Polygonal Billiards

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1 Introduction

Billiards in mathematics is not like the leisure sport of billiards. The difference is that in mathematical billiards things like friction and spin are neglected, the ball is considered a point with no dimensions, and if the point strikes a vertex (like a pocket) the game is over. The only factors in mathematical billiards are the table, the initial position of the point, and the initial direction of the point. This means that when the point is set in motion it can continue indefinitely if it has a path where it never strikes a vertex.

First we will define polygonal billiards, then we will explain the method of unfoldings and its uses, then we will show which polygons tile the plane with reflections, then we will discuss periodic orbits in billiards in two sections, finally we will mention some open questions in the field of polygonal billiards.

2 Definitions for Polygonal Billiards

In this paper we will be focusing on polygonal billiard tables and their periodic orbits.

Definition 1 *A polygonal billiard table is any closed bounded region bounded by a convex polygon (a polygon whose interior is a convex set) in the Euclidean plane with a point inside the polygon that has an initial position and direction associated with it.*

To explain how billiards works we must define how orbits react when they strike a boundary. To do that we need two more definitions.

Definition 2 *The angle of incidence is the acute angle between an incoming ray and the tangent line, if it exists, of a boundary where the ray strikes.*

Definition 3 *Given an incoming ray and outgoing ray (with vertex on the boundary), the angle of reflection is the acute angle created from the exiting ray and the line tangent to where the ray struck the boundary.*

Now we can define polygonal billiards.

Definition 4 *Given a polygonal billiard table, a billiard orbit is the curve obtained from the path of point inside the billiard table moving in its given direction where whenever it strikes a boundary it reflects off the boundary and the angle of incidence is congruent to its corresponding angle of reflection. If the orbit strikes a vertex on the polygon, the orbit ends.*

There are some special orbits which are of particular interest.

Definition 5 *A periodic orbit is an orbit in which the point returns to its initial position with the same initial direction.*

Definition 6 *A perpendicular periodic orbit is a periodic orbit with some angle of incidence equal to* $\frac{\pi}{2}$.

3 Method of Unfoldings

In a billiard table an orbit can be described using unfoldings. When an orbit strikes a boundary, the method of unfoldings doesn't have the orbit reflected back into the billiard table. Instead we reflect the billiard table across the boundary where the ray struck. The orbit then crosses the boundary into the reflected image of the billiard table. The same process happens within the reflected image of the billiard table when the ray strikes another boundary. For example, lets take a rectangle to be our billiard table. When the ray strikes a boundary of the rectangle we reflect the rectangle across that side and have the ray continue its path. A diagram of the example is given below.

So we can see that unfoldings is just another way of viewing our orbits. The method of unfoldings is very helpful when the billiard table tiles the plane with reflections. If a polygon tiles the plane using reflections then when the polygon is copied by reflection across all of its sides the original polygon and copied polygons do not overlap any of each other. All subsequent reflected copies of the reflected copies must not overlap each other either. Description of unfoldings referenced in [3, pp. 3524-3523].

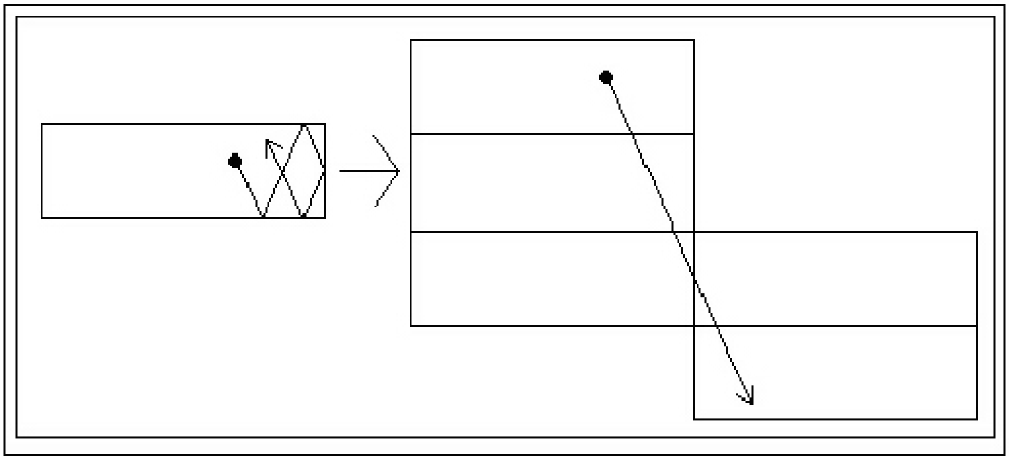

Figure 1: A billiard orbit expressed as a line through unfoldings.

When using unfoldings any orbit becomes expressed as a straight line. Also if we have a point in a table then in any other table with the same orientation as that first table, there exists a point that is identified with the original point. That means they represent the same point on the original table. This is useful for proving that an orbit is periodic because it will retain the same initial direction if it is in a table with the same orientation as the original table and is at a point that is identified with the initial positions in the original table. In later sections we use these facts to prove that certain tables have periodic orbits.

4 Polygons that Tile the Plane with Reflections

In this section we will significantly reduce the number of possible polygons that tile the plane with reflections, then we will prove which ones do.

Definition 7 *A polygon tiles the plane with reflections if given any sequence of polygons, x_n, that is generated by reflections across any side of the preceding polygon, then for all $m, n \in \mathbb{N}$, the intersection of the interiors of polygon x_n and x_m, is either empty or the full interior of x_n.*

To show which polygons tile the plane we need a well known lemma.

Lemma 8 *The sum of interior angles in a polygon is given by*

$$\sum_{i=1}^{s} a_i = (s-2)\pi = \left(\frac{s-2}{2}\right) 2\pi$$

where s is the number of sides in the polygon and a_i is an interior angle.

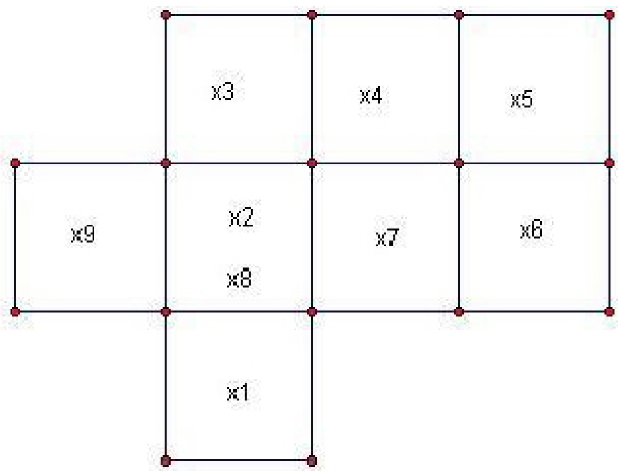

Figure 2: A polygon that tiles the plane with reflections.

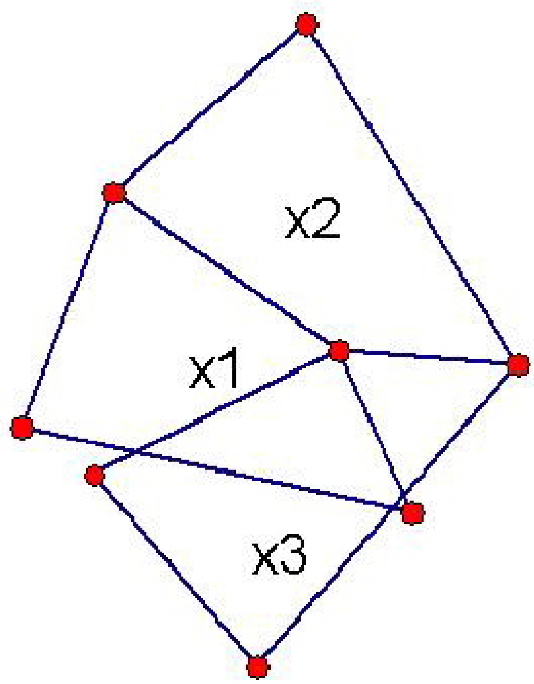

Figure 3: A polygon that does not tile the plane with reflections.

Theorem 9 *A polygon that tiles the plane with reflections must have angles in the form*

$$\alpha_i = \frac{2\pi}{k_i}$$

where α_i is any angle and $k_i \in \mathbb{Z}$ and $k_i \geq 3$.

Proof. In order for a polygon to tile the plane the plane must be completely covered with that polygon with no overlaps. Because of this if we take a vertex we see that all of the angles around the vertex must be congruent because they are created through reflections. The angles all add up to 2π because they complete a rotation around the vertex. Since all of the angles add up to 2π and all of the angles are equal, the angle must be of the form

$$\alpha_i = \frac{2\pi}{k_i}$$

where k_i is the number of angles around the vertex. See that k_i has to be greater than or equal to 3 because if it is not then the point that the angles are around would not be a vertex. ■

Theorem 10 *A polygon that tiles the plane with reflections has between three and six sides.*

Proof. It is obvious that a polygon can't exist if it has less than three sides, therefore a polygon that tiles the plan must have three or more sides. Using our previous theorem and lemma we find that

$$\sum_{i=1}^{s} \frac{2\pi}{k_i} = \sum_{i=1}^{s} a_i = (s-2)\pi = \left(\frac{s-2}{2}\right) 2\pi$$

which implies

$$\sum_{i=1}^{s} \frac{1}{k_i} = \frac{s-2}{2}.$$

Since $k_i \geq 3$ then

$$\sum_{i=1}^{s} \frac{1}{k_i} \leq \frac{s}{3}.$$

If $s > 6$ we find since

$$\frac{s}{3} = \frac{2s}{6}$$

and

$$\frac{s-2}{2} = \frac{3s-6}{6}$$

that

$$\frac{s}{3} = \frac{2s}{6} = \frac{3s-s}{6} < \frac{3s-6}{6} = \frac{s-2}{2}$$

Therefore if $s > 6$ then

$$\sum_{i=1}^{s} \frac{1}{k_i} \leq \frac{s}{3} < \frac{s-2}{2}$$

and so the equality

$$\sum_{i=1}^{s} \frac{1}{k_i} = \frac{s-2}{2}$$

won't hold. So a polygon that tiles the plane with reflections has between three and six sides. ■

Since we know then that the only polygons that tile the plane have rational angles and between three to six sides we can actually prove which specific polygons tile the plane.

