AN EXPLANATION OF PERIOD THREE IMPLIES CHAOS

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1. Introduction

In this project we will prove the main theorem in "Period Three Implies Chaos" by Tien-Yien Li and James A. Yorke. The paper describes an example of a discrete dynamical system where a sequence $\{x_n\}$ can be described by iterating a continuous function F such that $x_{n+1} = F(x_n)$ for all $n \in \mathbb{N}$. This sequence can be used to describe many different real world problems from trends in insect population to certain types of fluid flow. Li and Yorke use real analysis to describe the exact behavior of certain functions without approximating anything. What I found after reading their paper is that in proving the theorem there are many times when they skip over steps in their logic and ask the reader to figure out the missing steps on their own. What I will to do is prove, using Real analysis (Math 450), the theorem below but also expand the areas that Li and Yorke have skipped over and put more detail in them. I will also provide examples to help explain this theroem's applications. First we will state necessary definitions and the main theorem we will be proving. Next we will prove the main theorem. Then we will talk about the application of this theorem. Finally we will give the conclusion.

2. Main Theorem

First we must define a few terms that will be used to prove the theorem. Let J be an interval and let $F : J \to J$. For $x \in J$, let $F^0(x)$ denote x and $F^{n+1}(x)$ denote $F(F^n(x))$ for $n \in \mathbb{N}$.

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Definition 1. [LY75]Let $n \in \mathbb{N}$. We say that p is a **periodic point of period** n if $p \in J$ and $p = F^n(p)$ and $p \neq F^k(p)$ for $1 \leq k < n$.

Definition 2. [LY75]We say p is **periodic** or is a **periodic point** if p is periodic for some $n \geq 1$.

Definition 3. [LY75]We say q is **eventually periodic** if for some positive integer m, $p = F^m(q)$ is periodic.

Definition 4. [Rud64]Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers $x \in \mathbb{R} \cup \{\pm\infty\}$ such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. Let

$$\begin{aligned} s^* &= \sup E, \\ s_* &= \inf E. \end{aligned}$$

The numbers s^* and s_* are called the upper and lower limits of $\{s_n\}$ respectively; we use the notation

$$\begin{aligned} \limsup_{n\to\infty} s_n &= s^*, \\ \liminf_{n\to\infty} s_n &= s_*. \end{aligned}$$

Notice that for a set $\{x_n\}$ if for all $\varepsilon > 0$ there exist infinitely many x_n with $0 \leq x_n < \varepsilon$, then $\lim\inf(x_n) = 0$.

Theorem 5. [BS11]*(Heine-Borel) A subset K of $\mathbb{R}$ is compact if and only if K is closed and bounded.*

With this we can state the theorem that we will be proving.

Theorem 6. [LY75]*Let J be an interval and let $F : J \to J$ be continuous. Assume there are points $a, b, c, d \in J$ for which $F(a) = b$, $F^2(a) = c$, and $F^3(a) = d$ and either (Case 1)*

$$d \leq a < b < c$$

or (Case 2)

$$d \geq a > b > c.$$

Then

T1: *for every $k \in \mathbb{N}$ there exists a periodic point in J with period k.*

Furthermore,

T2: *there exists an uncountable set S in J (containing no periodic points) which satisfies the following conditions:*

(A) For every $p, q \in S$ with $p \neq q$

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0 \tag{2.1}$$

and

$$\liminf_{n \to \infty} |F^n(p) - F^n(q)| = 0. \tag{2.2}$$

(B) For every $p \in S$ and periodic point $q \in J$

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0. \tag{2.3}$$

Remark. Note that if $d = a$ then $F^3(a) = a$. Thus a is a periodic point of period three. So the hypothesis is satisfied by the existence of a periodic point of period three.

3. Proof of T1

To prove T1 we will first need to give the necessary lemmas.

Lemma 7. *Let $G : I \to \mathbb{R}$ be continuous, where I is an interval. For any compact interval $I_1 \subseteq G(I)$ there is a compact interval $Q \subseteq I$ such that $G(Q) = I_1$.*

Proof. Let $I_1 \subseteq G(I_1)$ and I_1 is compact. Then $\exists p, q \in I$ such that $I_1 = [G(p), G(q)]$. If $p < q$, let

$$r = \max\{x \in [p, q] | G(x) = G(p)\}$$

and let

$$s = \min\{x \in [r,q] | G(x) = G(q)\}.$$

Then by the intermediate value theorem $G[r,s] = I_1$. If $p > q$, let

$$r = \max\{x \in [q,p] | G(x) = G(q)\}$$

and let

$$s = \min\{x \in [r,p] | G(x) = G(p)\}.$$

Then by the intermediate value theorem $G[r,s] = I_1$. Therefore there is a compact interval $Q \subseteq I$ such that $G(Q) = I_1$. □

It is important to note that since G is not necessarily one to one then $I_1 \subseteq G[p,q]$ and perhaps not $G[p,q] \subseteq I_1$.

