Inflating the Platonic Solids while Preserving Distance

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In $\mathbb{R}^2$, it can be shown by the isoperimetric inequality that, of all simple closed curves with length L, the circle encloses the largest area. Similarly, in $\mathbb{R}^3$, we find that the sphere encloses the largest volume under a specified surface area. I am interested in increasing the volumes of the Platonic solids while preserving the distance between points on the surface of the solid. The process of increasing the volume of the solid will be referred to as inflating. Imagine that you are holding a hollow cube with a mylar surface. This cube has a known volume dependent on its side length. We will now pump air into the mylar cube. We ask ourselves how will this inflated solid look, and how does the volume of the inflated solid compare to the volume of the Platonic solid. In this paper, I will outline crucial definitions and theorems, demonstrate that an upper bound exists for the volumes of the inflated solids, provide a construction proof for increasing the volume, and present my constructed solids of increased volumes isometric to its respective Platonic solid. However, before we jump head first into the mathematics, we must first pave our path and develop some essential tools and nomenclature that will assist us later.

1 Definitions and Theorems

The goal of this project is to inflate the Platonic solids while preserving the distance between points on the surface. Hence, we need some way to characterize curves on the surface of the Platonic solids. This is where we will yield to geodesics.

Definition 1.1. [Gal03] *A* geodesic *on the surface of a polyhedron P is a curve γ of locally minimal length which does not pass through the vertices of the polyhedron. It means for that any two points $A, B \in \gamma$ lying close to each other, the length of the part $[A, B]$ of the curve γ with the endpoints A and B is less than the length of any other curve σ on the polyhedron surface with the same endpoints A, B*

Since we do not want to increase or decrease the distance between points on the surface of the solid, we want to make sure that our inflations do not increase or decrease the length of the geodesic segments connecting points.

Definition 1.2. [MP77] *An* isometry *from surface M to surface N is a one-to-one, onto, differentiable function $f : M \to N$ such that for any rectifiable curve $\gamma : [c, d] \to M$, the length of γ equals the length of rectifiable curve $(f \circ \gamma)$. M and N are isometric if such an isometry exists.*

Later in this paper, I will construct a solid of increased volume that is isometric to a Platonic solid. This construction utilizes a hypothesis characterized by the following definition.

Definition 1.3. *A set $C \subseteq \mathbb{R}^n$ is called* convex *if for all $x, y \in C$, $(1-t)x + ty \in C$ for $t \in [0, 1]$.*

We will show that the inflated solid is isometric to a Platonic solid, but it would also be interesting to understand what other properties does this solid maintain.

Definition 1.4. [Gal03] *The* defect of a vertex v on a polyhedron *is*

$$\Delta_v := 2\pi - \sum_{i \in v} \alpha_i$$

where the $\{\alpha_i\}$ *are all the plane angles of the polyhedral angle with vertex* v*.*

Definition 1.5. [Gal03] *The* total curvature *or* defect *of a polyhedron* P *is the sum of the vertices' defects,*

$$\Delta_P = \sum_{v \in P} \Delta_v.$$

Russian mathematician Aleksandr Danilovich Aleksandrov was able to shed some light on the properties of the inflated polyhedra with the following theorem that is used to characterize our inflation. This theorem utilizes the concept of total curvature of polyhedra.

Theorem 1.6 (Alexandrov's Uniqueness Theorem)**.** [DAKS06] *If the total curvature of a convex polyhedron* P_1 *is* 4π*, then every isometric mapping* ϕ *of* P_1 *onto a convex polyhedron* P_2 *can be realized by a motion or a motion and a reflection.*

I will now elaborate on the important theorems that motivate our construction because without them, we are not guaranteed the existence of an isometric surface with greater volume. The most important of these is Theorem 1.11 proven by Pak. Later on in this paper, I will provide a proof of Theorem 1.11 for a restricted case. However, we will need to review the following definitions that are helpful in understanding Pak's construction.

Definition 1.7. [Pak06] *We say that a surface* S *of a polyhedron* P *in* $\mathbb{R}^3$ *is* simplicial *if all of the faces of* P *are triangles.*

The concept of the distance between two points is important when discussing isometry. From this point on, $||\cdot||$ will denote the standard Euclidean metric, and $|\cdot|_S$ will denote the metric on the surface of the solid. Namely, $|\cdot|_S$ is the infimum of the length of all curves between two points on the surface.

Definition 1.8. [Mun00] *Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a* homeomorphism.

Definition 1.9. [Pak06] *We say that a surface $S' \subseteq \mathbb{R}^{\mathrm{d}}$ is* submetric *to S, write $S' \preceq S$, if there exists a homeomorphism $\phi : S \to S'$ which does not increase the geodesic distance: $|x, y|_S \geq |\phi(x), \phi(y)|_{S'}$ for all $x, y \in S$.*

In Pak's construction, he yields to a submetry. This means that the distance between the points on the surface of the inflated solids are less than or equal to the distance between their pre-images. This submetry is later corrected to become an isometry. We are guaranteed the existence of such an isometry by the following theorem.