Theorem 11 *The class of billiard tables that tile the plane with reflections is {the regular hexagon; the rectangle; any rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{6}$; any kite with angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$; any triangle an angle $\frac{2\pi}{3}$ and two angles of $\frac{\pi}{6}$; any triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle of $\frac{\pi}{6}$; any triangle with a right angle, and two angles of $\frac{\pi}{4}$; and any triangle with all angles $\frac{\pi}{3}$}*

Proof. Case 1: Six sides

In a hexagon without loss of generality we write our angles as

$$a_1 \geq a_2 \geq ... \geq a_6.$$

We know that the sum of all the angles is 4π so we can write

$$\sum_{i=1}^{6} a_i = 4\pi$$

which after dividing both sides by 2π we obtain

$$\sum_{i=1}^{6} \frac{1}{k_i} = 2.$$

because of how our angles are organized we have

$$\frac{1}{k_1} \geq \frac{1}{k_2} \geq ... \geq \frac{1}{k_6}$$

which means

$$k_1 \leq k_2 \leq ... \leq k_6.$$

We find then that if we take $k_1 = 4$ then at most we find

$$\sum_{i=1}^{6} \frac{1}{k_i} \leq \frac{6}{4} \neq 2.$$

This shows us that $k_1 = 3$ because if it were 4 or any number higher, then the sum will never reach the value 2. If $k_1 = 3$ then all the $k's$ must be 3 because

$$\sum_{i=1}^{6} \frac{1}{3} = 2.$$

and changing one of the $\frac{1}{3}'s$ to a value less would simply make the sum not 2. So we know that $k_i = 3$ for all the angles so all the angles are $\frac{2\pi}{3}$. So there is one possible hexagon that tiles the plane. It is obvious that the only the regular hexagon tiles the plane with reflections. This is because if any angle is $\frac{2\pi}{3}$ then the segments constructing that angle must be equivalent or else the polygon won't tile the plane with reflections. See figure 4.

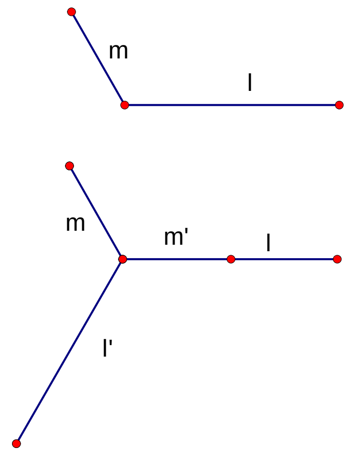

Figure 4: Sides l and m must be equivalent because if they are not then after a reflection across m then another across l' shows that m' and l will make give the interior of whatever polygon this set of segments belongs to be different and so the polygon will not tile the plane with reflections.

Case 2: Five sides

In a pentagon without loss of generality we write

$$a_1 \geq a_2 \geq ... \geq a_5.$$

The sum of a pentagons sides is given by

$$\sum_{i=5}^{5} a_i = \left(\frac{3}{2}\right) 2\pi.$$

We can divide both sides by 2π to obtain

$$\sum_{i=1}^{5} \frac{1}{k_i} = \frac{3}{2}.$$

Using the same concept from our hexagon if we take $k_1 = 4$ then at most we find

$$\sum_{i=1}^{5} \frac{1}{k_i} \leq \frac{5}{4} \neq \frac{3}{2}.$$

So $k_1 = 3$. If we take $k_2 = 4$ then at most we find

$$\frac{1}{3} + \sum_{i=2}^{5} \frac{1}{k_i} \leq \frac{4}{3} \neq \frac{3}{2}.$$

So $k_2 = 3$. If we take $k_3 = 4$ then at most we find

$$\frac{1}{3} + \frac{1}{3} + \sum_{i=3}^{5} \frac{1}{k_i} \leq \frac{17}{12} \neq \frac{3}{2}.$$

So $k_3 = 3$. If we take $k_4 = 5$,

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{k_i} \leq \frac{21}{15} \neq \frac{3}{2}.$$

So k_4 is either 4 or 3.

Case 2a: $k_4 = 4$

If $k_4 = 4$ then k_5 only has one solution. That is

$$\begin{aligned} \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{k_5} &= \frac{3}{2} \\ \frac{1}{k_5} &= \frac{1}{4}. \end{aligned}$$

So we have one possibility for a pentagon that tiles the plane so far. That pentagon is one with angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent. There are two possible combinations of angles.

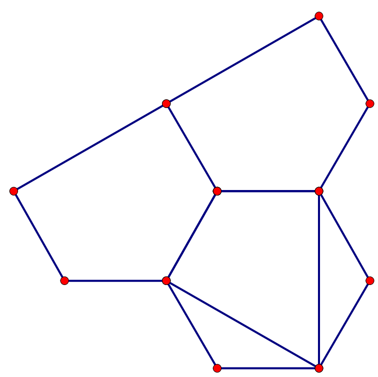

Figure 5: A pentagon with angles of the order $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent.

It is obvious then that a pentagon with angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$ in any order does not tile the plane with reflections because like in the case of the hexagon the polygons adjacent sides to an angle of $\frac{2\pi}{3}$ must be equivalent and the general pentagons of that type do not tile the plane with reflections.

Case 2b: $k_4 = 3$

Again we can see that there is only one possible solution for k_5 which is

$$\begin{aligned} \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{k_5} &= \frac{3}{2} \\ \frac{1}{k_5} &= \frac{1}{6}. \end{aligned}$$

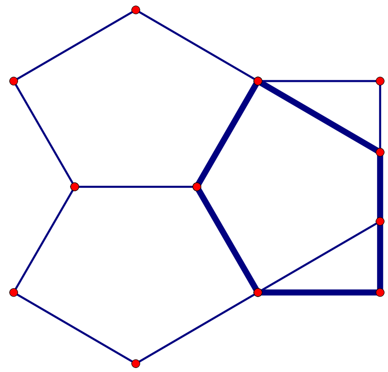

Figure 6: A pentagon with angles in the order $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent.

Another pentagon that can tile the plane with reflections is one with angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent. This must be checked graphically.

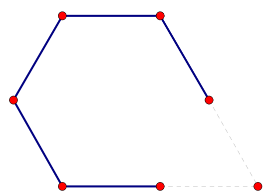

Figure 7: A pentagon with angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent cannot be a closed polygon.

It is obvious this pentagon does not tile the plane with reflections because it cannot be constructed such that it is closed.

Case 3: Four sides

In a quadrilateral without loss of generality we write

$$a_1 \geq a_2 \geq a_3 \geq a_4.$$

The sum of the angles in a quadrilateral is 2π so we write

$$\sum_{i=1}^{4} a_i = 2\pi.$$

Dividing both sides of the equality by 2π we obtain

$$\sum_{i=1}^{4} \frac{1}{k_i} = 1.$$

Since

$$k_1 \leq k_2 \leq k_3 \leq k_4.$$

If $k_1 = 5$ then

$$\sum_{i=1}^{4} \frac{1}{k_i} \geq \frac{4}{5} \neq 1.$$

So k_1 must be 4 or 3.

Case 3a: $k_1 = 4$

If $k_1 = 4$ then

$$\sum_{i=1}^{4} \frac{1}{k_i} = 1,$$

so all the other $k_i's$ must be 4 as well. This gives us a possible quadrilateral that tiles the plane with reflections, that is the rectangle. It is obvious that the rectangle tiles the plane with reflections.

Case 3b: $k_1 = 3$

If $k_2 = 5$ then

$$\sum_{i=1}^{4} \frac{1}{k_i} \geq \frac{14}{15} \neq 1$$

so k_2 must be either 3 or 4.

Case 3b(i): $k_2 = 4$

If $k_3 = 5$ then

$$\sum_{i=1}^{4} \frac{1}{k_i} \geq \frac{59}{60} \neq 1$$

so $k_3 = 4$. When $k_1 = 3$, $k_2 = 4$, and $k_3 = 4$ there is only one solution for k_4 and that is 6. So we have a second possibility for a quadrilateral that tiles the plane with reflection, that is a quadrilateral with angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent. We must check both orders of the angles graphically to see if it tiles the plane with reflections.

A kite is a quadrilateral with two sets of congruent adjacent sides. It is obvious that this polygon only tiles the plane.

Case 3b(ii): $k_2 = 3$ if $k_3 = 7$ then at most

$$\sum_{i=1}^{4} \frac{1}{k_i} = \frac{20}{21} \neq 1.$$

So when $k_1 = 3$ and $k_2 = 3$, k_3 is $6, 5, 4$, or 3.

If $k_3 = 6$ then $k_4 = 6$. So the quadrilateral with angles $\frac{2\pi}{3}$, $\frac{2\pi}{3}$, $\frac{\pi}{3}$, and $\frac{\pi}{3}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent is eligible to tile the plane with reflections. There are two orders of angles that have to be considered.