Lemma 8. *Let $F : J \to J$ be continuous and let $\{I_n\}_{n=0}^{\infty}$ be a sequence of compact intervals with $I_n \subseteq J$ and $I_{n+1} \subseteq F(I_n)$ for all n. Then there is a sequence of compact intervals Q_n such that $Q_{n+1} \subseteq Q_n \subseteq I_0$ and $F^n(Q_n) = I_n$ for $n \in \mathbb{N}$. Define $Q = \bigcap_{n=0}^{\infty} Q_n$ then for any $x \in Q$, we have $F^n(x) \in F^n(Q_n) = I_n$ for all n.*

Proof. Define Q_0 such that $F^0(Q_0) = I_0$, $Q_0 = I_0$. Now by lemma 7, $\exists Q_1 \subseteq Q_0$ such that $F(Q_1) = I_1 \subseteq F(I_0)$. Now suppose $\exists Q_n \subseteq Q_{n-1}$ such that $F^n(Q_n) = I_n \subseteq F(I_{n-1})$. Then, by lemma 7 $\exists Q_{n+1} \subseteq Q_n$ such that $F^{n+1}(Q_{n+1}) = I_{n+1} \subseteq F(I_n)$. This completes the induction.

Suppose $x \in Q = \bigcap_{n=0}^{\infty} Q_n$. This implies that $F^0(x) = x \in Q_0 = I_0$. Suppose $F^n(x) \in I_n$. Then $x \in Q \Rightarrow x \in Q_{n+1}$. Since $x \in Q_{n+1}$, $F^{n+1}(x) \in F^{n+1}(Q_{n+1}) = I_{n+1}$. Therefore $F^n(x) \in I_n$ for all $n \in \mathbb{N}$. □

Lemma 9. *Let $G : J \to \mathbb{R}$ be continuous. Let $I \subseteq J$ be a compact interval. Assume $I \subseteq G(I)$. Then there is a point $p \in I$ such that $G(p) = p$.*

Proof. Let $I = [\beta_0, \beta_1]$, since $I \subseteq G(I)$ we can choose $\alpha_0, \alpha_1 \in I$ such that $G(\alpha_0) = \beta_0$ and $G(\alpha_1) = \beta_1$. Then $\alpha_0, \alpha_1 \in G(I)$. Then $\alpha_0 - G(\alpha_0) = \alpha_0 - \beta_0 \geq 0$.

$\alpha_1 - G(\alpha_1) = \alpha_1 - \beta_1 \leq 0$. Thus the intermediate value theorem implies that there exists a $\beta \in I$ such that $\beta - G(\beta) = 0$. □

Proof of T1. Case 1: Assume $d \leq a < b < c$, write $K = [a, b]$ and $L = [b, c]$. Since $F(a) = b$, $F^2(a) = c$, and $F^3(a) = d$, the intermediate value theorem implies that $F(K) = F[a, b] \supseteq [F(a), F(b)] = [b, c] = L$ and $F(L) = F[b, c] \supseteq [F(c), F(b)] = [d, c]$. Since $L \subseteq [d, c]$ and $K \subseteq [d, c]$ then $L \subseteq F(L)$ and $K \subseteq F(L)$. Case2: Assume $d \geq a > b > c$, define $K = [b, a]$ and $L = [c, b]$. Again, since $F(a) = b$, $F^2(a) = c$, and $F^3(a) = d$ the intermediate value theorem implies that $F(K) = F[b, a] \supseteq [F(b), F(a)] = [c, b] = L$ and $F(L) = F[c, b] \supseteq [F(b), F(c)] = [c, d]$. Since $L \subseteq [c, d]$ and $K \subseteq [c, d]$ then $L \subseteq F(L)$ and $K \subseteq F(L)$. So for both case 1 and case 2 we know that

$$
\begin{aligned}
K &\subseteq F(L) \\
L &\subseteq F(L) \text{ and} \\
L &\subseteq F(K).
\end{aligned}
$$

Figure 3.1 is an example of what F could look like in case 1.

Let $k \in \mathbb{N}$. For $k > 1$ let $\{I_n\}$ be the sequence of intervals $I_n = L$ for $n = 0, \ldots, k-2$, and $I_{k-1} = K$, and define I_n to be periodic inductively by $I_{n+k} = I_n$ for $n \in \mathbb{N}$. If $k = 1$ let $I_n = L$ for all n. Recall that in order for Lemma 8 to apply to the function $F : J \to J$ and the sequence of compact intervals $\{I_n\}_{n=0}^{\infty}$ then $I_n \subseteq J$, and $I_{n+1} \subseteq F(I_n)$ for all $n \in \mathbb{N}$. Since $\{I_n\}$ is defined so that either $I_n = K$ or $I_n = L$ for all n, then the only time that I_n wouldn't satisfy Lemma 8 would be when $I_n = I_{n+1} = K$ because this requires that $K \subseteq F(K)$ which is not necessarily true. In other words whenever we have two intervals equal to K then we have an interval equal to L between them or it will not be sufficient to satisfy Lemma 8.

