# Finite and Infinite Hat Problems

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## 1 Introduction

This project will discuss hat problems and the Axiom of Choice. A basic example of a finite hat problem is helpful in introducing the concept. Assume that there are two players and two possible hat colors, red and blue. The players will be given a certain amount of time before the game begins to communicate and create a strategy for determining their own hat colors. After this initial strategy session, the game begins and no communication of any kind is allowed between the players. A third party will place a red or a blue hat on each of the players. Each player will be able to see his teammate's hat color, but will not be able to see his own hat color. The object of the game is for a player to correctly guess his own hat color. Each player's guess must be independent of his teammate's guess. Both players win the game if at least one of them guesses correctly. We want to know if there exists a strategy which ensures that at least one player correctly guesses the color of his own hat. Surprisingly, the answer to this question is yes. If one player guesses that his hat is the same color as his teammate's hat and the other player guesses that his hat color is the opposite of his teammates hat color, then exactly one of the players will guess correctly. Although this is a very basic example, the solution is not immediately apparent. The question which this project deals with is whether a strategy exists which guarantees a win for the players under various scenarios. The articles by Christopher S. Hardin and Alan D. Taylor [3], [4] along with the text by Thomas Q. Sibley [5] will be used in the discussion of these problems. Before we can explore these more complex problems and solutions, we must define a number of terms and concepts which will be used throughout this paper.

## 2 Definitions

Every hat problem can be described by the number of players, the number of possible hat colors, a directed graph which describes the visibility of each player to every other player, and a rule that determines the requirements for a win. Before we can formally define a hat problem, we must define a term from graph theory.

**Definition 1 (Directed Graph)** *A* ***directed graph*** *is a set $V$ and a set $E$ consisting of ordered pairs of distinct elements of $V$.*

Since we defined an edge as an ordered pair of distinct elements of $P$, we eliminate the possibility of loops in $V$. In terms of a hat problem this means that no player is able to see his own hat color. The following definition is the formal description of a hat problem.

**Definition 2 (hat problem)** *A* ***hat problem*** *is a tuple $(P, C, V, W)$, where $P$ and $C$ are any sets, $V$ is a directed graph with $P$ as the set of vertices, and $W$ is a family of subsets of $P$. We think of $P$ as the set of all players, and $C$ as the set of all possible hat colors. We call the directed graph $V$ the visibility graph of our hat problem. We say that player $a$ can see the hat of player $b$ if there exists a directed edge $(a, b)$. We let $V_a$ denote the set of all vertices adjacent to vertex $a$ and note that $V_a$ will not include $a$. Let $W$ denote the winning family. We say that all of the players win if and only if the set of players who guess their own hat colors correctly is in $W$ [4].*

One example of a winning family is as follows. Say that $W$ is the family of subsets of $P$ where each subset contains an even number of players. In this case, all of the players win if and only if an even number of players guess correctly.

We will also be using several terms which describe specific scenarios within a given hat problem and how such a hat problem can be solved. We note that our use of the term *coloring* is specific to this project and not consistent with the use of the term in graph theory.

**Definition 3 (coloring)** *A* ***coloring*** *is a function $h : P \longrightarrow C$ [4].*

Thus each player may be assigned only one color, but the same color may be assigned to multiple players. Now that we have defined the term coloring, we must define the notation used to describe a collection of every such coloring.

**Definition 4 ($^{P}C$)** *We define $^{P}C$ to be the set of all possible mappings from $P$ to $C$ [4].*

**Definition 5 (strategy)** *For a given hat problem $(P, C, V, W)$, we define a* ***strategy*** *to be a function $S : P \times {}^{P}C \longrightarrow C$ such that for all $a \in P$ and any two colorings $g, h \in {}^{P}C$, if $g\mid_{V_a} = h\mid_{V_a}$, then $S(a, g) = S(a, h)$. Furthermore, for a given coloring $g$ we let $S(a, h)$ denote player $a$'s guess as dictated by $S$.*

Note that player $a$'s guess, $S(a, g)$, given by $S$ is the same for both distinct colorings $g$ and $h$. In other words, a player's guess under any strategy is determined only by the hats visible to that player. We have defined a strategy which determines individual color guesses for a player or group of players, but we will also use a notion of strategy which gives an entire coloring as a guess and can be better defined as a global strategy. Since we will only use this notion of a strategy in our final theorem, the definition of a global strategy will be reserved for later. Now that we have defined the term strategy, we must define some terms which describes the effectiveness of such a strategy.

**Definition 6 (winning strategy)** *We say that a strategy which ensures that the set of players who guess correctly is in $W$ is a* ***winning strategy*** *or solution for the hat problem [4].*

**Definition 7 (minimal solution)** *We call a winning strategy which ensures that at least one player guesses his hat color correctly a* ***minimal solution*** *[4].*

Once we begin our discussion of infinite hat problems, we will require more sophisticated machinery such as the Axiom of Choice and the concept of a null set.