Theorem 1.10. [Pak06] *Let S_1 be a surface submetric to surface S in $\mathbb{R}^3$, and let $\epsilon > 0$ be any given constant. Then there exist a surface S_2 isometric to S, such that S_2 is in ϵ-neighborhood of S_1, that is, for all $x \in S_1$ there exist $y \in S_2$ such that $||x - y|| < \epsilon$.*

Theorem 1.11. [Pak06] *For every convex simplicial surface S in $\mathbb{R}^3$, there exists a volume increasing bending of S.*

The volume increasing bending utilized by Pak is what we will call an inflation. This process of inflating a solid is a function. Moreover, it is a special type of function called a homotopy.

Definition 1.12. [Mun00] *If f and f' are continuous maps of the space X into the space Y, we say that f is* homotopic *to f' if there is a continuous map $F : X \times I \to Y$ such that*

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$

for each x. (Here $I = [0, 1]$.) The map F is called a homotopy *between f and f'.*

Theorem 1.11 implies that there exist a homotopy that takes a Platonic solid to its inflation. H is a homotopy such that $H : I \times S^2 \to \mathbb{R}^3$ where I is an interval $[0, a]$ and a is dependent upon the Platonic solid with which we are working. Theorem 1.11 is an important result because it is a restriction of our overarching goal since not every Platonic solid is simplicial. In Pak's paper, he is able to extend this to a general case for convex polyhedra.

Theorem 1.13. [Pak06]*For every convex surface S in $\mathbb{R}^3$, there exist a submetric convex surface $S' \sim S$ of greater volume: $vol(S') > vol(S)$.*

Notice that this theorem tell us that for every Platonic solid, we have a submetric convex solid of greater volume. We are not interested in a submetric solid, so will utilize Theorem 1.10 that states an isometry to the Platonic solid is in an ϵ-neighborhood of the convex submetry of greater volume. Now referring back to the Alexandrov's Uniqueness Theorem, this inflated solid is isometric to its respective Platonic solid, but we will see that our bending is not a rigid motion, reflection, or composition of the two. Thus, we can conclude that our inflated solid is non-convex. We will come to find that our construction is an isometry in each face of the polyhedron. However, we will use the following theorem to guarantee that our construction is an isometry.

Theorem 1.14. *If $f_1 \colon U_1 \to \mathbb{R}^3$ and $f_2 \colon U_2 \to \mathbb{R}^3$ are isometries with $f_1(x) = f_2(x)$*

for all $x \in U_1 \cap U_2$*, then* $f\colon U_1 \cup U_2 \to \mathbb{R}^3$ *defined by*

$$f(x) = \begin{cases} f_1(x) & : x \in U_1 \\ f_2(x) & : x \in U_2 \end{cases}$$

is an isometry.

Proof. In order to show that f is an isometry, it is sufficient to show that f is bijective and differentiable. Let $x, y \in U_1 \cap U_2$. Assume that $f(x) = f(y)$. By assumption, $f_1(U_1 \cap U_2) = f_2(U_1 \cap U_2)$, so $f_1(x) = f_2(x) = f_1(y) = f_2(y)$ where f_1 and f_2 are isometries. The cases where $x, y \in U_1$ and $x, y \in U_2$ follow from assumption, so f is injective. We will now show that f is surjective by analyzing two cases.

Case 1: Let $x \in R(f_1)$ and $u \in U_1$ such that $u = f_1^{-1}(x)$ then $f(u) = f_1(u) = f_1 f_1^{-1}(x) = x$

Case 2: Let $x \in R(f_2)$ and $v \in U_2$ such that $v = f_2^{-1}(x)$ then $f(v) = f_2(v) = f_2 f_2^{-1}(x) = x$

We now must show that f is differentiable. We know f_1 is differentiable on U_1 and f_2 is differentiable on U_2, then f_1 and f_2 are both differentiable on $U_1 \cap U_2$ so f is differentiable on $U_1 \cup U_2$. Hence, f is an isometry since f_1 and f_2 preserve the lengths of curves and we showed that f is a differentiable bijection. □

Lastly, we need to guarantee that our inflations do in fact have larger volume. To assist with the volume calculations, we would like to subdivide the volume of our inflated solid into solids of easily calculable volumes. Hence, we will use the following definition.

Definition 1.15. [Mun00] *A subset A of $\mathbb{R}^n$ is said to be* star shaped *if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.*

This definition will allow for us to subdivide the solid into pyramids with the faces of the inflation as bases and a point a_0 as the origin. This concludes all of the necessary definitions and theorems that will be referenced in the construction of inflations for the Platonic solids, and we are now ready to begin our construction of isometric inflations of the Platonic solids, but in order to develop the upper bounds for the inflations we must include one more theorem. This theorem is related to the isoperimetric inequality in $\mathbb{R}^2$. According to Osserman, the isoperimetric inequality can be extended to $\mathbb{R}^n$, but since we are only concerned with the Platonic solids we will only work with its extension to $\mathbb{R}^3$.