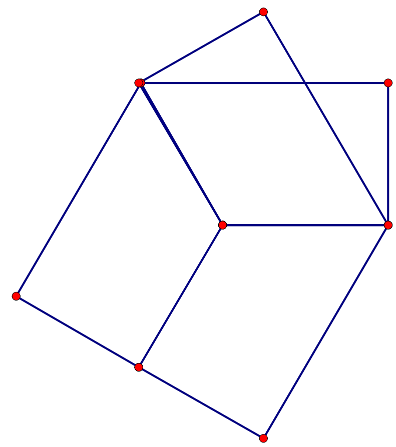

Figure 8: A quadrilateral with angles of the order $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent.

It is obvious that only the rhombus with these angles will tile the plane because it forms a hexagon which tiles the plane and so it must as well.

If $k_3 = 5$ then $\frac{1}{k_4} = \frac{2}{15}$ and since k_4 must be an integer the case where $k_3 = 5$ is not an eligible polygon.

If $k_3 = 4$ then $k_4 = 12$. We must check graphically if this polygon tiles the plane with reflections. There are two orders of angles this polygon can have.

It is obvious that this polygon does not tile the plane with reflections because it cannot be closed and have two sets of equivalent adjacent sides. The last case when $k_3 = 3$ is not eligible because there is no solution for k_4 that is a positive integer such that the sum of all the angles will not be greater than 2π.

Case 4: Three sides

The sum of angles in a triangle is π. Without loss of generality we take $a_1 \geq a_2 \geq a_3$ where a_i is an angle in the triangle. We can state

$$a_1 + a_2 + a_3 = \pi$$

which after dividing by 2π gives

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{2}.$$

We then take the reciprocal to find

$$k_1 \leq k_2 \leq k_3.$$

If we take $k_1 = 7$ then at most $\sum_{i=1}^{3} \frac{1}{k_i} = \frac{3}{7}$ which is less than $\frac{1}{2}$ so $k_1 \leq 6$.

Case I: $k_1 = 6$

Given the constraints

$$k_1 \leq k_2 \leq k_3$$

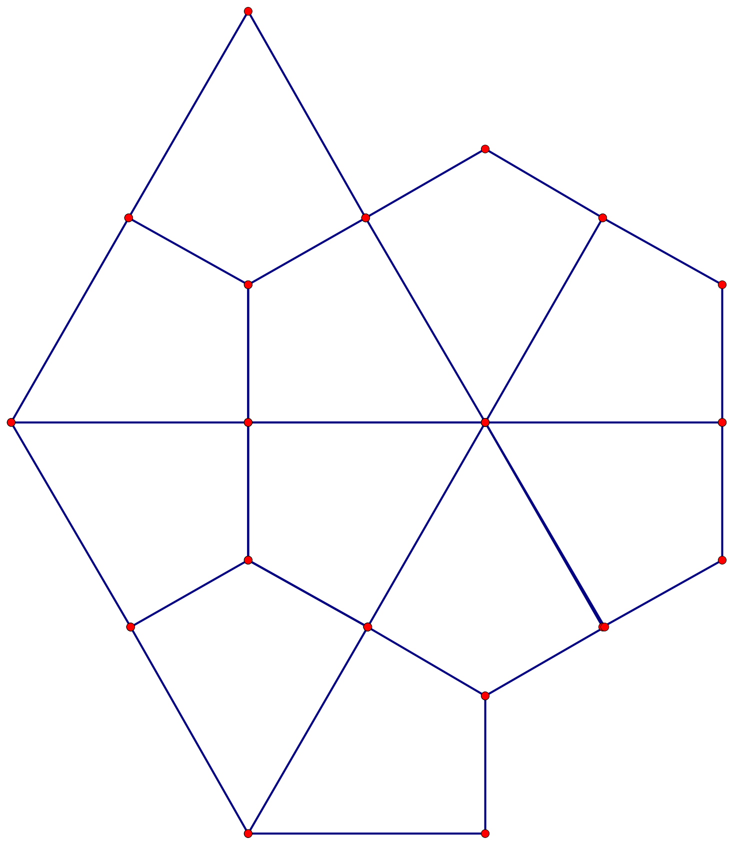

Figure 9: A kite with angles of the order $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}$

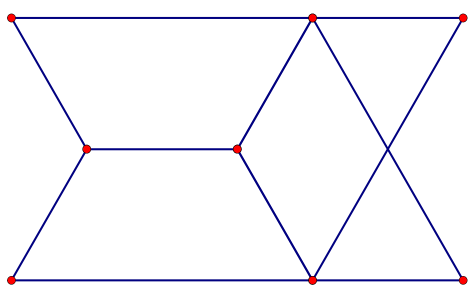

Figure 10: A quadrilateral with angles in the order $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent.

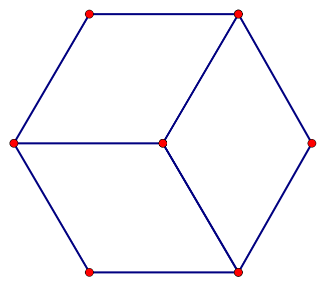

Figure 11: An rhombus with angles of the order $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent.

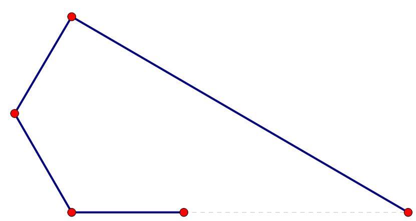

Figure 12: A closed quadrilateral cannot be constructed with the angles of the order $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{6}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent.

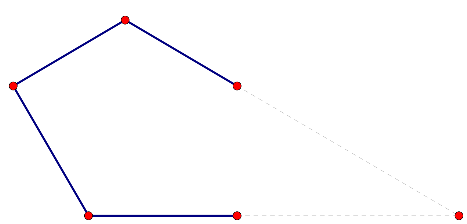

Figure 13: A closed quadrilateral with angles of the order $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{6}$ where any segments creating a angle of $\frac{2\pi}{3}$ are equivalent cannot be constructed.

and

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2}$$

can only be satisfied if both k_2 and $k_3 = 6$. This triangle is the equilateral triangle. It is obvious that it tiles the plane with reflections without overlapping.

Case II: $k_1 = 5$

Given the constraints

$$\begin{aligned} k_1 &\leq k_2 \leq k_3, \\ k_i &\in \mathbb{Z}, \end{aligned}$$

and

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2}$$

can only be satisfied when $k_2 = 5$ and $k_3 = 10$. If you take $k_2 > 5$ then the only possible values for the summation constraint for k_3 are not integers or leave the summation short of $\frac{1}{2}$. After sketching the reflections it is apparent that they overlap. This case yields no triangles that tile the plane with reflections.

Case III: $k_1 = 4$

Given the constraints

$$\begin{aligned} k_1 &\leq k_2 \leq k_3, \\ k_i &\in \mathbb{Z}, \end{aligned}$$

and

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2}.$$

The value for k_2 must be greater than 4 because if it is 4 then there is no possible value for k_3. With that in mind the possible values to satisfy the constraints are if $k_2 = 5$ and $k_3 = 20$, $k_2 = 6$ and $k_3 = 12$ and if both k_2 and $k_3 = 8$. No values for k_2 can be over 8 otherwise the summation will fall short of $\frac{1}{2}$. When k_2 is 7 the only possible value for k_3 is the non-integer $\frac{28}{3}$. When the triangle where $k_2 = 5$ is sketched with its reflections we find overlaps.

When the other two possible triangles are sketched with their reflections is apparent that they do not overlap. This gives us two more triangles that tile the plane with reflections.

Case IV: $k_1 = 3$

Given the constraints

$$\begin{aligned} k_1 &\leq k_2 \leq k_3, \\ k_i &\in \mathbb{Z}, \end{aligned}$$

and

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2},$$

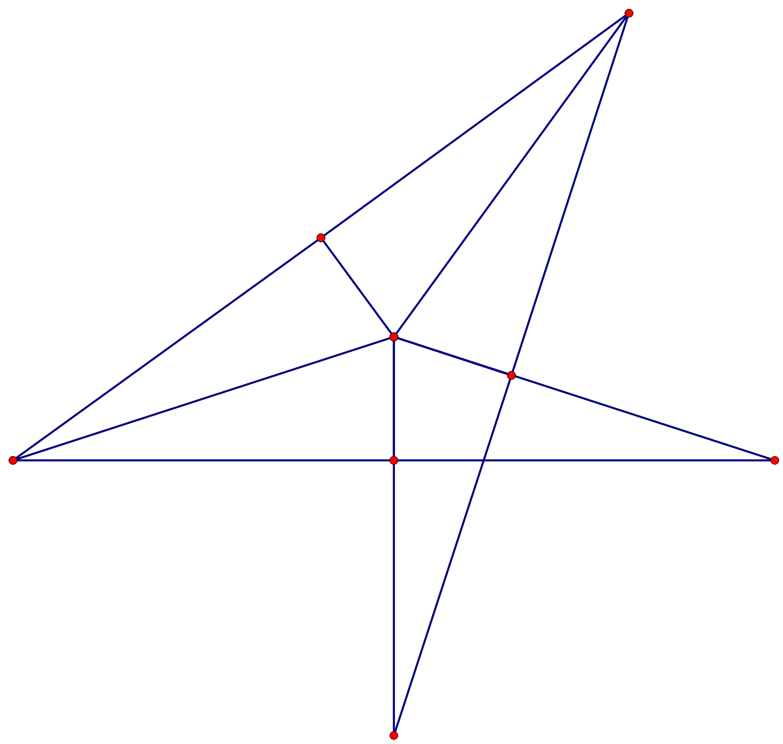

Figure 14: The triangle with angles $\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{10}$

