

$$\beta''_a[\mathbf{W}] = \frac{\eta_a[\mathbf{W}]}{3^{n-1}}$$

Now we can calculate a Banzhof measure of voting power using Example 6.11.

Example 6.14. Let $\mathbf{W}$ be the TVR from Example 6.11. We have $\eta_a[\mathbf{W}] = 7$. So,

$$\begin{aligned} \beta''_a[\mathbf{W}] &= \frac{\eta_a[\mathbf{W}]}{3^{n-1}} \\ &= \frac{7}{3^3} = \frac{7}{27}. \end{aligned}$$

6.1 Presidential Power

A real life example in which TVRs are used is the United States Federal System. When a bill is going to be passed it must go through the Senate, the House and the President. There needs to be a majority of the senators, a majority of the House that approve, and the President must approve [Tay95]. Or the President may vote no, veto, and then there must be at least 2/3 of the senators and 2/3 of the House that approve [Tay95]. When the President abstains, it is counted as if the President has voted yes. Every voter is allowed to abstain in this system, although only 49 out of 100 senators and 217 out of 435 House members can abstain at once. We can use TVRs to calculate the power of a voter. We will go through the steps in order to calculate the power of the President using TVRs. But first, we will go through and calculate the power of President as if abstentions were not allowed. Then we will be able to see how the power of the President changes when abstentions are allowed.

When we ignore abstentions that means that everyone only has 2 choices: yes or no. Let p be the President and let $\mathbf{U}$ be the United States Federal system. So, let's look at the numerator, $\eta_p[\mathbf{U}]$, first. This is the number of coalitions in which when p decreases their vote by one, from yes to no, the coalition changes from winning to not winning. So, let's look at the four disjoint sets of winning coalitions when p votes yes. The first set would be when there are 51-66 senators and 218-289 House members voting yes. The second set would be when there are 51-66 senators and 290-435 House members voting yes. The third set is when 67-100 senators and 218-289 House members vote yes. The fourth set is when 67-100 senators and 290-435 House members are voting yes. The first three sets are the only sets that will change from winning to not winning when p changes from yes to no. This is because the fourth set has both the Senate and the House at least 2/3, which will override the veto.

Now, we have to count these three sets. Let k be the number of senators voting yes and let j be the number of House members voting yes. We need to count how many ways there are to choose k senators and j House members from each of the three sets. First of all, we know that there are $\binom{100}{k}$ ways to choose k senators and there are $\binom{435}{j}$ ways to choose j senators. Then we need to sum these ways over each of the three sets and then add them together. This is shown in the following equation.

$$\begin{aligned}\eta_p[\mathbf{U}] = &\left[\sum_{k=51}^{66}\binom{100}{k}\right]\left[\sum_{j=218}^{289}\binom{435}{j}\right]\\ &+\left[\sum_{k=51}^{66}\binom{100}{k}\right]\left[\sum_{j=290}^{435}\binom{435}{j}\right]\\ &+\left[\sum_{k=67}^{100}\binom{100}{k}\right]\left[\sum_{j=218}^{289}\binom{435}{j}\right]\end{aligned}$$

Now we can calculate the denominator. We know that there are 100 senators, 435 House members and 1 President. So, we get the denominator to be as follows.

$$L = 2^{100+435+1-1} = 2^{535}$$

Now we have the numerator, $\eta_p[\mathbf{U}]$, and denominator of the Banzhof measure of voting power of p, $\beta'_p[\mathbf{U}]$. Therefore, we can calculate $\beta'_p[\mathbf{U}]$.

$$\beta'_p[\mathbf{U}] = \frac{\eta_p[\mathbf{U}]}{2^{535}} \approx 0.230103 \approx 23.0\%.$$

So, when we ignore abstentions we calculate the Banzhof power of the President of the United States to be 23.0%.