**Axiom 8 (Axiom of Choice)** *Given any collection of nonempty sets $\mathcal{A}$, there exists a function $F : \mathcal{A} \longrightarrow \bigcup_{A \in \mathcal{A}} A$ such that for every $A \in \mathcal{A}$, $F(A) \in A$ [2].*

**Definition 9 (null set)** *A set $Z \subset \mathbb{R}$ is said to be a* ***null set*** *if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that*

$$Z \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k) \quad \textit{and} \quad \sum_{k=1}^{\infty}(b_k - a_k) \leq \varepsilon \text{ [1].}$$

# 3 Results

## 3.1 Finite Hat Problems and Their Solutions

Now that we have laid the necessary groundwork, we can begin our discussion of more complex hat problems and their possible solutions.

**Lemma 10** *In a $k$-player, $n$-color hat problem, for any particular strategy, the number of players who guess correctly is on average $\frac{k}{n}$. (This average is taken over all colorings) [4].*

**Proof (Lemma 10)** Suppose that there are $k$ players and $n$ colors and let $a$ be an arbitrary player. Also let $S$ be some strategy. Now for any coloring of hats other than player $a$'s hat, player $a$'s guess is determined by $S$. Since there are $n$ ways to color $a$'s hat, player $a$ will guess correctly for exactly 1 out of $n$ color assignments. Since there are $k$ players, the average number of players who guess correctly is $k(\frac{1}{n}) = \frac{k}{n}$. ■

We have already described a minimal solution for a 2-player, 2-color hat problem, but what happens if we allow any finite number of players in our hat problem? Can we still find a minimal solution to this more generalized problem? The following theorems answers these questions.

**Theorem 11** *A $k$-player, 2-color hat problem has a minimal solution if and only if the visibility graph has a cycle [4].*

**Proof (Theorem 11)** Assume that $|P| = k$ and $|C| = 2$, where $k \in \mathbb{N}$. In other words, suppose that we have a $k$-player, 2-color hat problem. Without loss of generality, assume the colors are red and blue.

($\Longleftarrow$) Suppose that the directed visibility graph $V$ has a cycle. We let $W = \mathcal{P}(P) - \{\varnothing\}$. Choose a player on the cycle to be the first player. Call this player 1. Define $S$ to be the strategy such that player 1 guesses that his hat is the same color as the hat of the player immediately after him on the cycle, and every subsequent player on the cycle guesses that his hat is the opposite color of the hat of the player next on the cycle. Since we are only concerned with players on the cycle, we will say that every player not on the cycle guesses that his hat is red. Now suppose without loss of generality that player 1 has a red hat and suppose to the contrary that every player guesses incorrectly under $S$. Since every player guesses incorrectly and player 1 has a red hat, we know that player 1 will guess that his hat is blue. We also know that under strategy $S$ player 1 will guess that his hat is the same color as the hat of the player immediately following him on the cycle, thus the second player in the cycle must have a blue hat. Similarly, since player 2 guesses incorrectly and under $S$ guesses that his hat is the opposite color of the hat of the player immediately following him on the cycle, player 3 will have a blue hat as well. Continuing this line of logic we conclude that each subsequent player on the cycle must have a blue hat until we get back to the first player on the cycle. We must conclude that the first player has a blue hat as well, but this contradicts our original assumption that player 1 has a red hat. Thus, at least one player on the cycle will guess correctly under $S$ and $S$ is a minimal solution for a $k$-player, 2-color hat problem. See Figure 1 below for an example of a 10-player, 2-color hat problem which illustrates our previous contradiction.