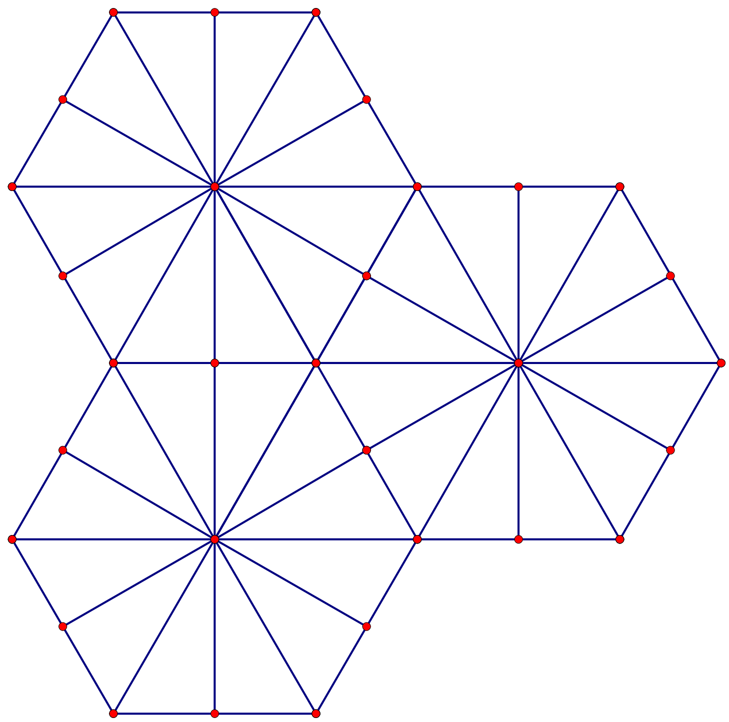

Figure 15: The triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$.

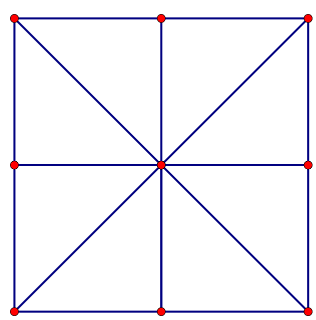

Figure 16: The triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$.

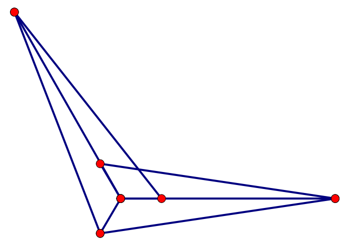

Figure 17: The triangle with angles $\frac{2\pi}{3}, \frac{2\pi}{7}, \frac{\pi}{21}$.

our k_2 must be greater than 6. This is because if it is 5, 4, or 3 then the summation goes over $\frac{1}{2}$, and if it is 6 then our k_3 has no possible value. If $k_2 = 7$ then our k_3 must be 42, and when we sketch this triangle with reflections we find it overlaps.

If $k_2 = 8$ then our $k_3 = 24$, again this triangle sketched with reflections overlaps.

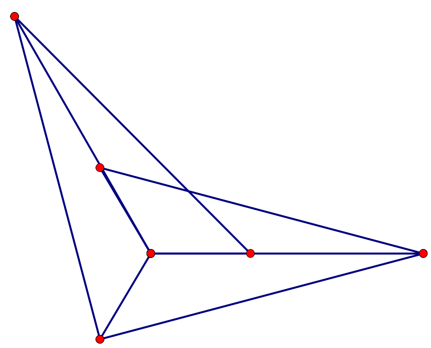

Figure 18: The triangle with angles $\frac{2\pi}{3}, \frac{\pi}{4}, \frac{\pi}{12}$.

If $k_2 = 9$ then our $k_3 = 18$, and once again when sketched with reflections this triangle will overlap.

If $k_2 = 10$ then $k_3 = 15$, but sketching this triangle with reflections we find overlaps.

If $k_2 = 11$ then the only solution for our summation constraint for k_3 would be the non-integer $\frac{66}{5}$. If $k_2 = 12$ then $k_3 = 12$. No values for k_2 can be greater than 12 because if they are the summation will fall short of $\frac{1}{2}$. After sketching the other possible triangle with reflections it is apparent that they do not overlap because they can be reflected in a way to form the hexagon, which does not overlap with reflections.

This gives us our last triangle to tile the plane with reflections.

So the class of billiard tables that tile the plane with reflections is {the regular hexagon; the rectangle; any rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{6}$; any kite with angles $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$, and $\frac{\pi}{3}$; any triangle an angle $\frac{2\pi}{3}$ and two angles of $\frac{\pi}{6}$; any triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle

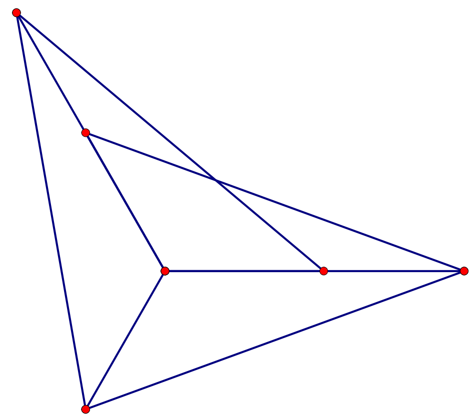

Figure 19: The triangle with angles $\frac{2\pi}{3}, \frac{2\pi}{9}, \frac{\pi}{9}$.

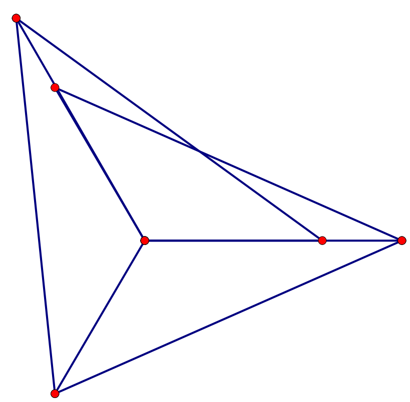

Figure 20: The triangle with angles $\frac{2\pi}{3}, \frac{\pi}{5}, \frac{2\pi}{15}$.

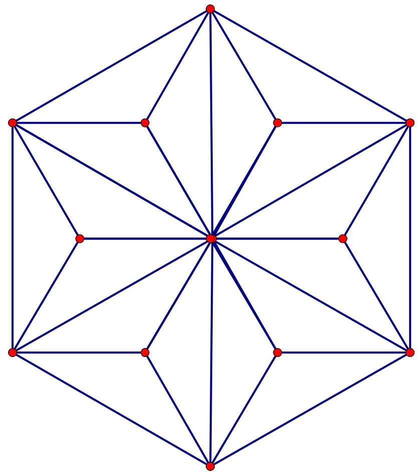

Figure 21: The triangle with angles $\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}$.

of $\frac{\pi}{6}$; any triangle with a right angle, and two angles of $\frac{\pi}{4}$; and any triangle with all angles $\frac{\pi}{3}$} ■

Now we have a class of billiard tables that are of much use to us when we use the method of unfoldings. We will prove things about this class of tables in later sections.

5 Fagano Orbit

If we drop an altitude from each vertex, the triangle made from the points where the altitudes intersect the sides of the triangle, we call the orbit that follows that triangle the Fagano orbit. An example of the Fagano orbit is triangle $\triangle DEF$ illustrated below.

We wish to show that the Fagano orbit is a periodic orbit. We use the extended law of sines to prove this.

Lemma 12 (The Extended Law of Sines) *In a triangle $\triangle ABC$ with sides a, b, and c opposite angles A, B and C respectively*

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R$$

where R is the radius of the circle about triangle $\triangle ABC$.

This is cited from [1, p. 2]

Theorem 13 *In an acute triangle there exists at least one non-perpendicular periodic orbit called the Fagano orbit.*

Proof. Let $\triangle ABC$ be an arbitrary acute triangle and drop the altitudes (as seen in Figure 22) from each vertex A, B, and C to points D, E, and F respectively. Note that the altitudes all pass

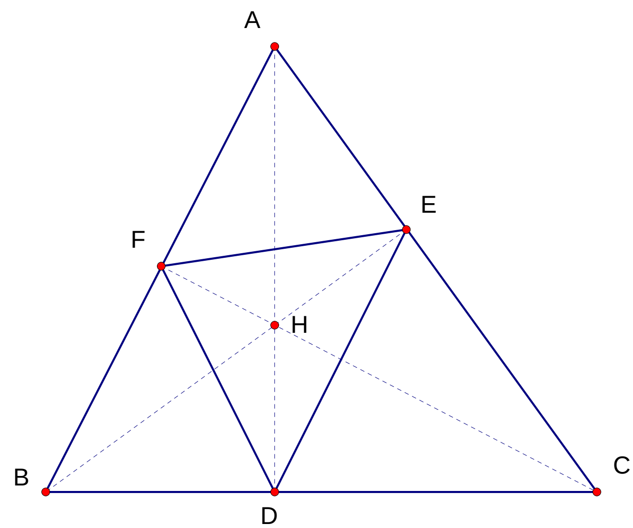

Figure 22: An example of a Fagano orbit.

through a common point called the orthocenter which we label H. Since the angles $\angle HFB$ and $\angle HEC$ are right, and since angles $\angle FHB$ and $\angle EHC$ are congruent vertical angles, then angles $\angle FBH$ and $\angle ECH$ are congruent.