Now, let's include abstentions in the United States Federal system and calculate the power of the President. In order to find the power of p we need to find the Banzhof score, $\eta_p[\mathbf{U}]$, the numerator. Then also we have to find the denominator, L, the possible number of tripartitions not including the President. Let's deal with $\eta_p[\mathbf{U}]$ first. Then we have to find the coalitions in which p is positively $\mathbf{U}$-critical, the coalitions in which when p decreases their vote by one, the coalition goes from winning to not winning. There are three different sets we can look at, either p is voting yes, no or abstaining. But, if p is voting yes, when they decrease their vote to abstention, the coalition will still be winning because the United States Federal System treats the President's abstention as a vote of yes. So, none of the coalitions in which the President is voting yes are positively **U**-critical. Also, when p is voting no, they cannot decrease their vote anymore, so none of the coalitions in which p votes no are positively **U**-critical. Therefore, the only coalitions that are possibly positively **U**-critical are when p is abstaining.

Now, when p is abstaining we have four different disjoint winning coalition sets. The first set is a majority of the senators not abstaining to one less than 2/3 of the senators not abstaining and a majority of the House not abstaining to one less than 2/3 of the House not abstaining. The second is a majority of the senators not abstaining to one less than 2/3 of the senators not abstaining and 2/3 of the House not abstaining to all of the members of the House not abstaining. The third is 2/3 of the senators not abstaining to all of the senators not abstaining and a majority of the House not abstaining to one less than 2/3 of the House not abstaing. The fourth is 2/3 of the senators not abstaining to all of the senators not abstaining and 2/3 of the House not abstain to all of the House not abstaining. Now, if we look at all four of these sets of coalitions only the first three will be positively **U**-critical when p decreases their vote from abstention to no. This is because for the fourth set both the senate and the House have at least 2/3 voting yes, which will override the President's veto, therefore the winning coalition will not change to losing coalitions. So, now we will only be using the first three sets.

Let m be the number of voters abstaining in the Senate, and let n be the number of voters abstaining in the House for $0 \leq m \leq 49$ and $0 \leq n \leq 217$. Also, let k be the number of senators voting yes, and let j be the number of House members voting yes.

First we have to choose the senators and House members that are abstaining. So we have $\binom{100}{m}$ and $\binom{435}{n}$. Then we have to count the possible coalitions in the set which will be $\binom{100-m}{k}$ and $\binom{435-n}{j}$. We will multiply these together and sum them over the corresponding of the three sets, and then sum them from $m = 0$ to 49 and $n = 0$ to 217. We will do this for each of the three sets and add them together as we can see in the following equation for the numerator of the Banzhof measure of voting power of p. Note that the symbol $|\lceil x \rceil|$ is the smallest integer so that $|\lceil x \rceil| > x$.

$$\begin{aligned}
\eta_p[\mathbf{U}] = &\left[\sum_{m=0}^{49} \sum_{k=|\lceil \frac{100-m}{2} \rceil|}^{|\lceil \frac{2(100-m)}{3} \rceil|-1} \binom{100}{m}\binom{100-m}{k}\right]\left[\sum_{n=0}^{217} \sum_{j=|\lceil \frac{435-n}{2} \rceil|}^{|\lceil \frac{2(435-n)}{3} \rceil|-1} \binom{435}{n}\binom{435-n}{j}\right] \\
&+ \left[\sum_{m=0}^{49} \sum_{k=|\lceil \frac{100-m}{2} \rceil|}^{|\lceil \frac{2(100-m)}{3} \rceil|-1} \binom{100}{m}\binom{100-m}{k}\right]\left[\sum_{n=0}^{217} \sum_{j=|\lceil \frac{2(435-n)}{3} \rceil|}^{435-n} \binom{435}{n}\binom{435-n}{j}\right] \\
&+ \left[\sum_{m=0}^{49} \sum_{k=|\lceil \frac{2(100-m)}{3} \rceil|}^{100-m} \binom{100}{m}\binom{100-m}{k}\right]\left[\sum_{n=0}^{217} \sum_{j=|\lceil \frac{435-n}{2} \rceil|}^{|\lceil \frac{2(435-n)}{3} \rceil|-1} \binom{435}{n}\binom{435-n}{j}\right].
\end{aligned}$$

So, now we have the Banzhof score, $\eta_p[\mathbf{U}]$, or the numerator of the Banzhof Measure of voting power of p.