Theorem 1.16. [Oss78] *Define $V(A)$ to be the volume of set A, $B_r^n(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$ and $\omega_n = V(B_1^n(0))$. If D is a connected open set in $\mathbb{R}^n$, its volume* V *and surface area* A *are related by*

$$A^n \geq n^n \omega_n V^{n-1}$$

with equality if and only if $D = B_r^n(a)$ for some r and a.

In $\mathbb{R}^2$, the relationship between the length of a closed curve, L, and the area, A, that L encloses can be expressed as $4\pi A \leq L^2$.

For Theorem 1.16, in $\mathbb{R}^3$, $\omega_3 = V(B_1^3(0))$ represents volume of the unit sphere, so $\omega_3 = \frac{4\pi}{3}$. The isoperimetric inequality can then be expressed in terms of surface area, A, as

$$V \leq \frac{A^{\frac{3}{2}}}{6\sqrt{\pi}}. \tag{1.1}$$

The isoperimetric inequality extension to $\mathbb{R}^3$ will serve as an upper bound to the volume of the Platonic solids after applying any bending function. We only have equality in Equation 1.1 when the surface area is 4π. We can bound each unit platonic solid by calculating the surface area and using the isoperimetric inequality extension to $\mathbb{R}^3$. I determined that the inflations of the unit tetrahedron, cube, octahedron, dodecahedron, and icoshedron are bounded above by $\frac{1}{2\sqrt[4]{3}\sqrt{\pi}}$, $\sqrt{\frac{6}{\pi}}$, $\frac{\sqrt{\frac{2}{\pi}}}{\sqrt[4]{3}}$, $\frac{5\sqrt{3}(85+38\sqrt{5})^{\frac{1}{4}}}{2\sqrt{\pi}}$, and $\frac{5\sqrt{\frac{5}{\pi}}}{2\sqrt[4]{3}}$ respectively. Hence, we know that the inflation constructions have a supremum. We will now use to following construction to find lower bounds for the volume inflations.

2 Construction of Isometric Inflations

Pak's construction of inflations is more general then the following proof that restricts his construction to simple, convex, regular polyhedra. It should be noted that Pak's construction requires that P only be simplicial and convex. Since the Platonic solids are regular, this proof is simpler than Pak's because we can omit superfluous details that are not necessary due to their symmetry. I will remark on these details in the following subsection.

Restriction of Theorem 1.11 to the case of simple, convex, regular polyhedron. Let P be a convex simple regular polyhedron in $\mathbb{R}^3$. This means each vertex v of P has degree 3. P consists of vertices v, faces F, and edges e. Let $\epsilon > 0$ be a parameter. A vertex v in a face F is the intersection of two edges e and e'. For each vertex v and edges e and e' containing v, denote $x_{v,F}$ to be the point in the face F of e and e' such that $x_{v,F}$ is ϵ away from e and e'. We know such a point exist because the edges e and e' intersect at v. A line segment from vertex v to $x_{v,F}$ should be created on the surface. This line will be used later in the construction. Let X_F be the polygon consisting of the convex hull of $x_{v,F}$ in a face F. As of this moment in the construction, each face F consists of X_F, two triangles per vertex, and one rectangle per edge defined by an edge and two $x_{v,F}$. This subdivision of P can be seen in Figure 2.1.

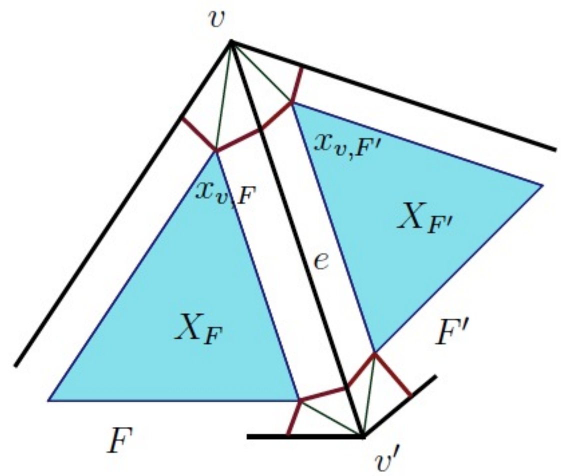


Figure 2.1: Subdivision of the Surface of P [Pak06]

Let T_ϵ be the union of two adjacent rectangles in F and F' that share an edge e. Think of T_ϵ as a creased rectangle. We wish to continuously deform T_ϵ until it is coplanar. After deforming T_ϵ, we see that the $x_{v,F}$'s are mapped to new vertices since they comprise the vertices of T_ϵ. In $\mathbb{R}^3$, using the standard Euclidean metric, the distance, $||x_{v,F} - x_{v,F'}|| \leq 2\epsilon$. This follows from the triangle inequality. We know that there exist line segments that connect $x_{v,F}$ to edge e and $x_{v,F'}$ to edge e of length ϵ. Say that these lines segments intersect e at $e(t_0)$. We will draw a line segment from $x_{v,F}$ to $x_{v,F'}$. Hence we have

$$||x_{v,F} - x_{v,F'}|| \leq |x_{v,F} - e(t_0)|_S + |x_{v,F'} - e(t_0)|_S = 2\epsilon.$$

Thus, we want to push X_F in each face such that the images of $x_{v,F}$'s defined by adjacent faces F and F' that share an edge e have a distance of 2ϵ.