Circumscribe a circle around the triangle $\triangle HDC$ we can use the extended law of sines to find the length of $\overline{HC}$ with respect to the circle about $\triangle HDC$. Where R_1 is the radius of the circle about $\triangle HDC$, then

$$\frac{\overline{HC}}{\sin(\angle HDC)} = 2R_1.$$

But since $\angle HDC$ is right, $\sin(\angle HDC) = 1$, so $\overline{HC} = 2R_1$. Using the same idea for a circle circumscribed around triangle $\triangle HEC$, knowing $\angle HEC$ is right and labeling R_2 as the radius of the circle about $\triangle HEC$, then

$$\frac{\overline{HC}}{\sin(\angle HEC)} = \overline{HC} = 2R_2.$$

Therefore we see $R_1 = R_2$. If we draw a circle around the point C and a circle around the point H both with radius R_1, we see that they will intersect at one point which we will call O. The point O is the center of both of the circles circumscribed around triangles $\triangle HDC$ and $\triangle HEC$. Since both of these circles have the same radius, they must be the same circle. Therefore we see if two distinct right triangles share a hypotenuse, the quadrilateral formed by the composition of the two triangles can be inscribed in a circle. If the quadrilateral $CEHD$ is inscribed in a circle then we can see that the angles $\angle ECH$ and $\angle HDE$ are congruent because they sweep out the same arclength in the circle.

Likewise we can see that since triangles $\triangle HFB$ and $\triangle HDB$ are right triangles that share a hypotenuse they can be inscribed in a circle and the angles $\angle FBH$ and $\angle HDF$ sweep out the same

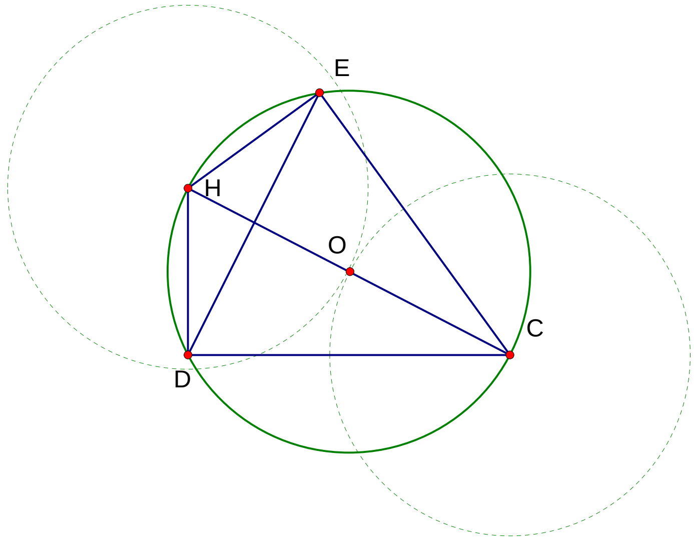

Figure 23: A diagram of quadrilateral $CEHD$ inscribed in a circle.

arclength in that circle so they are congruent.

Therefore we have

$$\angle HDF \tilde{=} \angle FBH \tilde{=} \angle ECH \tilde{=} \angle HDE.$$

Since angles $\angle HDC$ and $\angle HDB$ are both right and the angles $\angle HDF$ and $\angle HDE$ are congruent, we can say that their complements, $\angle FDB$ and $\angle EDC$ are congruent.

Analogously we find angles $\angle DEC$ and $\angle FAE$ are congruent, and angles $\angle DFB$ and $\angle AFE$ are congruent.

The triangle $\triangle DEF$ is a periodic orbit orbit because we have shown that its angles of incidence are congruent to their corresponding angle of reflection. No angle of incidence can be a right angle because triangle $\triangle ABC$ is acute and so the altitudes intersect a point between the other two vertices of triangle $\triangle ABC$ that is not a vertex. (Note: if an angle of incidence were a right angle the area of triangle $\triangle DEF$ would be zero) This means that the orbit is non-perpendicular. Taking any point on the boundary of the triangle $\triangle DEF$ and using it as the initial position for our orbit it is apparent that if we take our initial direction to be the same as the direction of the boundary of triangle $\triangle DEF$, the orbit will eventually end up in the same initial position, with the same initial direction. Therefore it is a periodic orbit. ■

The proof was introduced in reference [2] and the given proof was generalized from a proof in reference [1, p. 17]. This Fagano orbit is useful in proving things about polygons that can be unfolded into larger acute triangles. An example would be how we can break apart a hexagon into six acute triangles. Since there is a periodic orbit in the acute triangle and a hexagon can be represented as acute triangles, it must have a periodic orbit.

The logic behind the Fagano orbit fails when looking at a obtuse triangle because some altitudes are dropped outside of the triangle where the orbit cannot reach.

6 Periodic orbits

We can prove many things about periodic orbits, like if we have one periodic orbit, there are infinitely many.

Theorem 14 *Any periodic orbit in the class of polygons that tile the plane with reflections belongs to a strip (neighborhood) of periodic orbits.*

Proof. Expressing our periodic orbit through unfoldings, we find that there are a finite number of vertices in the corridor that is created by the reflections to point that is identified with the initial position. We take the distances between every segment perpendicular to the vertices and the line that represents our orbit. We label these distances $(x_1, x_2, x_3, ..., x_n)$ and take $M = \min(x_1, x_2, x_3, ..., x_n)$, where $M \in \mathbb{R}$. We see that $M \neq 0$ because if it did, the orbit would not be periodic because that would mean that the orbit would be passing through one of the vertices.

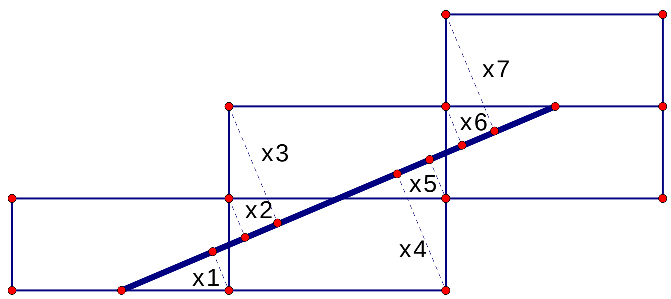

Figure 24: An example of distance aquired from verticies and our periodic orbit expression.

Taking $\varepsilon = \frac{M}{2}$ we see that we can translate our periodic orbit a distance of ε in the perpendicular direction of the orbit without intersecting another vertex anywhere in the expression. This gives us an ε-neighborhood of initial positions that using the same initial direction as our original orbit will give us a periodic orbit. If we take this neighborhood of initial positions and the periodic orbits they create using the same initial direction as the original periodic orbit, a strip of periodic orbits will be made. They are periodic because they all maintain their direction in the correct orientation and if the initial position is translated an amount in a direction then if an identified point is translated the same amount in the same direction, the translated points will be identified as well. This means there exists a strip of periodic orbits that any periodic orbit is a part of.

■

Corollary 15 *There are infinitely many periodic orbits in a billiard table given the fact that it has one periodic orbit.*

This is because there are infinitely many periodic orbits in a billiard table given the fact that there is one periodic orbit because it belongs to a strip of periodic orbits. Because there are infinitely many initial positions in that ε-neighborhood (because there are infinitely many numbers between any two real numbers) that will give a periodic orbit.

We can also take a look at perpendicular periodic orbits.

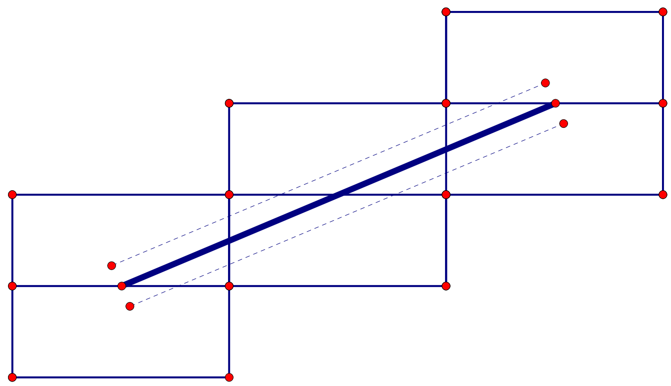

Figure 25: Example of the epsilon neighborhood obtained from the previous example.

Theorem 16 *In the class of billiard tables that tile the plane with reflections, {the regular hexagon; the rectangle; any rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{3}$; any kite with angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$; any triangle an angle $\frac{2\pi}{3}$ and two angles of $\frac{\pi}{6}$; any triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle of $\frac{\pi}{6}$; any triangle with a right angle, and two angles of $\frac{\pi}{4}$; and any triangle with all angles $\frac{\pi}{3}$}, every table admits a perpendicular periodic orbit.*

Proof. Given any regular hexagon, or any rectangle it is obvious that a perpendicular periodic orbit can be obtained by simply creating an orbit that is perpendicular to any two edges of the polygon that are parallel where the orbit does not strike a vertex.

A rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{3}$ can be expressed as a regular hexagon with unfoldings. Since we have a hexagon we see that a perpendicular periodic orbit can be obtained by creating an orbit that is perpendicular to any two parallel edges of the regular hexagon expression of the rhombus, where the orbit does not pass through any vertices.

In the triangle with angle $\frac{2\pi}{3}$ and two other angles of $\frac{\pi}{6}$, if we take any orbit which does not start at a vertex, with the angle of incidence on the longest edge equal to $\frac{\pi}{3}$ we see the the angle of reflection will also be $\frac{\pi}{3}$ and since both the acute angles in this obtuse triangle are $\frac{\pi}{6}$ when the orbit strikes the shorter sides they must be right angles because the degree measure of a triangle is π. This orbit is periodic because it will end up in the same initial position with the same initial direction because it has a perpendicular angle of incidence and will not strike a vertex. So it is a perpendicular periodic orbit.