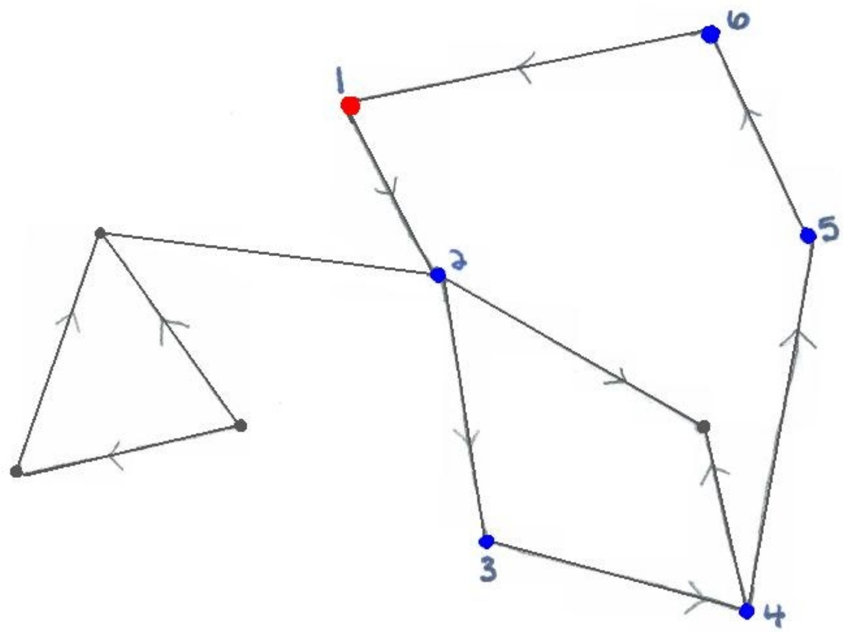


Figure 1: A visibility graph with a cycle

($\Longrightarrow$) Now we will show that if there is a minimal solution $S$ for a $k$-player, 2-color hat problem, then the directed visibility graph $V$ has a cycle. We will show that this statement holds using the contrapositive. Suppose that the directed visibility graph $V$ does not have a cycle. Since $V$ has no cycles, we know that no vertices are repeated along any path. We also know that the set of vertices is finite since the number of players is some finite $k \in \mathbb{N}$. Therefore, the length of any directed path is finite. Thus we can assign a rank to each player. We assign a rank to a given player $a$ by determining the length of the longest directed path from the vertex representing player $a$ to any other vertex representing another player. The length of this longest path is the rank of player $a$. Now if a player $b$ has a rank of $i$ and player $b$ is seen by player $a$, then player $a$ has a rank of at least $i + 1$. In other words, if player $a$ has rank $j$ and there is a directed edge from vertex $a$ to some vertex $b$, then the length of the longest directed path from vertex $b$ to any other vertex is at most $j - 1$. Therefore player $b$ has rank strictly less than $j$. We know that a directed edge from vertex $a$ to vertex $b$ means that player $a$ can see the hat worn by player $b$. Thus, a player can only see the hats of players with rank strictly less than his own. Now given any strategy we can create a coloring which ensures that every player will guess incorrectly. We know that once we assign hat colors to all players which have a rank strictly less than $k$, the guesses of all the players of rank $k$ are determined by the given strategy since a player's guess under a given strategy is determined only by the hats that player can see. Once we know the guesses of the players of rank $k$, we color their hats so that they guess incorrectly. This strategy is further demonstrated in the example given below. Thus if the directed visibility graph of a $k$-player 2-color hat problem has no cycles, then for any strategy there exists a coloring so that every player will guess incorrectly. This statement is logically equivalent to the statement that if a $k$-player 2-color hat problem has a minimal solution, then the directed visibility graph for this hat problem must have a cycle.■

**Example** Suppose that there is a $k$-player 2-color hat problem with visibility graph $V$. Also assume that $V$ has no cycles. We can assign a rank to each player as described in the proof given above. If there are players who cannot see any other players, these players are assigned a rank of 0, since there are no directed edges coming from these vertices and therefore no directed paths beginning at their vertices. The guess of each of these players is determined by the given strategy. Therefore we know the guess of each of these players. Now we color the hats of the players of rank 0 so that they all guess incorrectly. Now that we have colored the hats of the players of rank 0, the guess of each player of rank 1 is determined by the strategy. This is because players of rank 1 can only see players of rank 0 and the guess determined by the strategy is based strictly on what the player can see. Since we know the guesses of the players of rank 1, we color the hats of these players so that they all guess incorrectly. We continue this process, coloring the hats of players of rank 2 and 3 and so on until we have colored all the hats so that every player guesses incorrectly.

We can generalize the hat problem further by allowing the number of possible colors to be any finite number, as long as the number of possible colors is equal to the number of players. We want to know if a solution exists for this type of problem and what if any other conditions must be met in order to ensure its existence. Our next theorem provides and answer to both of these questions.

**Theorem 12** *A $k$-player, $k$-color hat problem has a minimal solution if and only if the visibility graph is complete [4].*

**Proof (Theorem 12)** Assume that $|P| = k$ and $|C| = k$, where $k \in \mathbb{N}$. In other words, suppose that we have a $k$-player, $k$-color hat problem.

($\Longleftarrow$) Assume that the directed visibility graph $V$ is complete. Since $V$ is complete we know that every player can see every other player. Now we will find a strategy so that for any given coloring at least one player will guess correctly. To begin, we number the players 0, 1, 2, ..., $k-1$ and the colors 0, 1, 2, ..., $k-1$. Now, let $i \in \mathbb{N}$ and for each $i$ define $s_i$ to be the $\bmod k$ sum of the colors of the hats seen by player number $i$, that is the hats of every player except $i$. Also let $c_i$ denote the color of the hat worn by player $i$. Then

$$s_i = (c_0 + c_1 + ... + c_{i-1} + c_i + c_{i+1} + ... + c_{k-1} - c_i) \bmod k,$$

for some player $i$. We now show that a minimal solution can be found by describing such a solution. Our strategy is for each player $i$ to guess that his hat is color $(i - s_i) \bmod k$. Since the $\bmod k$ sum of all the hat colors will always be equal to one of the elements of the set $\{0, 1, ..., k-1\}$, we know that the sum of all the hat colors $c_0 + c_1 + c_2 + ... + c_{k-1} = i (\bmod k)$, for some player $i$. Now, by performing simple algebra on the previous equation, we see that the color of the hat worn by some player $i$,

$$\begin{aligned} c_i &= (i - (c_0 + c_1 + ... + c_{i-1} + c_i + c_{i+1} + ... + c_{k-1} - c_i)) \bmod k \\ &= (i - s_i) \bmod k. \end{aligned}$$

Thus given any coloring, the guess of some player $i$ determined by our strategy is correct when the some of the hat colors of all the players is equal to $i (\bmod k)$. So our strategy is a minimal solution for any $k$-player $k$-color hat problem with a complete visibility graph.