Push each X_F a distance $\delta > 0$ in the direction of the normal to X_F away from the center of P. The new vertices after pushing each $x_{v,F}$ will be denoted $y_{v,F}$ and similarly, we will denote the pushed polygonal faces as Y_F. Let Q be the convex hull of $\{y_{v,F} : v \subset F\}$. Q consists of Y_F, rectangles T'_ϵ, and polygonal faces determined by vertices of $y_{v,F}$ for each F that shares a vertex v. We want the height of T'_ϵ to be less than or equal to the height of T_ϵ. Thus we choose δ such that $||y_{v,F} - y_{v,F'}|| = 2\epsilon$. δ is function of ϵ. Consider two adjacent edges a and b of X_F. We will consider a and b as vectors both sharing the same initial point. The lengths of a and b are a function of epsilon since we required each $x_{v,F}$ be ϵ away from the two edges of F that intersect at v. The normal to F is the cross-product of vectors a and b since both a and b are elements of F. δ is scalar quantity. Hence we have $\delta = ||a \times b|| = ||a|| \cdot ||b|| \sin\theta$ where θ is the angle between a and b. Thus, we have shown that δ is a function of ϵ.

Since P is simple, we note that Q has a triangular surface defined by three adjacent $y_{v,F}$ which are derived from three $x_{v,F}$ sharing a vertex v, we will call this triangle U_v. Consider the union of triangles around a vertex v. This is a triangular cone shape and we want to transform this cone into a pyramid with U_v as its base. This transformation can be seen in Figure 2.2. The lines connecting v to $x_{v,F}$ will serve as

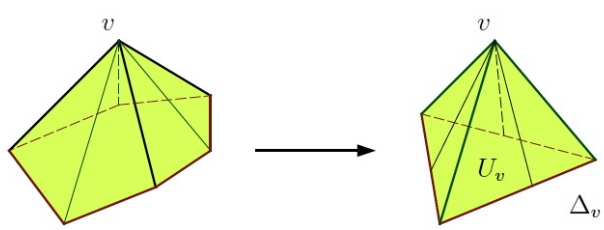


Figure 2.2: Transformation of the Cone [Pak06]

edges to the pyramid. Hence as we push X_F, we see that triangular cone is stretched over P. This construction provides a pyramid with U_v as its base. We have preserved the distances between adjacent $x_{v,F}$'s through this transformation, but we have not preserved the length of the line segment connecting v to the pushed $x_{v,F}$. Since regular polyhedra have rotational symmetry and we are performing the same deformation in every face, we should be able to find a vertex v' such that the pyramid defined by v' and all of the $y_{v,F}$'s around an initial vertex v is isometric to the union of triangles around the vertex v on the surface of P. The new vertex v' for the pyramid is located somewhere on the line connecting the initial vertex v to the center of the solid P. Thus using length of the segments connecting v to $x_{v,F}$, we are able to find a scalar that we can multiply to the vector defining v from the origin to define the new vertex v' that characterizes and isometry. The inflated solids consist of three components, an pushed internal polygonal face, Y_F, from F, flattened rectangles, and pyramids. The distance between points on the pushed faces are preserved as are the distances between points on the rectangles. We have just shown that the pyramid with vertex v' and base U_v is isometric to the union of triangles around a vertex of P, so by Theorem 1.14, we can conclude that the constructed solid is isometric to the initial convex, simple, regular polyhedron. □

2.1 Remark on Theorem 1.11

The theorem proven above is a restriction of the theorem proven by Pak to the case where P is a convex, simple, regular polyhedron. Pak's generalization works for all convex simplicial polyhedra, but I imposed the extra condition of regularity since arguments made by Pak are superfluous for the inflations of the Platonic solids. However, I will discuss Pak's construction starting with the development of the triangular pyramid whose base is U_v. The preliminary construction of subdividing the surface and pushing the internal polygons is the same for the general case. A triangular cone shape can be obtained from the union of triangles around a vertex v. We wish to map this cone onto U_v. We will stretch the cone such that the vertices of the pushed $x_{v,F}$ serve as vertices for a new triangular cone. From the construction, $|x_{v,F} - x_{v,F'}|_S = 2\epsilon$, however for Pak's generalization, we will push each face of X_F such that $|y_{v,F} - y_{v,F'}| \leq 2\epsilon$. We now will shrink each side of the cone from $|x_{v,F} - x_{v,F'}|_S$ to $|y_{v,F} - y_{v,F'}|$. By shrinking the cone we obtain a triangular pyramid that can be seen in Figure 2.2. We are not guaranteed a submetry by shrinking the sides of a triangle, but we will later show that this map can be corrected to become a submetry.