The kite with angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$ can be expressed as a hexagon, and since a hexagon has a perpendicular periodic orbit so must the kite.

A triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle of $\frac{\pi}{6}$ can be expressed as a rhombus through unfoldings. A rhombus has a perpendicular periodic orbit and then so must the right triangle that is expressed as a rhombus.

A triangle with a right angle, and two angles of $\frac{\pi}{4}$ can be expressed as a square through unfoldings. Since a square is a rectangle and a rectangle has a perpendicular periodic orbit, so must the right triangle expressed as a square.

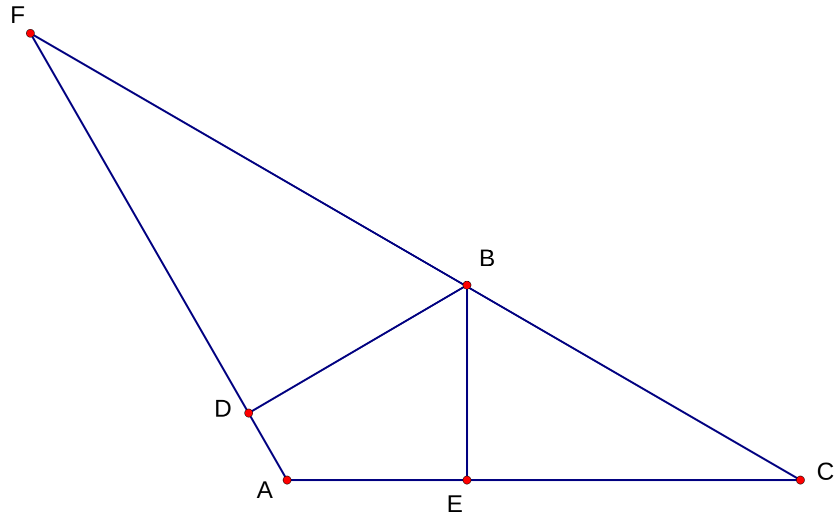

Figure 26: If $\angle C$ and $\angle F$ are $\frac{\pi}{6}$, and angles $\angle CBE$ and $\angle FBD$ are $\frac{\pi}{3}$ then angles $\angle BEC$ and $\angle BDF$ must be right because they belong to triangles $\triangle BEC$ and $\triangle BDF$ respectively.

Finally, a triangle with all angles $\frac{\pi}{3}$ is equilateral, and using unfoldings we can express it as a hexagon. Since the hexagon has a perpendicular periodic orbit, so must the equilateral triangle. ■

Using the same class of billiard tables we can prove that all of those tables have a non-perpendicular periodic orbit.

Theorem 17 *In the class of billiard tables that tile the plane with reflections, {the regular hexagon; the rectangle; any rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{3}$; any kite with angles $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$, and $\frac{\pi}{3}$; any triangle an angle $\frac{2\pi}{3}$ and two angles of $\frac{\pi}{6}$; any triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle of $\frac{\pi}{6}$; any triangle with a right angle, and two angles of $\frac{\pi}{4}$; and any triangle with all angles $\frac{\pi}{3}$}, every table admits a periodic orbit.*

Proof. Without loss of generality we want to scale the regular hexagon so that its apothem is $\frac{1}{2}$, the rectangle so that its height is 1 and its length is g, the rhombus so that its height is 1 and its length is $\sqrt{3}$, the kite whose longest side is 2, the obtuse triangle so that its longest edge is $\frac{\sqrt{3}}{2}$, the $\frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}$ triangle so that its hypotenuse is 2, the isosceles right triangle so that its shorter sides equal 1, the our equilateral triangle so that its altitude equals 1.

Take e_1 to be a translation of $(0, 2)$ in the coordinate plane, $e_2 := (\sqrt{3}, 1)$, $e_3 := (0, 3)$, $e_4 :$ $(2\sqrt{3}, 0)$, $e_5 := (0, 2\sqrt{3})$, $e_6 := (1 + \sqrt{3}, \sqrt{3})$, $e_7 := (2, 0)$, and $e_8 := (0, 2g), e_9 := (8, 0), e_{10} :=$ $(0, 2\sqrt{3} + 2)$. Note that these translations are specifically taken in relation to our generalized tables to find points in the unfolded tables that identify with the origin. It is illustrated below how some of these distances for the translations were found. In the images the polygons that are grey have points that are identified with the original table that contains the origin.

In the hexagon all the points on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_2$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0, 0)$ such that

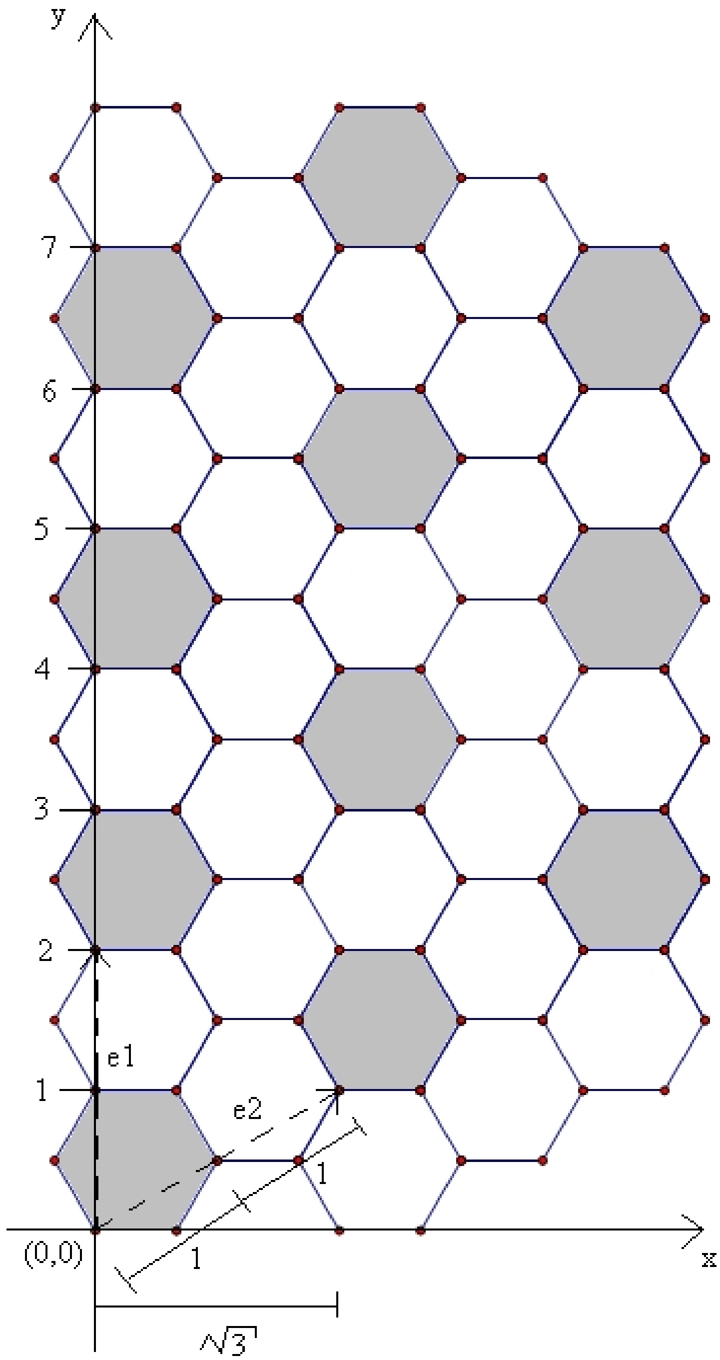

Figure 27: It is obvious how $e_1's$ dimensions are obtained, $e_2's$ dimensions are obtained using the pythagorean theorem.

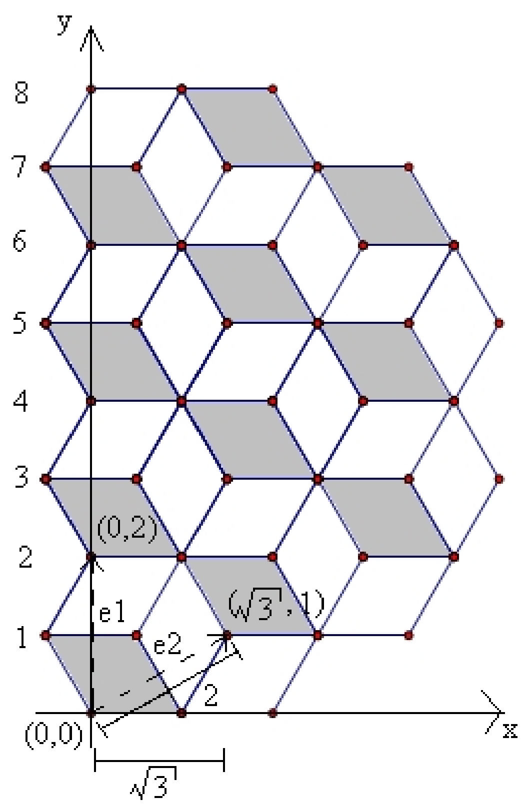

Figure 28: The dimensions are found analogously to how they are found in the hexagon.