($\Longrightarrow$) Now we will show that if a $k$-player $k$-color hat problem has a minimal solution, then the visibility graph of that problem is complete. We will show this statement using the contrapositive. Now suppose that the directed visibility graph $V$ is not complete and let $S$ be any strategy. Since $V$ is not complete, there is at least one player that does not see every other player. Without loss of generality, assume that player $a$ cannot see the hat worn by player $b$, and assume that $a \neq b$. Now, we pick some coloring in which player $a$ will guess correctly using $S$. Now, if necessary, we change that color of player $b$'s hat so that player $b$ guesses correctly. Since player $a$ cannot see the hat worn by player $b$, player $a$'s guess does not change when we change the color of player $b$'s hat. Thus, both players will guess correctly under this new coloring, but, by Lemma 2, we know that given any strategy the average number of players who guess correctly for a $k$-player, $k$-color hat problem is $\frac{k}{k} = 1$. Since we have two players guessing correctly for this coloring, we know that there must be a coloring in which less than one player, or no players, guess correctly in order to achieve an average of 1. Thus, for any $k$-player, $k$-color hat problem if $V$ is not complete, then for any strategy there exists a coloring in which every player guesses incorrectly. This statement is the logical equivalent of the statement that if a $k$-player, $k$-color hat problem has a minimal solution, then the visibility graph $V$ of this hat problem is complete. ■

In our last finite case, we remove another restriction on the initial set up of our hat problem. In this case we allow an arbitrary finite number of players and an arbitrary finite number of colors. The number of possible colors need not equal the number of players, but the number of colors must be less than or equal to the number of players in order for there to exist a winning strategy. The following theorem and its proof show why this must be the case and show that if the given condition is met a minimal solution must exist. This theorem will also show the limit of how accurate such a strategy will be.

**Theorem 13** *Consider the hat problem with $|P| = k$ and $|C| = n$, and a complete visibility graph $V$. Then there exists a strategy such that $\left\lfloor \frac{k}{n} \right\rfloor$ players guess correctly, but there does not exist a strategy which ensures that $\left\lfloor \frac{k}{n} \right\rfloor + 1$ players guess correctly [4].*

**Proof (Theorem 13)** Let $|P| = k$ and $|C| = n$ where $k, n \in \mathbb{N}$. Also assume that visibility graph $V$ is complete. The first thing we need to show is that there is a strategy so that $\lfloor \frac{k}{n} \rfloor$ players guess correctly. We will now describe such a strategy and show that it works. We first choose $n(\lfloor \frac{k}{n} \rfloor)$ players from $P$ and divide these players into $\lfloor \frac{k}{n} \rfloor$ pairwise disjoint groups of size $n$. Since these groups are pairwise disjoint, none of the pairs share a common player with any other pair. Then we can consider each of these pairs as a separate hat problem with $n$-players and $n$-colors in each. Since the number of players is equal to the number of possible colors and the visibility graph is complete for each of these smaller hat problems, we can apply the second part of Theorem 12 to find a minimal solution for each one of these smaller problems. We can in fact find a solution such that exactly one player from each group guesses correctly. This strategy is the strategy used in the right to left direction of the proof of Theorem 12. Recall that we numbered the players 0, 1, 2, ..., $n-1$ and the colors 0, 1, 2, ..., $n-1$. Now let $i \in \mathbb{N}$ and for each $i$ define $s_i$ to be the $\bmod n$ sum of the colors of the hats seen by player number $i$, that is the hats of every player except $i$. Also let $c_i$ denote the color of the hat worn by player $i$. Then the strategy is for each player $i$ to guess that his hat is color $(i - s_i) \bmod n$. We will call this strategy $S$. Now suppose that both player $i$ and player $j$ guess correctly under $S$. Then

$$c_i = (i - s_i) \bmod n$$

and

$$c_j = (j - s_j) \bmod n.$$

We also know that

$$s_i = \left( \left( \sum_{\alpha=0}^{n-1} c_\alpha \right) - c_i \right) \bmod n$$

and

$$s_j = \left( \left( \sum_{\alpha=0}^{n-1} c_\alpha \right) - c_j \right) \bmod n.$$