If the vertex v of this pyramid projects inside of U_v, we have a submetry, however there is a possibility for v to project outside of U_v. If v does project outside of U_v, we know that at most two of the dihedral angles at the base of the pyramid are obtuse. I will discuss the case where only one dihedral angle is obtuse. Assume that the dihedral angle of edge e of U_v is obtuse. We can then find a hyperplane H_e through e such that v projects onto the intersection of the hyperplane with the triangular pyramid. Project the remainder of the pyramid that is on the same side of H_e as v onto the intersection of H_e and the pyramid. This projection yields a new pyramid called $\Delta_{v'}$, with a vertex v' that projects onto U_v. Hence, we have a submetry. I will not discuss the case when there are two obtuse dihedral angles.

Let Q_ϵ be the union of all of the $\Delta_{v'}$'s with Q. We will now show that the construction can be corrected to become a submetry while increasing the volume. Assume that Q_ϵ is a submetry of S, the surface of P, and let P' be the convex hull of all of the $x_{v,F}$'s. We then know that the volume of Q_ϵ is bounded below by

$$\mathrm{vol}(Q_\epsilon) \geq \mathrm{vol}(P') + \sum_{F \subset S} \delta(\epsilon)\mathrm{area}(X_F) \tag{2.1}$$

where δ is a function of ϵ because we can say $\delta \leq c\epsilon$ for some c. Now let us look at X_F. The length of each side of X_F is a function of ϵ. Hence, we can say

$$\text{area}(X_F) = \text{area}(F) - O(\epsilon).$$

We can now rewrite Equation 2.1 such that

$$\text{vol}(Q_\epsilon) = \text{vol}(P') + \delta(\epsilon)(\text{area}(S) - O(\epsilon)). \tag{2.2}$$

We will now subdivide P into regions by using hyperplanes. These planes will cut through each U_v and each rectangle defined by $\{x_{v,F}, x_{v',F}, x_{v',F'}, x_{v,F'}\}$. Thus we have regions of three types: the region P', a triangular pyramid for each vertex, and triangular prism for each edge. The length of every edge of the pyramid is a function of ϵ, thus we can conclude the area of U_v is $O(\epsilon^2)$. Also, the height of the pyramid is $O(\epsilon)$. Hence the volume of each pyramid is $O(\epsilon^3)$. The area of the triangular base of the triangular prism is $O(\epsilon^2)$ and each prism has height $|e|$, so the volume of the triangular prism is $O(\epsilon^2)$. Thus, $\text{vol}(P') = \text{vol}(P) - O(\epsilon^2)$, and referring back to Equation 2.2, we conclude

$$\text{vol}(Q_\epsilon) = \text{vol}(P) + \delta(\epsilon)\text{area}(S) - O(\epsilon^2).$$

Hence, the volume is increased for small enough ϵ, and we have shown that a volume increasing submetry of a solid exist. Later in Pak's paper, he is able to remove the required condition of being simplicial for Theorem 1.11 and reduce it to a general case for convex polyhedra. I will not discuss this generalization.

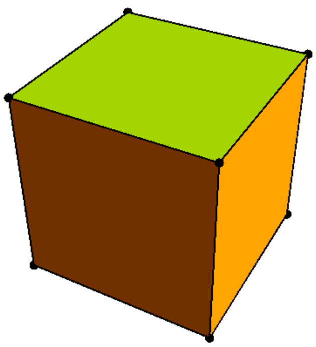
3 Inflations of the Platonic Solids

In this section, we will consider the unit Platonic solids centered at the origin. We will use the construction above to create a non-convex solid that is isometric to the Platonic solid but with increased volume. We know that the inflated solid is isometric to the initial solid by Theorem 1.14. Also, these solids are non-convex by the Alexandrov Uniqueness Theorem (Thm. 1.6) since our inflation is not a motion, reflection, or composition of the two. Now, we must develop a function of ϵ that represents the volume of the solid. This will be done by dividing the solid into pyramids whose bases are the faces of the inflated Platonic solid. The origin will serve as a vertex for every pyramid as well. Since all of the inflated solids are star shaped with the origin as the star point, we are able calculate the volume of the solid by finding the volume of the pyramids with each of face types as bases. The face types are the pushed internal polygonal faces, the rectangles, and the triangular faces the of the pyramids from the construction. Since the Platonic solids are regular, we are able to count the number of pyramids of each type, and hence we can describe the volume of the inflated solid. Now that we have a method of calculating the volume of the solid, we must determine how we are going to find the volume of the pyramids. The volume of a pyramid is $\frac{1}{3}Bh$ where B represents the area of the base and h is the height of the pyramid. The height h is found by finding a unit vector normal to the face and taking the dot product of this unit normal with a point in the face serving as the base for our pyramid of interest.