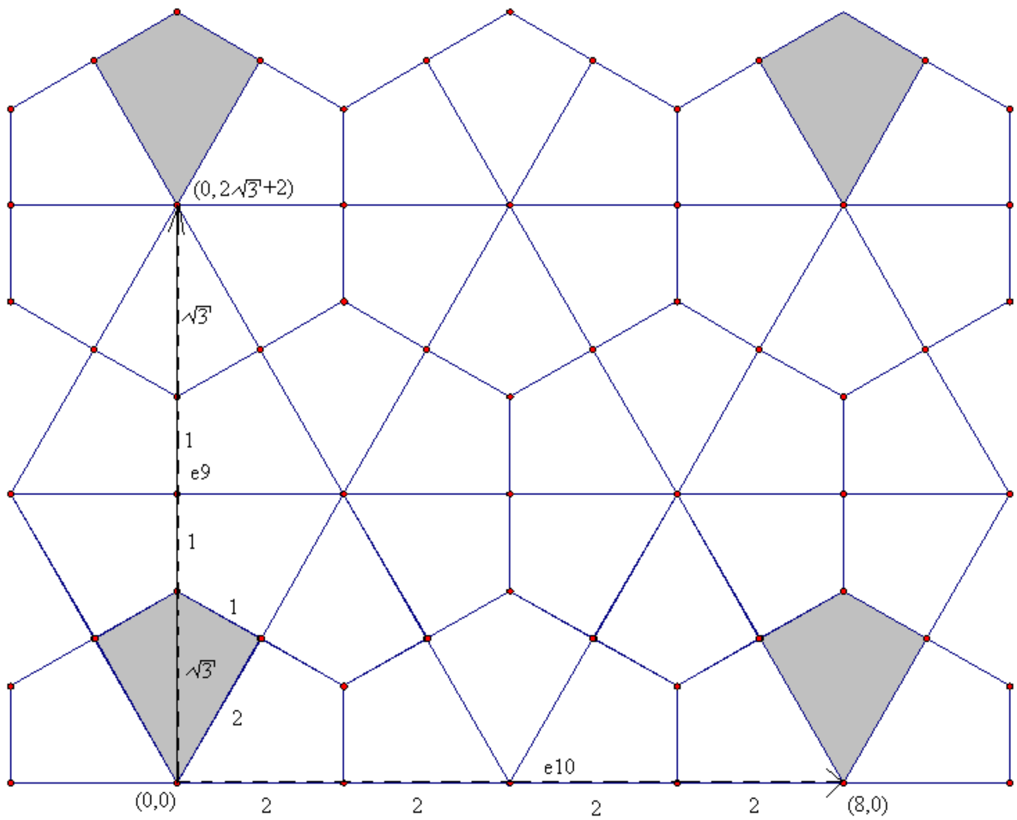

Figure 29: Since the kite can be broken into two right triangles, the longer diagonal can be found using the pythagorean theorem. Both $e_9's$ and $e_{10}'s$ dimensions can be found by just adding the distances between the identified points.

when we give the orbit the slope $\frac{2m+n}{n\sqrt{3}}$ because the orbit extends to $(n\sqrt{3}, 2m+n)$ which is on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_2$. Since the point $(n\sqrt{3}, 2m+n)$ is on the lattice that is points identified with the origin, if the billiard table containing point $(n\sqrt{3}, 2m+n)$ is the same orientation as the original table then the orbit is periodic. If it isn't the same orientation, continue the orbit until it until it reaches the point $(2(n\sqrt{3}), 2(2m+n))$ which is also on the lattice of identified points. The orientation has to be different than the last table because if it isn't it would contradict the fact that the last table wasn't the same as the original. If the table containing point $(2(n\sqrt{3}), 2(2m+n))$ is again not the same orienting as the original table then continue the orbit to point $(3(n\sqrt{3}), 3(2m+n))$. Continue this process until a table with the same orientation as the original is found. There if a finite amount of orientations (six in the hexagon) that are possible in a polygon so the correct orientation has to be found eventually. Take $(d(n\sqrt{3}), d(2m+n))$ to be the point that is identified with the origin and that table containing $(d(n\sqrt{3}), d(2m+n))$ has the same orientation as the original table. If the orbit strikes a vertex we shift the initial position ε over where $0 < \varepsilon <$ min(all of the distances from the orbit to all the vertices created through the path of unfoldings that are not zero) so that the orbit will not strike a vertex. We still label this new initial position $(0,0)$ and give it the same slope of $\frac{2m+n}{n\sqrt{3}}$. The orbit returns to the same initial position because the points $(0,0)$ and $(d(n\sqrt{3}), d(2m+n))$ are identified and the orbit has the same initial direction because it has the same slope in tables with the same orientation. Therefore the orbit is periodic.

The rest are done analogously except they have different lattices of identified points and different slopes are needed to reach a point on those lattices.

In the rectangle all the points on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_8$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0,0)$ such that when we give the orbit the slope $\frac{m}{ng}$ the orbit extends to $(2m, 2ng)$ which is on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_8$. Similarly to the hexagon case then there exists a periodic orbit in the rectangle.

In the case of the rhombus all the points on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_2$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0,0)$ such that when we give the orbit the slope $\frac{2m+n}{n\sqrt{3}}$ the orbit extends to $(n\sqrt{3}, 2m+n)$ which is on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_2$. Similarly to the hexagon case then there exists a periodic orbit in the rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{3}$.

In the case of the kite with angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$ all the points on the lattice $\mathbb{Z}e_9$ x $\mathbb{Z}e10$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0,0)$ such that when we give the orbit the slope $\frac{(2\sqrt{3}+2)m}{8n}$ the orbit extends to $(8n, (2\sqrt{3}+2)m)$ which is on the lattice $\mathbb{Z}e_9$ x $\mathbb{Z}e10$. Similarly to the hexagon case then there exists a periodic orbit in the kite with angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{\pi}{3}$.

In the case of the obtuse triangle all the points on the lattice $\mathbb{Z}e_3$ x $\mathbb{Z}e_4$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0,0)$ such that when we give the orbit the slope $\frac{3m}{2n\sqrt{3}}$ the orbit extends to $(2n\sqrt{3}, 3m)$ which is on the lattice on the lattice $\mathbb{Z}e_3$ x $\mathbb{Z}e_4$. Similarly to the hexagon case then there exists a periodic orbit in the triangle with an angle $\frac{2\pi}{3}$ and two angles of $\frac{\pi}{6}$.

In the case of the triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle of $\frac{\pi}{6}$ all the points on the lattice $\mathbb{Z}e_5$ x $\mathbb{Z}e_6$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0,0)$ such that when we give the orbit the slope $\frac{m2\sqrt{3}+n\sqrt{3}}{n(1+\sqrt{3})}$ the orbit extends to $(n(1+\sqrt{3}), m2\sqrt{3}+n\sqrt{3})$ which is on the lattice $\mathbb{Z}e_5$ x $\mathbb{Z}e_6$.Similarly to the hexagon case then there exists a periodic orbit in the triangle with one right angle, one angle of $\frac{\pi}{3}$, and one angle of $\frac{\pi}{6}$.

In the case of the isosceles right triangle all the points on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_7$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0, 0)$ such that when we give the orbit the slope $\frac{m}{n}$ the orbit extends to $(2n, 2m)$ which is on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_7$. Similarly to the hexagon case then there exists a periodic orbit in the triangle with one right angle and two angles of $\frac{\pi}{4}$.

In the case of our equilateral triangle all the points on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_2$ are identified with the origin. Given any $m, n \in \mathbb{Z}$ there exists an initial position of an orbit not on a vertex which we label $(0, 0)$ such that when we give the orbit the slope $\frac{2m+n}{n\sqrt{3}}$ the orbit extends to $(n\sqrt{3}, 2m + n)$ which is on the lattice $\mathbb{Z}e_1$ x $\mathbb{Z}e_2$.Similarly to the hexagon case then there exists a periodic orbit in the equilateral triangle. ■

Corollary 18 *Given any rectangle whose ratio of dimensions is rational there exists an initial position for an orbit such that given that the orbit has a rational slope, then the orbit is periodic.*

Proof. Therefore all the billiard tables that are polygons that tile the plane with reflections admit a periodic orbit. ■

There is also a strong statement we can make about all right triangle billiard tables.

Theorem 19 *Given a billiard table that is any arbitrary right triangle, there exists a perpendicular periodic orbit.*

Proof. Take triangle $\triangle ABC$ to be an arbitrary right triangle with the right angle at angle $\angle ABC$. Using the previous theorem when $\angle BAC = \angle ACB$ there exists a perpendicular periodic orbit. Without loss of generality take angle $\angle BAC < \angle ACB$ to examine all the other cases. If we use an unfolding across segment $\overline{AB}$ to create a new triangle $\triangle ABC'$ then the larger triangle $\triangle ACC'$ is acute because since $\angle BAC < \angle ACB$ then $\angle BAC < \frac{\pi}{4}$ so the larger angle of $\angle CAC' = 2\angle BAC < \frac{\pi}{2}$ and both angles $\angle ACC' \cong \angle AC'C < \frac{\pi}{2}$. Considering the large triangle $\triangle ACC'$ is acute it has a periodic Fagano orbit with initial position at B. We label the other points where the Fagano orbit strikes the boundaries D and E.

If we shift our initial position over ε we obtain another orbit. We label this new initial positions F and the subsequent points where the orbit strikes the boundaries of the large triangle to be G, H, I, J, and K.