Then we have

$$\begin{aligned}
i &= (c_i + s_i) \bmod n \\
&= \left( c_i + \sum_{\alpha=0}^{n-1} c_\alpha - c_i \right) \bmod n \\
&= \left( \sum_{\alpha=0}^{n-1} c_\alpha \right) \bmod n \\
&= \left( \sum_{\alpha=0}^{j-1} c_\alpha + c_j + \sum_{\alpha=j+1}^{n-1} c_\alpha \right) \bmod n
\end{aligned}$$

but $c_j = j - s_j$. We use this fact and simplify the previous equation to get

$$\begin{aligned} i &= \left(\sum_{\alpha=0}^{j-1} c_\alpha + j - \left(\sum_{\alpha=0}^{n-1} c_\alpha - c_j\right) + \sum_{\alpha=j+1}^{n-1} c_\alpha\right) \bmod n \\ &= \left(\sum_{\alpha=0}^{j-1} c_\alpha + j - \sum_{\alpha=0}^{n-1} c_\alpha + c_j + \sum_{\alpha=j+1}^{n-1} c_\alpha\right) \bmod n \\ &= \left(\sum_{\alpha=0}^{j-1} c_\alpha + c_j + \sum_{\alpha=j+1}^{n-1} c_\alpha + j - \sum_{\alpha=0}^{n-1} c_\alpha\right) \bmod n \\ &= \left(\sum_{\alpha=0}^{n-1} c_\alpha + j - \sum_{\alpha=0}^{n-1} c_\alpha\right) \bmod n \\ &= j. \end{aligned}$$

Thus, strategy $S$ ensures that exactly one player guesses correctly. Since exactly one player guesses correctly from each of the $\left\lfloor \frac{k}{n} \right\rfloor$ groups, we have a total of $\left\lfloor \frac{k}{n} \right\rfloor$ players who will guess correctly. Now we need to show that there does not exist a strategy which ensures that $\left\lfloor \frac{k}{n} \right\rfloor + 1$ players guess correctly. To do this we use Lemma 10 which states that in a $k$-player, $n$-color hat problem, for any particular strategy, the number of players who guess correctly is on average $\frac{k}{n}$. We recall that this average is taken over every possible coloring. We also know that $\frac{k}{n} < \left\lfloor \frac{k}{n} \right\rfloor + 1$, since $0 \leq \frac{k}{n} - \left\lfloor \frac{k}{n} \right\rfloor < 1$. Since the average number of players who guess correctly is $\frac{k}{n}$ for any strategy and $\frac{k}{n} < \left\lfloor \frac{k}{n} \right\rfloor + 1$ we know that there cannot exist a strategy which guarantees that $\left\lfloor \frac{k}{n} \right\rfloor + 1$ players guess correctly for any coloring.■

## 3.2 Infinite Hat Problems and Their Solutions

We have discussed a number of interesting results in the finite case, but none of these results are as counterintuititive as those given for the infinite case. When we refer to a hat problem as being in the infinite case, we mean that the number of players is infinite. The number of possible colors is arbitrary and may be finite or infinite. The following theorems are given for arbitrary player and color sets and are not limited strictly to the infinite case; however, they are much more surprising when seen in that context. As a result of the increased complexity posed by the infinite case, we must use the Axiom of Choice to prove all of the following results. As in the finite case, we wish to determine not only the existence of a minimal solution, but discover how often such a strategy can correctly predict a player's hat color. The conditions presented in the Gabay-O'Connor theorem are the least restrictive of any that we have presented so far.

**Theorem 14 (Gabay-O'Connor)** *Consider the situation in which the set $P$ of players is arbitrary, the set $C$ of colors is arbitrary, and every player can see* *all but finitely many of the other hats. Then there exists a strategy under which all but finitely many players guess correctly. Moreover, the strategy is robust in the sense that each player's guess is unchanged if the colors of finitely many hats are changed [4].*

**Proof (Gabay-O'Connor's Thm)** Let the set of players be arbitrary and the set of possible hat colors be arbitrary. Also, assume that each player can see all but finitely many of the other players' hats. Now let $h, g \in {}^P C$, and define an equivalence relation by $h \approx g$ if $\{a \in P \mid h(a) \neq g(a)\}$ is finite. In other words, $h$ is related to $g$ if the values of $h$ and $g$ are different for only finitely many players. It is clear that for any $h \in {}^P C$ the set $\{a \in P \mid h(a) \neq h(a)\}$ is empty and therefore finite. So $h \approx h$. Similarly, for every $h, g \in {}^P C$ if $h \approx g$, then $\{a \in P \mid h(a) \neq g(a)\}$ is finite, but

$$\{a \in P \mid h(a) \neq g(a)\} = \{a \in P \mid g(a) \neq h(a)\}.$$

Therefore, $\{a \in P \mid g(a) \neq h(a)\}$ is finite as well, and $g \approx h$. Also for any $h, g, f \in {}^P C$ such that $h \approx g$ and $g \approx f$ we know that $\{a \in P \mid h(a) \neq g(a)\}$ is finite and $\{a \in P \mid g(a) \neq f(a)\}$ is finite. The set

$$\{a \in P \mid h(a) \neq f(a)\} \subseteq \{a \in P \mid h(a) \neq g(a)\} \cup \{a \in P \mid g(a) \neq f(a)\}.$$