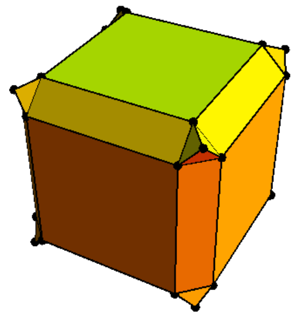
3.1 Cube

The unit cube has a volume of 1 cubic unit. Refer to Table 1 for inflations of the cube with different values of ϵ. The inflation of the cube with the largest volume can be seen in Figure 3.1.

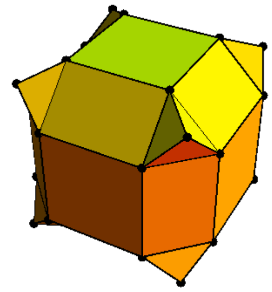
Table 1: Inflations of the Cube



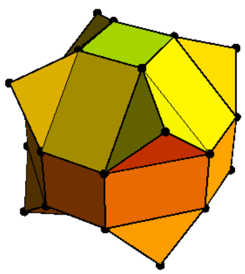
$\epsilon = 0$



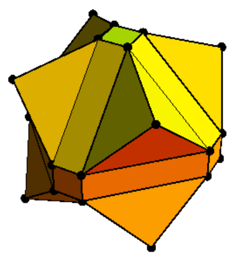
$\epsilon = 0.1$



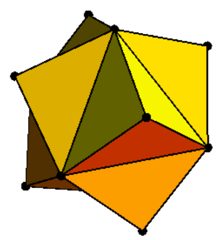
$\epsilon = 0.2$



$\epsilon = 0.3$



$\epsilon = 0.4$



$\epsilon = 0.5$

Now we will determine the volume of the inflated cube as a function of ϵ. This will be done by utilizing the star shaped property of the inflated solid, and subdividing the solid into pyramids. We see there are three types of pyramids in the subdivision of the inflated cube. We have 6 pyramids defined by the internal square face. There are 12 pyramids defined by the rectangles of length $1 - 2\epsilon$ and width 2ϵ. Lastly, each vertex has three pyramids with congruent triangular bases. Hence, there are 24 pyramids with triangular bases. The area of the triangular bases can be determined using Heron's formula.

The distance between the vertices of the pushed X_F's from faces F and F' sharing and edge e needs to be 2ϵ. We pushed each face of X_F some distance δ and the normal

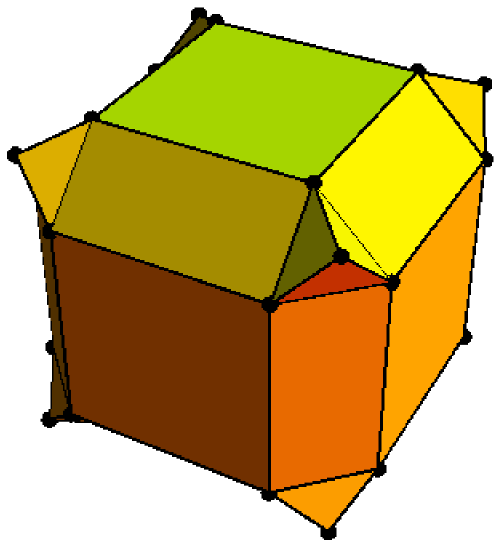


Figure 3.1: Optimum Inflation of the Cube

vectors for adjacent faces on the cube are orthogonal. Hence by the Pythagorean theorem, $(2\epsilon)^2 = 2(\delta+\epsilon)^2$. Thus we arrive at the conclusion that $\delta = (\sqrt{2}-1)\epsilon$. By utilizing the Mathematica code for the cube similar to the one found in the appendix for the octahedron, we conclude that the volume of the inflated cube as a function of ϵ is

$$v(\epsilon) = 1 + 6(\sqrt{2}-1)\epsilon + 24(\sqrt{2}-1)\epsilon^2 + \left(\frac{88\sqrt{2}}{3} - 32\right)\epsilon^3.$$

By plotting this function with the the volume of the cube (Figure 3.2), it is apparent that an interval exists where the volume of the inflated cube is greater than the volume of the cube, and the volume of the inflated cube is maximized at 1.18205 cubic units when $\epsilon \approx 0.163036$. Previously, we bounded the inflation volume above by $\sqrt{\frac{6}{\pi}} \approx 1.38198$ cubic units. Hence there may exist different inflation constructions for the cube that yield greater volume than the volume we obtained from our construction. However, we know that the volume is bounded above from