Now we have to find the denominator, let's call it L. Before we can calculate this we must first realize that everyone cannot abstain, so everyone does not have three choices. Let m and n be as stated previously. We know that at most 49 senators and 217 House members can abstain at once. We need to count how many ways there are to choose m and n. As before, this will be 100 choose m and 435 choose n. Then we have the rest of the voters, $100 - m$ and $435 - n$ who either will vote yes or no, they have two choices. So, we count these as 2^{100-m} and 2^{435-n}. Then we sum these from $0 \leq m \leq 49$ and $0 \leq n \leq 217$. The equation is as follows.

$$L = \left[\sum_{m=0}^{49} \binom{100}{m} 2^{100-m} \right] \left[\sum_{n=0}^{217} \binom{435}{n} 2^{435-n} \right].$$

So, now we have the numerator and the denominator of the Banzhof measure of voting power. When we calculate the power of p using this equation we get the following.

$$\beta_p''[\mathbf{U}] = \frac{\eta_p[\mathbf{U}]}{L} \approx 0.232240046 \approx 23.2\%.$$

So, when we include abstention in the United States Federal system we calculate the Banzhof power of the President to be about 23.2%. When we ignored abstentions in the previous example we calculated the Banzhof power of the President to be about 23.0%. This means that when members of the United States Federal system are allowed to abstain they increase the President's Banzhof power by about 0.2%. The power of the President does not increase considerably, but it still has increased. One would assume that if the other members are given more choices to vote then the President's power will decrease. But, as we can see, the President's power does not decrease; it increases a small amount.

7 Appendix A

Presidential Calculations

These are the calculations of the President from section 6 using Wolfram Mathematica. The following are commands that we define in order to make the calculations simpler. superceil is the command for $|\lceil n \rceil|$, majority is the commmand for when we have a majority of the voters voting yes and supermaj is the command for when we have at least 2/3 of the voters voting yes.

```
superceil[n_] := Ceiling[n + 10^-10];
majority[n_] := superceil[n/2];
supermaj[n_] := superceil[2 n/3];
```

The following is the calculation for the Banzhof measure of voting power of the President when abstentions are not ignored.

```
AbsoluteTiming[(Sum[
      Sum[Binomial[100, m] Binomial[100 - m, k] Binomial[435,
         n] Binomial[435 - n, j], {k, majority[100 - m],
        supermaj[100 - m] - 1}, {j, majority[435 - n],
        supermaj[435 - n] - 1}], {m, 0, 49}, {n, 0, 217}]
+ Sum[Sum[Binomial[100, m] Binomial[100 - m, k] Binomial[435,
         n] Binomial[435 - n, j], {k, majority[100 - m],
        supermaj[100 - m] - 1}, {j, supermaj[435 - n], 435 - n}], {m,
       0, 49}, {n, 0, 217}]
    + Sum[Sum[Binomial[100, m] Binomial[100 - m, k] Binomial[435,
         n] Binomial[435 - n, j], {k, supermaj[100 - m],
        100 - m}, {j, majority[435 - n], supermaj[435 - n] - 1}], {m,
       0, 49}, {n, 0, 217}])
/(Sum[Binomial[100, m] 2^(100 - m), {m, 0, 49}] Sum[
      Binomial[435, m] 2^(435 - m), {m, 0, 217}])]
// N
0.232240046
```

The following is the calculation for the Banzhof measure of voting power of the President when abstentions are ignored.

```
AbsoluteTiming[
[Sum[Sum[Binomial[100 - m, k] Binomial[435 - n, j],
 {k, majority[100 - m], supermaj[100 - m] - 1},
 {j, majority[435 - n],  supermaj[435 - n] - 1}],
 {m, 0, 0}, {n, 0, 0}]
+Sum[Sum[Binomial[100 - m, k] Binomial[435 - n, j],
{k, majority[100 - m], supermaj[100 - m] - 1},
 {j, supermaj[435 - n],  435 - n}],
{m, 0, 0}, {n, 0, 0}]
+Sum[Sum[Binomial[100 - m, k] Binomial[435 - n, j], {k,
      supermaj[100 - m], 100 - m}, {j, majority[435 - n],
      supermaj[435 - n] - 1}],
{m, 0, 0}, {n, 0, 0}]]
/2^(535)
// N
0.23010269065320496
```

References

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