Fix a coloring $h \in {}^P C$. Now, a result from Math 223 states that a subset of a finite set is finite. A related result states that the union of finite sets is also finite. Therefore, $\{a \in P \mid h(a) \neq f(a)\}$ is finite, and $h \approx f$. Thus, this relation is an equivalence relation on ${}^P C$. This equivalence relation partitions ${}^P C$ into a family of sets or equivalence classes where the elements of these sets are functions from $P$ to $C$. We know that these sets are nonempty since every element is at least related to itself by this equivalence relation. By the Axiom of Choice, we know that there exists a function $\Phi : {}^P C \longrightarrow {}^P C$ such that $\Phi(h) \approx h$, and so that if $h \approx g$, then $\Phi(h) \approx \Phi(g)$. In other words, there is a function $\Phi$ which allows us to choose one element from each equivalence class in ${}^P C$. For the purpose of this project, we will assume that the players can know this choice function. In fact, the players must agree on the choice function before the game begins. Since every element in a given equivalence class is related to every other element in that equivalence class, one element can be chosen from each equivalence class to represent every element in that class. Now, since player $a$ can see all but finitely many hats in a coloring, we know that for some coloring $h$ each player $a$ knows the equivalence class $[h]$. Also, since each player knows the equivalence class $[h]$ each player knows $\Phi(h)$ as well. So, the strategy $S$ is for each player to assume that $\Phi(h)$ is the correct coloring and guess that his hat is the color assigned to him by this coloring. In other words,

$$S(a, h) = \Phi(h)(a).$$

Now, we know that for any coloring $h$, the set $\{b \in P \mid \Phi(h)(a) \neq h(b)\}$ is finite. Then all but finitely many players guess correctly. We can also say that no player will change his guess if finitely many hats change color, since this new coloring will still be in the equivalence class and $\Phi(h)$ does not change. ■

We have just shown how often a strategy may correctly predict the hat colors of a set of players under the very generalized conditions of the Gabay-O'Connor theorem, but what happens if we place some restrictions on the set of possible colors? Will a strategy be more or less effective under such conditions? Surprisingly, the answer is both. Lenstra's theorem and its proof will show the existence of a completely effective strategy or another completely ineffective strategy.

**Theorem 15 (Lenstra)** *Consider the situation in which the set $P$ of players is arbitrary, $|C| = 2$, and assume the visibility graph is complete. Then there exists a strategy under which every player guesses correctly or every player guesses incorrectly [4].*

**Proof (Lenstra's Thm)** Let the set $P$ of players be arbitrary and $|C| = 2$, and let each player see every hat except his own. Now let $S$ be a strategy such that $S(a, h) = \Phi(h)(a)$, and let the function $\Phi$ be defined as in the proof of Gabay-O'Connor's theorem. Now, since each player $a$ knows the value of the choice function $\Phi(h)$ for any coloring $h$, each player $a$ knows each hat color predicted by $\Phi(h)$ and therefore the guess of every other player under $S$. More formally, $a$ knows the value of $S(b, h)$ for every player $b$. We also assumed that each player $a$ knows the value of $h(b)$ for every player $b$ where $b \neq a$. Thus each player $a$ knows if any other player $b$'s guess is incorrect under $S$. Now we let $T$ be a strategy such that a player will keep his guess under $S$ if and only if that player sees an even number of incorrect guesses under $S$. More formally, $T(a, h) = S(a, h)$ if and only if $|\{b \in P \mid b \neq a \text{ and } S(b, h) \neq h(b)\}|$ is even. If

$$|\{b \in P \mid \; S(b, h) \neq h(b)\}| \text{ is even,}$$

then every player guesses correctly under strategy $T$. We see that under $T$ every player whose guess is correct under strategy $S$ sees an even number of players with incorrect guesses and thus keeps the guess given by strategy $S$ and remains correct. This statement is true since every player whose guess is correct under $S$ sees every player whose guess is incorrect under $S$. Conversely, every player whose guess is incorrect under $S$ will see an odd number of players, since such a player counts every player whose guess is incorrect except himself. Thus, under $T$, every player whose guess is incorrect under $S$ will change his guess to the opposite, correct color. Then every player's guess is correct under $T$. Alternatively, if

$$\{b \in P \mid b \neq a \text{ and } S(b, h) \neq h(b)\} \text{ is odd,}$$

every player whose guess is correct under $S$ sees an odd number of incorrect guesses since such a player sees every incorrect guess. Under $T$, each of these players will change change his guess to the opposite, and thus incorrect, color. Every player whose guess is incorrect under $S$ sees an even number of incorrect guesses since such a player does not count his own incorrect guess. So, under $T$, every player whose guess is incorrect will keep his incorrect guess as given by $S$. Thus every player's guess is incorrect under $T$. ■

As demonstrated by the theorems given above, it is often more effective to look at the set of players who guess incorrectly under a given strategy than the set of players who guess correctly when evaluating the effectiveness of the given strategy. Our final theorem and its corollary are reinterpreted versions of Theorem 3.1 and Corollary 3.4 from [3]. These results provide insight into the set of incorrect players under a particular strategy defined below. Our approach to these proofs is slightly different than our previous results. Instead of assuming some set of conditions and proving the existence or nonexistence of a strategy which fulfills those conditions, we began define defining a strategy and proving statements about that strategy. The following results were originally presented in terms of time and the ability to predict the present for a given instant of time. We have reframed these results in terms of hat problems. As a result, we must redefine a few terms as well as introducing a couple of new items.