Since this new orbit has the same initial direction $\overline{BD}||\overline{FG}$ and so $\angle BDC \tilde{=} \angle FGC$. Since we are looking at orbits $\angle ADE \tilde{=} \angle BDC \tilde{=} \angle FGC \tilde{=} \angle AGH$ so then $\overline{DE}||\overline{GH}$. Repeating this method for each set of angles shows

$$\begin{aligned} \angle ADE &\tilde{=} \angle BDC \tilde{=} \angle FGC \tilde{=} \angle AGH \tilde{=} \angle AJK \tilde{=} \angle CJI, \\ \angle AKJ &\tilde{=} \angle AED \tilde{=} \angle AHG \tilde{=} \angle C'HI \tilde{=} \angle C'EB \tilde{=} \angle C'KL, \\ \text{and } \angle C'IH &\tilde{=} \angle C'BE \tilde{=} \angle C'LK \tilde{=} \angle CFG \tilde{=} \angle CBD \tilde{=} \angle CIJ. \end{aligned}$$

We know that triangle $\triangle AC'C$ is isosceles so $\angle AC'C \tilde{=} \angle ACC'$ and since angle $\angle CFB \tilde{=} \angle C'IH$ angles $\angle C'HI \tilde{=} \angle CGF$ because the sum of the angles in a triangle is the same for every triangle. So

$$\angle ADE \tilde{=} \angle BDC \tilde{=} \angle FGC \tilde{=} \angle AGH \tilde{=} \angle AJK \tilde{=} \angle CJI \tilde{=} \angle AKJ \tilde{=} \angle AED \tilde{=} \angle AHG \tilde{=} \angle C'HI \tilde{=} \angle C'EB \tilde{=} \angle C'KL$$

which means triangles $\triangle AJK$, $\triangle ADE$, and $\triangle AGH$ are all isosceles and similar to triangle $\triangle ACC'$. This also means $\overline{DE}||\overline{GH}||\overline{KJ}||\overline{CC'}$.

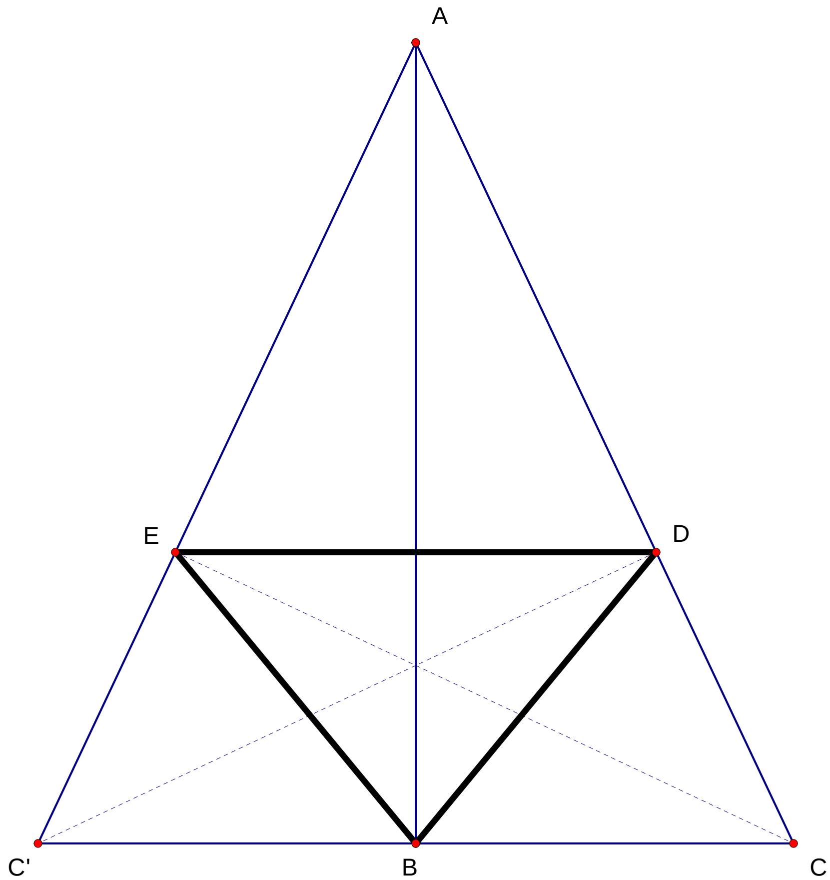

Figure 30: A Fagano orbit of an acute triangle formed from the reflection of an arbitrary right triangle.

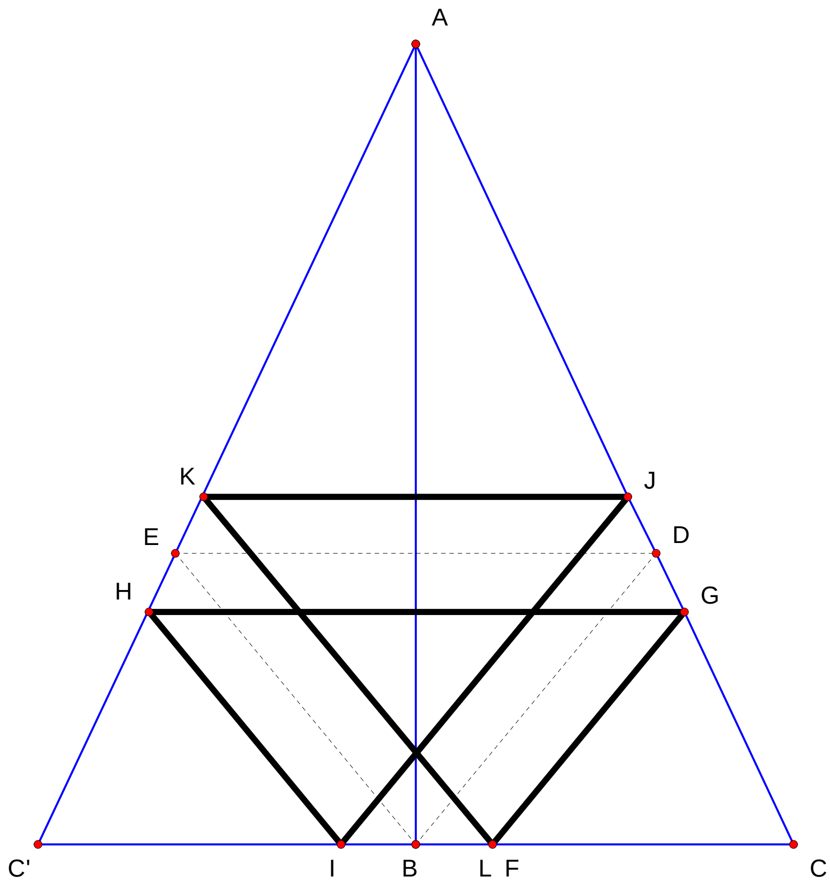

Figure 31: A Fagano orbit shifted over ε where $\varepsilon = \overline{BF}$.

Since the bases of the isosceles triangles $\triangle AGH, \triangle ACC'$, and $\triangle AKJ$ are parallel and the triangles share angle $\angle C'AC$ then the segments $\overline{GC} \tilde{=} \overline{HC'}$ and $\overline{KH} \tilde{=} \overline{JG}$. So triangles $\triangle FGC \tilde{=} \triangle IHC'$ because they have a congruent side adjacent to two congruent angles.

Since $\overline{BC'}$ is a reflected copy of $\overline{BC}$ then $\overline{BC} \tilde{=} \overline{BC'}$. Since $\overline{BC} = \overline{FC} + \varepsilon$, $\overline{BC'} = \overline{C'I} + \overline{IB}$ and $\overline{C'I} \tilde{=} \overline{FC}$ because corresponding parts of a congruent triangle are congruent then $\overline{IB} = \varepsilon$.

Since triangles $\triangle FGC$ and $\triangle IJC$ are similar and $\overline{IB} = \varepsilon$ the ratio of their sides gives

$$\begin{aligned} \frac{\overline{FC} + 2\varepsilon}{\overline{FC}} &= \frac{\overline{CG} + \overline{JG}}{\overline{CG}} \\ \frac{\overline{CG}(2\varepsilon)}{\overline{FC}} &= \overline{JG}. \end{aligned}$$

Since $\overline{IB} = \varepsilon, \overline{KH} \tilde{=} \overline{JG}, \overline{GC} \tilde{=} \overline{HC'}$, and $\overline{FC} \tilde{=} \overline{C'I}$ then viewing the ratio of sides of triangles $\triangle C'HI$ and $\triangle C'Kl$ gives

$$\begin{aligned} \frac{\overline{KH} + \overline{HC'}}{\overline{HC'}} &= \frac{\overline{C'I} + \varepsilon + \overline{BL}}{\overline{C'I}} \\ \frac{\overline{KH}}{\overline{HC'}} &= \frac{\varepsilon + BL}{C'I} \\ \frac{\overline{JG}}{\overline{GC}} &= \frac{\varepsilon + \overline{BL}}{\overline{FC}} \\ \frac{\frac{\overline{CG}(2\varepsilon)}{\overline{FC}}}{\overline{GC}} &= \frac{\varepsilon + \overline{BL}}{\overline{FC}} \\ \varepsilon &= \overline{BL}. \end{aligned}$$

So if $\varepsilon = \overline{BL}$ and $\varepsilon = \overline{BF}$ then $\overline{BL} \tilde{=} \overline{BF}$ and so L is F. That means the orbit returns to the initial point and since $\angle C'LK \tilde{=} \angle CFG$ it will have the same initial direction. That means that this is a periodic orbit. It is perpendicular because since $\overline{GH} || \overline{KJ} || \overline{CC'}$ and $\overline{AB}$ is perpendicular to $\overline{CC'}$ then it is perpendicular to $\overline{GH}$ and $\overline{KJ}$ as well. ■

7 Conclusion and Open Questions

In this paper we have defined polygonal billiards, explained the method of unfoldings and how it is useful, shown which polygons tile the plane with reflections, and explored different kinds of periodic orbits.

An open question in the field of billiards is *does every obtuse triangle have a periodic orbit?* Another question is *do all tables admit a periodic orbit*?

8 References

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