Let Q_n be a sequence of compact intervals defined as in Lemma 8; therefore $Q_{n+1} \subseteq Q_n$ and $F^n(Q_n) = I_n$ for all $n \in \mathbb{N}$. Notice that $Q_k \subseteq Q_o = L$ and by our

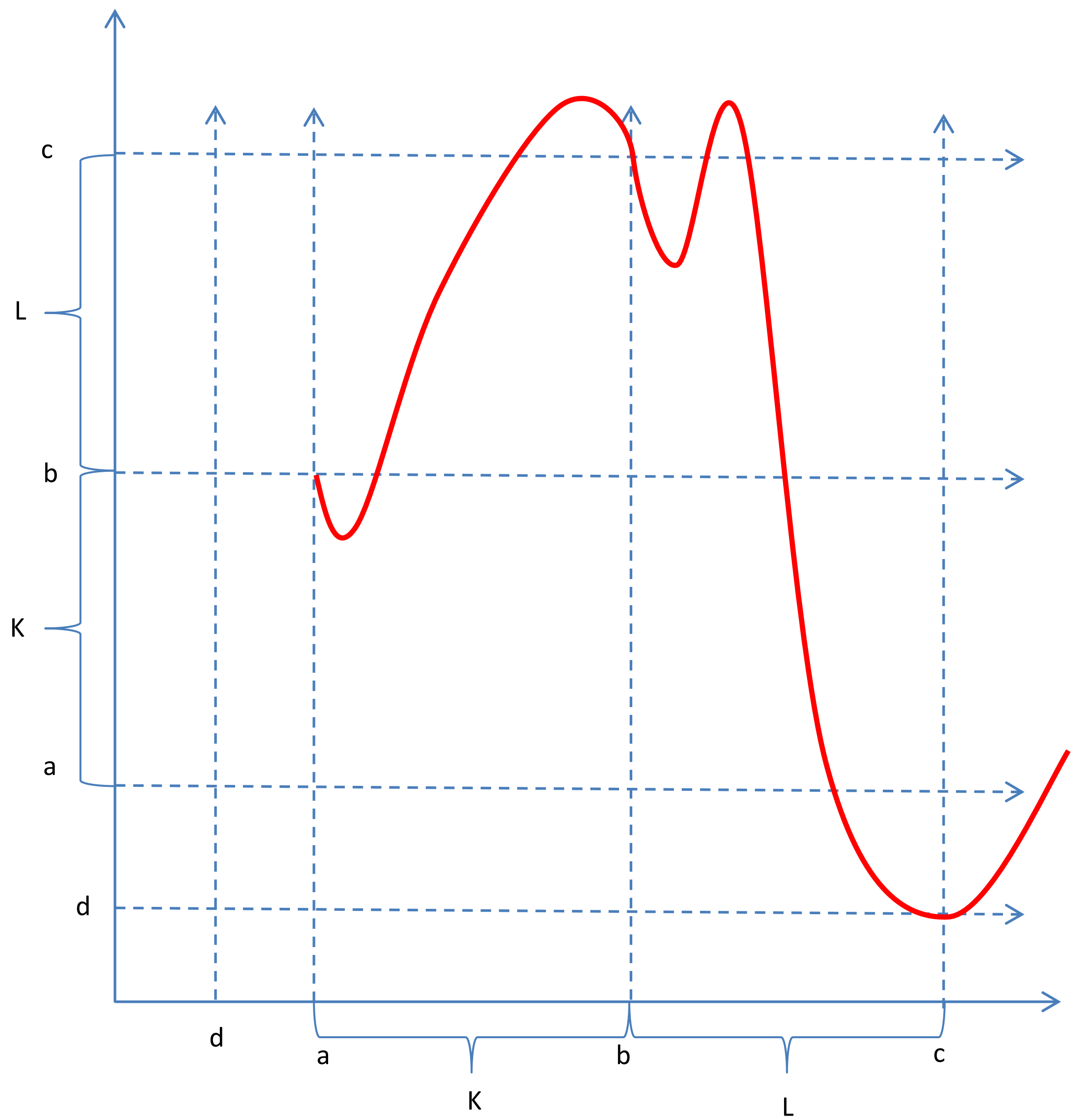

FIGURE 3.1. Plot F under the condition of case 1.

definition of I_n, $F^k(Q_k) = I_k = L = Q_0 \supseteq Q_k$. Then by Lemma 9,$Q_k \subseteq F^k(Q_k)$ and so with $G = F^k$ and we see that G has a fixed point p_k in Q_k.