**Definition 16 (global strategy)** *Let $\lhd$ be some binary relation on $P$. Now for each $x \in P$, we let the equivalence relation $\approx_x$ on ${}^PC$ given by $f \approx_x g$ if and only if $f$ and $g$ agree on $\lhd x = \{y \in P \mid y \lhd x\}$. Then let $[f]_x$ denote the equivalence class of $f$ under $\approx_x$. Now fix a coloring $v$. Let $\mathcal{O} = \{[v]_x \mid x \in P\}$. We define a **global strategy** to be a function $G : \mathcal{O} \longrightarrow {}^PC$ such that $G([v]_x) \in [v]_x$ for $x \in P$ [3].*

**Definition 17 (total order)** *A partial order, $\trianglelefteq$, on a set $S$ is a **total order** iff $\forall a, b \in S$, we have $a \trianglelefteq b$ or $b \trianglelefteq a$ [5].*

**Definition 18 (well-ordering)** *A well-ordering on a set $S$ is a total order on $S$ with the property that every non-empty subset of $S$ has a least element in this ordering.*

**Theorem 19** *The statement that "every set can be well ordered" is equivalent to the Axiom of Choice [5].*

**Definition 20 ($\mu$-strategy)** *Fix a well-ordering $\preceq$ of ${}^PC$. We know that a well-ordering of ${}^PC$ exists as a result of the Axiom of Choice. The **$\mu$-strategy** is the strategy $\mu : \mathcal{O} \longrightarrow {}^PC$ defined by letting $\mu([f]_x)$ be the $\preceq$-least element of $[f]_x$. We abbreviate $\mu([v]_x)$ by $\langle v \rangle_x$ [3].*

**Definition 21** *Fix a true scenario $v \in {}^PC$, and let $\mathcal{W}_0 = \{x \in P \mid \langle v \rangle_x (x) \neq v(x)\}$. In other words, $W_0$ is the set of all players who guess incorrectly using the $\mu$-strategy [3].*

Now that we have defined the necessary sets and terms, we can begin our discussion of our final results.

**Theorem 22** *If $\lhd$ is transitive, then $\mathcal{W}_0$ is well-founded in $\lhd$; that is, $\mathcal{W}_0$ has no infinite descending $\lhd$-chain [3].*

**Proof (Theorem 22)** Let $\lhd$ be a binary relation on $P$, $v$ be the given coloring and let $x_0$, $x_1$, $x_2$, ...be a sequence where $x_i \in \mathcal{W}_0$ and $x_i$ represents player $i$. Also suppose to the contrary that $... \lhd x_2 \lhd x_1 \lhd x_0$, an infinite descending $\lhd$-chain, exists. Thus, for any $i \in \mathbb{N}$, $x_{i+1} \lhd x_i$. We know that $\langle v \rangle_{x_i} \approx_{x_i} v$. This is true since $x_i$ will only choose colorings which are consistent with what he can see as demanded by the $\mu$-strategy. Now, since $x_i$ can see the hat color of every player whose index is greater than his own and since player $x_i$ will only guess colorings which are consistent with what he can see,

$$\langle v \rangle_{x_i} (x_{i+1}) = v(x_{i+1}),$$

but we know that $x_{i+1} \in \mathcal{W}_0$ so

$$\langle v \rangle_{x_i} (x_{i+1}) = v(x_{i+1}) \neq \langle v \rangle_{x_{i+1}} (x_{i+1}).$$

Thus, $\langle v \rangle_{x_i} \neq \langle v \rangle_{x_{i+1}}$. We define $(-\infty, x_i] = \{x \mid x \lhd x_i\}$. Since $\lhd$ is transitive, if for some $z \in \mathcal{W}_0$ we have $z \lhd x_{i+1}$ and $x_{i+1} \lhd x_i$, then $z \lhd x_i$. Since $x_{i+1}$ does not know his own hat color, he may choose colorings which do not color his hat with the same color as the true coloring; however, $x_i$ can see $x_{i+1}$'s hat color and therefore will not choose these colorings which are inconsistent with what he can see. Then $[v]_{x_i} \subseteq [v]_{x_{i+1}}$. Since $\langle v \rangle_{x_i}$ is the $\preceq$- least element of $[v]_{x_i}$ and $[v]_{x_i} \subseteq [v]_{x_{i+1}}$ we know that $\langle v \rangle_{x_{i+1}} \prec \langle v \rangle_{x_i}$, or else $\langle v \rangle_{x_i} = \langle v \rangle_{x_{i+1}}$ which contradicts our previous conclusion. Thus, we have

$$\langle v \rangle_{x_o} \succ \langle v \rangle_{x_1} \succ \langle v \rangle_{x_2} \succ ...,$$

which contradicts the fact that $\preceq$ well-orders ${}^{P}C$.■

We can now use Theorem 21 along with some knowledge of null sets and countable sets to prove the following result. We need Theorem 21 to describe how we may define $\mathcal{W}_0$ as a countable set.