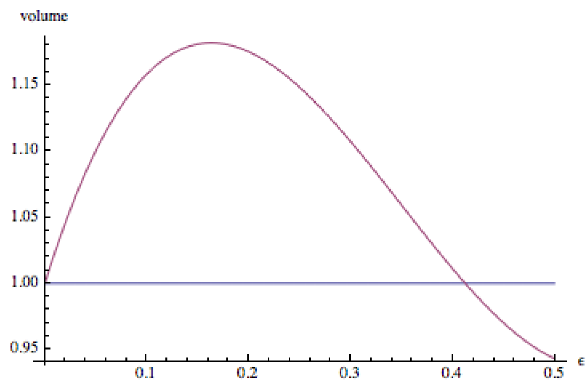
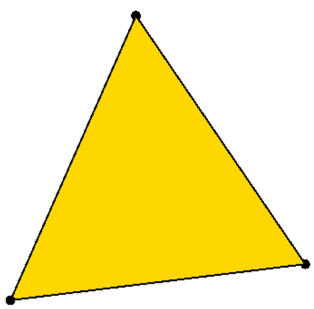
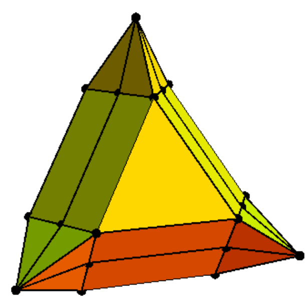
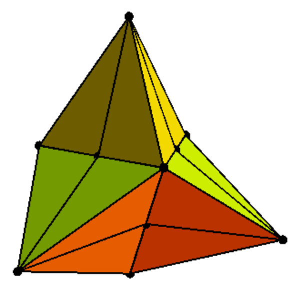


Figure 3.2: Volume of Inflated Cube vs. ϵ

Table 2: Inflations of the Tetrahedron

		
$\epsilon = 0$	$\epsilon = 0.15$	$\epsilon = \frac{1}{2\sqrt{3}}$

the isoperimetric inequality extension.

3.2 Tetrahedron

The area of one face of a tetrahedron with unit length is $\frac{\sqrt{3}}{4}$. Also the height of the tetrahedron is $\sqrt{\frac{2}{3}}$. Hence we conclude the volume of the unit side length tetrahedron is $\frac{1}{6\sqrt{2}}$ since the volume of a solid is $\frac{1}{3}A_B h$ where A_B is the area of the base, and h is the height. The optimum inflation of the tetrahedron can be seen in Figure 3.3, and the inflation of the tetrahedron for different values of ϵ can be seen in Table 2.

There are 4 pyramids with regular triangular bases, 6 pyramids with rectangular bases, and 12 with triangular bases. The volume function of the inflated tetrahedron can be seen in Figure 3.4. Using a numerical approximation, we find that the maximum volume for the inflated tetrahedron is around 0.156032 cubic units. This volume occurs when ϵ is approximately 0.0954153. We have bounded the volumes of the inflated tetrahedrons above by by $\frac{1}{2\sqrt[4]{3}\sqrt{\pi}} \approx 0.214346$, and so we conclude that other inflation constructions may exist that yield volumes larger than ours that is still bounded above by the isoperimetric inequality extension.

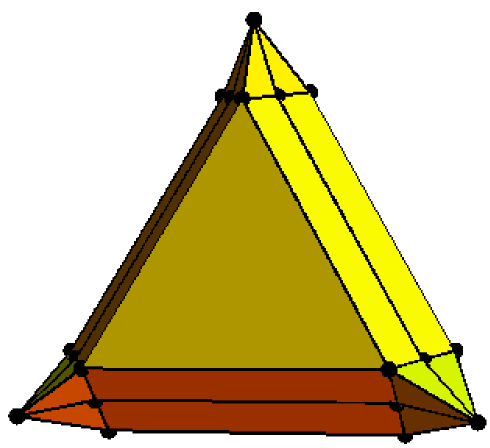


Figure 3.3: Inflated Tetrahedron

3.3 Octahedron

The volume of the unit octahedron is $\frac{\sqrt{2}}{3}$. If we cut the octahedron in half along the edges of unit length that define a square, we obtain two congruent square pyramids. For each of these two pyramids, the area of the base is 1, and the height is $\frac{\sqrt{2}}{2}$. Hence, we conclude that the volume of the unit length octahedron is $\frac{\sqrt{2}}{3}$. The inflations of the octahedron for different values of ϵ can be seen in Table 3. After optimally inflating

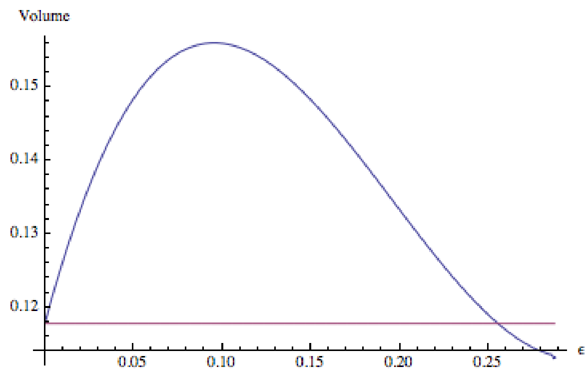
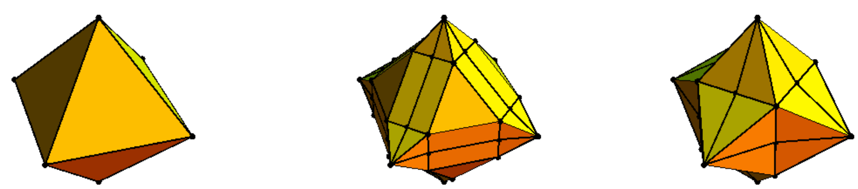