Now suppose p_k has period $n_0 \in \{0, \cdots, k-1\}$ or in other words $F^{n_0}(p_k) = p_k$. We know that $F^{k-1}(p_k) \in I_{k-1} = K$. Now, $F^m(p_k) \in I_m = L$ for $m < k-1$. Since

$k-1-n_0 < k-1$ this implies

$$\begin{aligned} F^{k-1-n_0}(p_k) &= F^{k-1-n_0}(F^{n_0}(p_k)) \\ &= F^{k-1-n_0+n_0}(p_k) \\ &= F^{k-1}(p_k) \end{aligned}$$

$\Rightarrow F^{k-1}(p_k) \in L$ and $F^{k-1}(p_k) \in K \Rightarrow F^{k-1`}(p_k) \in L \cap K \Rightarrow F^{k-1}(p_k) = b$. But $F^{k+1}(p_k) \in L$ and $F^{k+1}(p_k) = F^2(F^{k-1}(p_k)) = F^2(b) = d$. But $d \notin L$ a contradiction. Therefore p_k has period k. □

4. Proof of T2

(Proof of T2). Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals satisfying conditions

$$M_n = K \text{ or } M_n \subseteq L \text{ and, } M_{n+1} \subseteq F(M_n) \tag{4.1}$$

and if $M_n = K$ then

$$n \text{ is the square of an integer and } M_{n+1}, M_{n+2} \subseteq L. \tag{4.2}$$

For case 1, $K = [a, b]$ and $L = [b, c]$. For case 2, $K = [b, a]$ and $L = [c, b]$. As was proven earlier for both cases we have $K \subseteq F(L)$, $L \subseteq F(L)$, and $L \subseteq F(K)$. For $M \in \mathcal{M}$, let $P(M, n)$ denote the number of i's in $\{1, \dots, n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$ choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} P(M^r, n^2)/n = r. \tag{4.3}$$

To help understand how we can choose what M^r is, lets give an example of the case $r = 4/5$. First let every interval of M_n^r where n is the square of an integer be equal to K except for every fifth square of an integer. Otherwise let $M_n^r = L$. So the first 36 terms of M_n^r looks like

$$\begin{aligned} M_n^r &= \{K, L, L, \\ &\quad K, L, L, L, L, \\ &\quad K, L, L, L, L, L, L, \\ &\quad K, L, L, L, L, L, L, L, L, \\ &\quad L, L, L, L, L, L, L, L, L, L, L, L, \\ &\quad K, L, \ldots\}. \end{aligned}$$

Notice that the 25^{th} term in the sequence is equal to L. It should be easy to see that this example satisfies 4.3. Other examples of M_n^r where r is rational can be found in a similar way. For irrational values of r such as $\pi/4$ first we choose enough squares to be K such that the tens digit of r is satisfied 4.3. Then we choose enough of the K squares so that for the hundreds digit of r 4.3 is satisfied. Repeat this for every digit of r to get an example of M_n^r that satisfies 4.3. For example $\pi/4 \approx 0.7853981634\ldots$. So seven out of the first ten are K, then 78 out of the first 100 are K, then 785 out of the first 1000, and so on.

Notice that the number of i's in $\{1, \ldots, n^2\}$ for which i is the square of an integer is n. So this implies that $P(M^r, n^2) \leq n$ for all n. Thus $P(M^r, n^2)/n \leq 1$ for all n. So the choice that $r < 1$ makes sense. The reason we choose $r > 3/4$ will be made clear later in the proof. For now it is sufficient to say that since $r > 0$ then 4.3 implies that there are an infinite number of K's in M^r because otherwise the limit in 4.3 would be equal to zero.

Let $\mathcal{M}_0 = \{M^r : r \in (3/4, 1)\} \subseteq M$. Suppose $r_1, r_2 \in (3/4, 1)$ such that $r_1 \neq r_2$. Then $M^{r_1} \neq M^{r_2}$. This means that there are as many possible M^r as there are r values or for every unique value of r there a corresponding M^r. Now since $(3/4, 1)$ is an uncountable set then the set of possible values of r is uncountable then $\mathcal{M}_0$ is uncountable. For each $M^r \in \mathcal{M}_0$, by Lemma 8, there exists a point x_r with $F^n(x_r) \in M_n^r$ for all n. Let $S = \{x_r : r \in (3/4, 1)\}$. For $x_r \in S$, let $P(x_r, n)$ denote the number of i's in $\{1, \ldots, n\}$ for which $F^i(x_r) \in K$. Suppose $F^k(x_r) = b$. Then $F^{k+2}(x_r) = d$, but $F^{k+2}(x_r) \in M^r_{k+2} \subseteq (L \cup K)$. In order for this to be true $a = d$. Thus

$$\begin{aligned} F^{k+3}(x_r) &= F(F^{k+2}(x_r)) \\ &= F(a) \\ &= b \\ &= F^k(x_r) \end{aligned}$$

so x_r would eventually have period three and $F^{k+2}(x_r) = a$. Since $a \in K$, $a \notin L$, and $F^{k+2}(x_r) \in M^r_{k+2}$ then $F^{k+2}(x_r) \in K$. Then 4.2 implies that $F^{k+3}(x_r) \in L$, $F^{k+4}(x_r) \in L$, and

$$\begin{aligned} F^{k+3m+2}(x_r) &= F^{3m}(F^{k+2}(x_r)) \\ &= F^{3m}(a) \\ &= a \notin L. \end{aligned}$$