**Corollary 23** *Let $P = \mathbb{R}$ and let $C$ be arbitrary. Also let $V$ be the directed graph defined by the condition that $(x, y)$ is a directed edge in $V$ iff $x, y \in \mathbb{R}$ and $y < x$. Fix a coloring, $v \in {}^{P}C$. For any strategy $S$, define $\mathcal{W}_0 = \{a \in P \mid S(a, h) \neq h(a)\}$. Then if $S$ is the $\mu$-strategy, $\mathcal{W}_0$ is a null set [3].*

**Proof (Corollary 23)** Let $P = \mathbb{R}$ and let $C$ be arbitrary. Also let $V$ be the directed graph defined by the condition that $(x, y)$ is a directed edge in $V$ iff $x, y \in \mathbb{R}$ and $x > y$. Fix a coloring, $v \in {}^{P}C$. For the $\mu$-strategy, define $\mathcal{W}_0 = \{a \in P \mid \langle v \rangle_x (x) \neq v(x)\}$. In Theorem 19 we showed that $\mathcal{W}_0$ has no infinite descending $<$-chains. Thus, $\mathcal{W}_0$ has a least element. Choose some element in $\mathcal{W}_0$, say $x$. Now remove every element from $\mathcal{W}_0$ which is less than or equal to $x$. This new set, $\mathcal{W}_0 - (-\infty, x]$, also has a least element or is empty, call this element $y$. Now $\mathcal{W}_0 \subseteq \mathbb{R}$ and the set $\mathbb{Q}$ is dense in $\mathbb{R}$, so we know that there exists a unique rational number between every element of $\mathcal{W}_0$, specifically there exists some rational number between $x$ and $y$. Call this rational $r_x$. Since the elements of $\mathcal{W}_0$ are distinct we have the relationship $x < r_x < y < r_y$, so

$r_x \neq r_y$. This relationship holds for every element in $\mathcal{W}_0$, so the rational number between each element of $\mathcal{W}_0$ is unique. Thus there exists a bijection between the elements of $\mathcal{W}_0$ and a subset of $\mathbb{Q}$. We know that the set $\mathbb{Q}$ is countable and any subset of $\mathbb{Q}$ is countable, therefore $\mathcal{W}_0$ is countable. Since $\mathcal{W}_0$ is countable we can list its elements, $\mathcal{W}_0 = \{w_1, w_2, w_3, ...\}$. Now let $\varepsilon > 0$ and $n \in \mathbb{N}$. We can create an open interval, $(w_1 - \frac{\varepsilon}{4}, w_1 + \frac{\varepsilon}{4})$, which contains the point $w_1$ and has a length of $\frac{\varepsilon}{2}$. Similarly, we can create an open interval $(w_2 - \frac{\varepsilon}{8}, w_2 + \frac{\varepsilon}{8})$, which contains the point $w_2$ and has a length of $\frac{\varepsilon}{4}$. We can extend this process to get the countable collection of open intervals defined by

$$\left(w_n - \frac{\varepsilon}{2^{n+1}}, w_n + \frac{\varepsilon}{2^{n+1}}\right).$$

We know that $\mathcal{W}_0 \subseteq \bigcup\limits_{n=1}^{\infty} \left(w_n - \frac{\varepsilon}{2^{n+1}}, w_n + \frac{\varepsilon}{2^{n+1}}\right)$ since each open interval, $\left(w_n - \frac{\varepsilon}{2^{n+1}}, w_n + \frac{\varepsilon}{2^{n+1}}\right)$, contains $w_n \in \mathcal{W}_0$. Furthermore, each open interval $\left(w_n - \frac{\varepsilon}{2^{n+1}}, w_n + \frac{\varepsilon}{2^{n+1}}\right)$ has a length of $\frac{\varepsilon}{2^n}$. Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(w_n + \frac{\varepsilon}{2^{n+1}} - (w_n - \frac{\varepsilon}{2^{n+1}})\right) &= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\
&= \sum_{n=1}^{\infty} \varepsilon(\frac{1}{2^n}) \\
&= (1)\varepsilon \\
&\leq \varepsilon.
\end{aligned}$$

Therefore $\mathcal{W}_0$ is a null set.■

# 4 Conclusion

We have shown how changes in the player set, color set, or visibility graph can affect the existence of winning strategies for both finite and infinite hat problems. The results shown here are interesting within the context of hat problems, but they are even more surprising when we relate them to predicting the present. In fact, the last theorem and corollary were originally stated in reference to predicting the present. The phrase, predicting the present, is fairly vague and requires some further explanation. We can think of predicting the present as finding a strategy which can predict the value of an arbitrary function from $\mathbb{R}$ into some set of states for some value $t$ based solely on the values of the function which are strictly less than $t$. It is surprising to see that such a strategy exists and to find that such a strategy often predicts these values correctly. We note, however, that although such a strategy exists, its application is not practical. The Axiom of Choice guarantees the existence of the strategy, but it does not tell us how to formulate such a strategy.

## 5 References

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