But since the values where M^r can be K are non periodic then it follows that for some values of m $M^r_{k+3m+2} \neq K$ a contradiction since $a \in K$. So $F^k(x_r) \neq b$ for any $x_r \in S$ then either $x_r \in K$ or $x_r \in L$ but not both. Since $F(x_r) \neq b$, $r_1 \neq r_2 \Rightarrow x_{r_1} \neq x_{r_2}$. Therefore S is uncountable and $P(x_r, n) = P(M^r, n)$ for all n. Define ρ to be

$$\rho(x_r) = \lim_{n\to\infty} P(x_r, n^2)/n = r$$

for all r. Now 4.1 implies that $M^r_n = K$ or $M^r_n \subseteq L$ for all n. We claim that for $p, q \in S$ with $p \neq q$, there exist infinitely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$ or vice versa. Without loss of generality assume $\rho(p) > \rho(q)$. Then $\lim_{n\to\infty} P(p, n^2)/n > \lim_{n\to\infty} P(q, n^2)/n$. Therefore

$$\lim_{n\to\infty} \frac{P(p, n^2) - P(q, n^2)}{n} > 0.$$

Since the denominator of this expression approaches infinity then the numerator must approach infinity as well. But since the function P only increases by a value of $+1$ or 0 for every increment n and the only time $P(p, n^2)$ increases while $P(q, n^2)$ remains the same is when $F^{n^2}(p) \in K$ and $F^{n^2}(q) \in L$. Then there must be infinitely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$.

For case 1, $K = [a, b]$ and $L = [b, c]$. Since $F^2(a) = c$ and $F^2(b) = d \leq a$ and F^2 is continuous, there exists a $\delta > 0$ such that $F^2(x) - F^2(b) < (b - d)/2$ for all $x \in [b - \delta, b] \subset K$. This implies $F^2(x) > (b + d)/2$ for all $x \in [b - \delta, b]$. If $p \in S$ and $F^n(p) \in K$, then 4.2 implies $F^{n+1}(p) \in L$ and $F^{n+2}(p) \in L$. Therefore $F^2(F^n(p)) \in L$. Suppose $F^n(p) \in [b - \delta, b]$ then $F^2(F^n(p)) < (b + d)/2$; a contradiction since $d < b$ and so $(b + d)/2 < b$. Therefore $F^n(p) \notin [b - \delta, b]$. Since $F^n(q) \in L$, then

$$F^n(p) < b - \delta < b < F^n(q).$$

Therefore

$$\begin{aligned} |F^n(p) - F^n(q)| &= F^n(q) - F^n(p) \\ &> b - (b - \delta) \\ &= \delta. \end{aligned}$$

For case 2 $K = [b, a]$ and $L = [c, b]$, since $F^2(a) = c$ and $F^2(b) = d \geq a$ and F^2 is continuous, there exists a $\delta > 0$ such that $F^2(b) - F^2(x) > (d - b)/2$ for all $x \in [b, b + \delta] \subset K$. This implies $F^2(x) < (b + d)/2$ for all $x \in [b, b + \delta]$. If $p \in S$ and $F^n(p) \in K$, then 4.2 implies $F^{n+1}(p) \in L$ and $F^{n+2}(p) \in L$. Therefore $F^2(F^n(p)) \in L$. Suppose $F^n(p) \in [b, b + \delta]$ then $F^2(F^n(p)) < (b + d)/2$ a contradiction since $b < d$ and so $b < (b + d)/2$. Therefore $F^n(p) \notin [b, b + \delta]$. Since $F^n(q) \in L$, then

$$F^n(q) < b < b + \delta < F^n(p).$$

Therefore

$$\begin{aligned}|F^n(p) - F^n(q)| &= F^n(p) - F^n(q) \\ &> (b+\delta) - b \\ &= \delta.\end{aligned}$$

So for both case 1 and case 2, $|F^n(p) - F^n(q)| > \delta$ for the infinitely many cases where $F^n(p) \in K$ and $F^n(q) \in L$. Thus the set

$$\{|F^k(p) - F^k(q)| : \forall k \in \mathbb{N} \text{ and } F^k(p) \in K \text{ and } F^k(q) \in L\}$$

is an infinite set and all values in the set are greater than δ. Therefore by definition 4

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

We will now prove 2.3 since the proof is similar to the proof of 2.1. Now assume $p \in S$ and let q be a periodic point in J with period k. As was shown in the proof of 2.1 when $F^n(p) \in K$ then there exists a $\delta > 0$ such that for case 1 $F^n(p) < b - \delta$ and for case 2 $F^n(p) > b + \delta$. Let I_n be as defined in the proof of T1 such that $F^n(q) \in I_n$ for all $n \in \mathbb{N}$. Now, $I_n = K$ only when $n = jk - 1$ where $j \in \mathbb{N}$. Since the set of perfect squares is non periodic then the set of n's where $F^n(p) \in K$ and $F^n(q) \in L$ is infinite. Thus the set

$$\{|F^k(p) - F^k(q)| : \forall k \in \mathbb{N} \text{ and } F^k(p) \in K \text{ and } F^k(q) \in L\}$$

is an infinite set and all values in the set are greater than δ. Therefore by definition 4

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

We now prove 2.2. To prove 2.2 we must first modify our definition of M_n^r. First, assume case 1, so that $F(b) = c$ and $F(c) = d \leq a < b$. We can choose the set of intervals $[b_n, c_n]$, $n = 0, 1, 2, \ldots$ such that $b = b_0$ and $c = c_0$. since $F[b_0, c_0] \supseteq [d, c]$ then by the intermediate value theorem $\exists b_1 \in [b_0, c_0]$ such that $F(b_1) = b_0$. Since

$F[b_1, c_0] \supseteq [d, c_0] \ni b_0$ then by the intermediate value theorem that $\exists c_1 \in [b_1, c_0]$ such that $F(c_1) = b_0$. Then it follows by similair arguments that we can choose b_n and c_n inductively such that

$$b_{n+1} = \max\{x \in [b_n, c_n] | F(x) = c_n\}$$

and

$$c_{n+1} = \min\{x \in [b_{n+1}, c_n] | F(x) = b_n\}.$$

This implies that $[b, c] = [b_0, c_o] \supseteq [b_1, c_1] \supseteq [b_2, c_2] \supseteq \cdots \supseteq [b_n, c_n] \supseteq \cdots$, and $F(x) \in (b_n, c_n)$ for all $x \in (b_{n+1}, c_{n+1})$, and $F(b_{n+1}) = c_n$, $F(c_{n+1}) = b_n$. Now, because the definition of b_n and c_n are very similar for case 2 we will omit it in order to prevent confusion. Since $b_n < c_n$ for all n and since b_n is increasing then the limits $b^* = \lim b_n$ and $c^* = \lim c_n$ exist.

Now, in addition to 4.1 and 4.2 we assume that if $M_k = K$ for both $k = n^2$ and $k = (n+1)^2$ then $M_k = [b_{2n-(2j-1)}, b^*]$ for $k = n^2 + (2j - 1)$, $M_k = [c^*, c_{2n-2j}]$ for $k = n^2 + 2j$ where $j = 1, \cdots, n$. For all other values of k we assume $M_k = L$. What this means is that if $M_k = K$ for both $k = n^2$ and $k = (n+1)^2$ then for every value of k between n^2 and $(n+1)^2$, M_k are set to the intervals given previously.

Now we check if these conditions are consistent with 4.1 and 4.2. Since $b_n, b^*, c_n, c^* \in L$ then $[b_n, b^*] \subseteq L$ and $[c^*, c_n] \subseteq L$ for all $n \in \mathbb{N}$. Let $M_k = K$ for both $k = n^2$ and $k = (n+1)^2$. Then $M_{n^2+1} = [b_{2n-1}, b^*]$ and $M_{n^2+2} = [c^*, c_{2n-2}]$. Since n^2 and $(n+1)^2$ are both squares of integers then condition 4.2 is satisfied. Now, we know that $M_{n^2+2n} = [c^*, c^0]$. For 4.1 to be satisfied then $F(M_k) \supseteq M_{k+1}$ for all $k \in \mathbb{N}$. To show this is true first

$$\begin{aligned} F(M_{k^2}) &= F(K) \\ &\supseteq L \\ &\supseteq [b_{2k-1}, b^*] \qquad \text{since } b_k, b^* \in L \\ &= M_{k^2+1}. \qquad \text{by definition of } M \end{aligned}$$

Next,

$$\begin{aligned} F(M_{k^2+2j-1}) &= F[b_{2k-(2j-1)}, b^*] \\ &\supseteq [F(b^*), F(b_{2k-2j+1})] \qquad \text{since both } F(b_{2k-(2j-1)}), F(b^*) \in F[b_{2k-(2j-1)}, b^*] \\ &= [c^*, c_{2k-2j}] \qquad \text{by our choice of } c^* \text{and } c_k \\ &= M_{k^2+2j}. \end{aligned}$$

And

$$\begin{aligned} F(M_{k^2+2j}) &= F[c^*, c_{2k-2j}] \\ &\supseteq [F(c_{2k-2j}), F(c^*)] \qquad \text{since both } F(c_{2k-2j}), F(c^*) \in F[bc_{2k-2j}, c^*] \\ &= [b_{2k-2j-1}, b^*] \qquad \text{by our choice of } b^* \text{and } b_k \\ &= M_{k^2+2j+1}. \end{aligned}$$

Finally

$$\begin{aligned} F(M_{(k+1)^2-1}) &= F(M_{k^2+2k}) \\ &= F[c^*, c_0] \\ &\supseteq [F(c_0), F(c^*)] \\ &= [d, b^*] \\ &\supseteq K \\ &= M_{(k+1)^2}. \end{aligned}$$

Therefore the new conditions are compatible with 4.1. Since the new conditions do not affect the number of i's in $\{1, \ldots, n\}$ for which $M_i = K$ then for each $r \in (3/4, 1)$ we can choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} P(M^r, n^2)/n = r.$$

Therefore conditions 4.1, 4.2, and 4.3 are satisfied by our new conditions.

Let $r, r^* \in (3/4, 1)$ where $r \neq r^*$. Choose $M^r, M^{r*} \in \mathcal{M}$ such that

$$\lim_{n\to\infty} P(M^r, n^2)/n = r$$

and

$$\lim_{n\to\infty} P(M^{r^*}, n^2)/n = r^*.$$

Let the set $B = \{B_n\}$ be defined so that $B_k = 1$ if and only if $M^r_{k^2} = K$, $B_k = 0$ otherwise. Likewise let the set $B^* = \{B_n^*\}$ be defined so that $B_k^* = 1$ if and only if $M^{r^*}_{k^2} = K$, $B_k^* = 0$ otherwise. Then by 4.3 and the definition of M^r

$$r = \lim_{n\to\infty} \left(\left(\sum_{m=1}^{n} B_m \right) /n \right)$$

and

$$r^* = \lim_{n\to\infty} \left(\left(\sum_{m=1}^{n} B_m^* \right) /n \right).$$

Now define $\{Z_n\}$ such that $Z_k = a$ if and only if $B_k = B_k^* = 0$, $Z_k = b$ if and only if $B_k = 0$ and $B_k^* = 1$, $Z_k = c$ if and only if $B_k = 1$ and $B_k^* = 0$, and $Z_k = d$ if and only if $B_k = B_k^* = 1$. Note that $Z_k = Z_{k+1} = d$ if and only if $B_k = B_{k+1} = B_k^* = B_{k+1}^* = 1$. Now since $r > 3/4$ then $Z_k \in \{c, d\}$ more than three fourths of the time. This implies that $Z_k = b$ less than one fourth of the time. since $r^* > 3/4$ then $Z_k \in \{b, d\}$ more than three fourths of the time. This implies that $Z_k = c$ less than one fourth of the time. Suppose that $Z_k = d$ less than or equal to one half of the time. This implies that $Z_k = c$ more than one fourth of the time a contradiction. Therefore $Z_k = d$ more than one half of the time. Now suppose that if $Z_k = d$ then $Z_k \neq d$. This implies that for every time $Z_k = d$ then there is at least one time where $Z_k \neq d$. Then $Z_k = d$ for less than or equal to half the time a contradiction. This implies that $Z_k = Z_{k+1} = d$ happens infinitely many times, which implies that $B_k = B_{k+1} = B_k^* = B_{k+1}^* = 1$ happens infinitely many times.

Therefore there exist infinitely many n such that $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $k = (n+1)^2$. Let $x_r \in S$ and $x_{r^*} \in S$. Since $\lim(b_n) = b^*$ and $\lim(c_n) = c^*$, for any $\varepsilon > 0$ there exists N with $|b_n - b^*| < \varepsilon$, $|c_n - c^*| < \varepsilon$ for all $n > N$. Then, for any n with $n > N$ and $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$, we have

$$F^{n^2+1}(x_r) \in M_k^r = [b_{2n-1}, b^*]$$

with $k = n^2 + 1$ and $F^{n^2+1}(x_r)$ and $F^{n^2+1}(x_{r^*})$ both belong to $[b_{2n-1}, b^*]$. Therefore, $|F^{n^2+1}(x_r) - F^{n^2+1}(x_{r^*})| < \varepsilon$. And so by definition

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0.$$

□

5. APPLICATION OF THE THEOREM

The most commonly used example to demonstrate the behavior described by the theorem is the logistic equation

$$F(x) = \rho x(1-x)$$

where $F : [0,1] \to [0,1]$ and $\rho > 0$. In order to apply this theorem we will choose $\rho = 4$. First let's check and see if the example satisfies the hypothesis. Figure 5.1 shows the plots of x, $F(x)$, $F^2(x)$, and $F^3(x)$ versus x. In order for us to be able to apply the theorem there must exist a point $a \in [0,1]$ such that $F^3(a) \leq a < F(a) < F^2(a)$. From Figure 5.1 we can see that the points in the neighborhood of 0.15 will satisfy the hypothesis. If we solve for $F^3(x) = 0$ we find that the root that is closest to 0.15 is $(2-\sqrt{2})/4 \approx 0.146466094$. Now, let $a = (2-\sqrt{2})/4$, then $b = F(a) = 1/2$, $c = F^2(a) = 1$, and $d = F^3(a) = 0$. So $d < a < b < c$ which

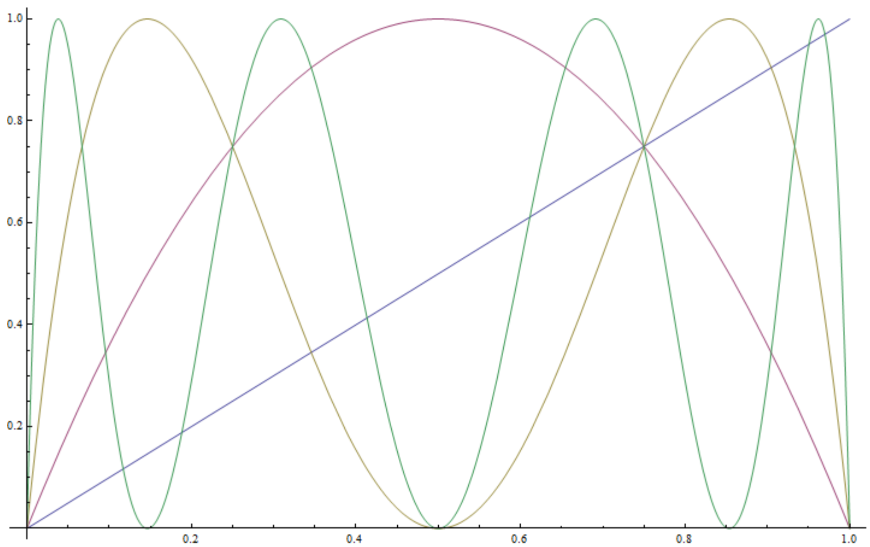

FIGURE 5.1. The blue plot is of x vs. x. The red plot is of $F(x)$ vs. x. The yellow plot is of $F^2(x)$ vs. x. The green plot is of $F^3(x)$ vs. x.

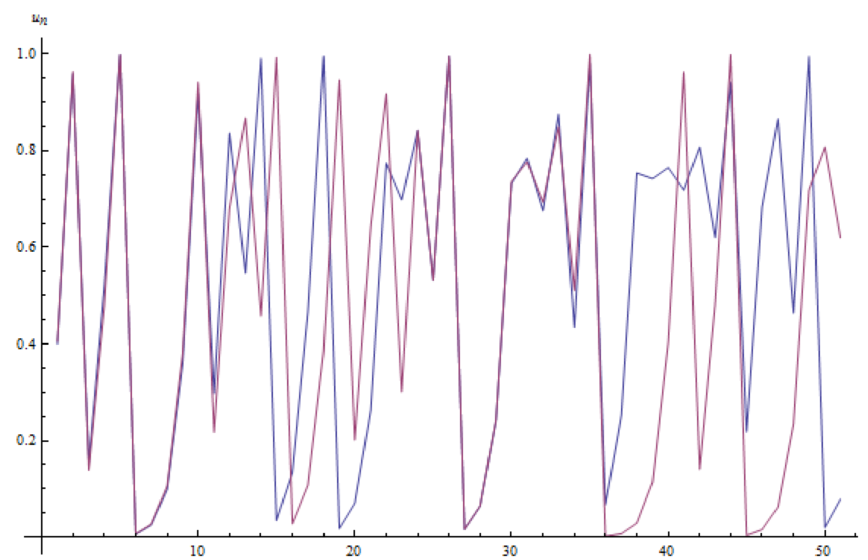

FIGURE 5.2. The blue plot is of x_n vs. n where $x_0 = 0.4$. The red plot is of $y_n$vs. n where $y_0 = 0.405$

satisfies our hypothesis. Figure 5.2 plots the sequences $\{x_n\}$ and $\{y_n\}$ vs. n where $x_{n+1} = F(x_n)$ and $x_0 = 0.4$ also $y_{n+1} = F(y_n)$ and $y_0 = 0.405$ for 50 iterations. We can see from Figure 5.2 that for about the first ten iterations the two plots remain very close together. Then after the tenth iteration the two plots start to diverge. For about 15 iterations, the plot's values, while all of them are still within J, the distance between these values is quite different. Then quite abruptly the plots seem to converge around the 24^{th} iteration and for about seven iterations the two plots appear to be almost identical. Finally they diverge once more at around the 32^{nd} iteration. Neither of these plots show any sign of being periodic or even show signs of being asymptotically periodic. Both plots show a general lack of a discernible pattern. It is important to note that this plot is only an approximation and we cannot attest to the accuracy of the values for larger and larger n. We can only say that this is similar to the behavior we would see if x_0 and y_0 exist in the set S from the proof of T2.

The $\limsup$ in Statement 2.1 states that for any two initial conditions in S then no matter how many iterations we do the difference between the respective sequences for certain iterations will never approach zero. Statement 2.2 implies that for other iterations these sequences get very close to each other. Finally 2.3 implies that for any initial condition in S then for certain iterations the corresponding sequence will never be close to any periodic point of F. This implies that none of the initial conditions in S will even be asymptotically periodic to any of the periodic points given by T1.

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