Figure 3.4: Volume of the Tetrahedron vs. ϵ

Table 3: Inflations of the Octahedron



$\epsilon = 0$	$\epsilon = 0.15$	$\epsilon = \frac{1}{2\sqrt{3}}$

the octahedron, we obtain the solid in Figure 3.5. Using a numerical approach, the volume as a function of ϵ is expressed in Figure 3.6.

Clearly, there is an interval in which the volume of the inflated octahedron is greater than that of the octahedron. The maximum inflation occurs at 0.0952238. The volume at this point is 0.518244 cubic units. From the upper bound on the volume of the octahedron we see that the optimum inflation of the octahedron yields a maximum volume between 0.518244 cubic units and $\frac{\sqrt{\frac{2}{\pi}}}{\sqrt[4]{3}} \approx 0.606261$, yet again there is still the possibility for inflation constructions yielding larger volumes bounded above by $\frac{\sqrt{\frac{2}{\pi}}}{\sqrt[4]{3}}$.

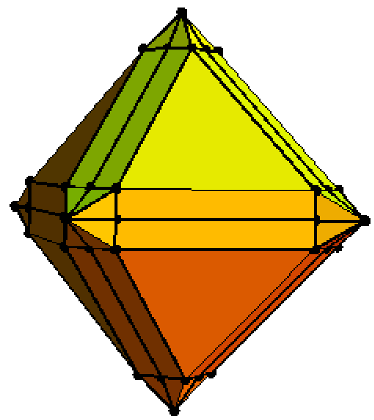


Figure 3.5: Inflated Octahedron

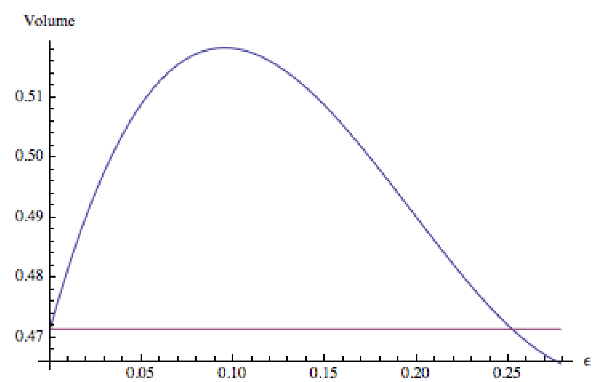
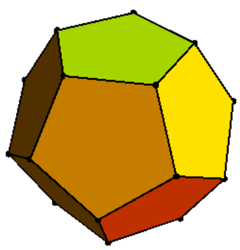


Figure 3.6: Volume of Inflated Octahedron vs. ϵ

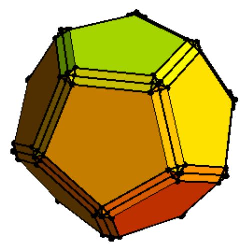
3.4 Dodecahedron

Inflations of the dodecahedron can be found in Table 4. The volume as a function of ϵ can be seen graphically in Figure 3.7. It is also apparent that there is a local extremum. The volume of the dodecahedron is $\frac{15+7\sqrt{5}}{4} \approx 7.66312$ cubic units. This

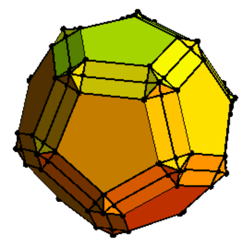
Table 4: Inflations of the Dodecahedron



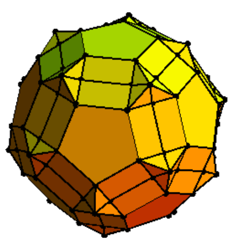
$\epsilon = 0$



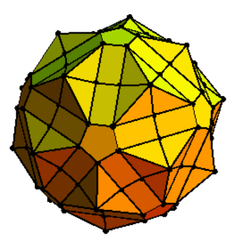
$\epsilon = 0.1$



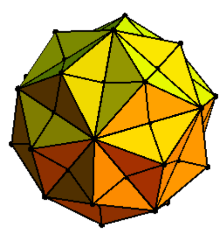
$\epsilon = 0.2$



$\epsilon = 0.3$



$\epsilon = 0.5$



$\epsilon = 0.69$

can be determined using the star shaped property of the dodecahedron. We have 12 regular pentagonal faces. The area of each face is $\frac{\sqrt{25+10\sqrt{5}}}{4}$, and we can determine the height by finding a unit normal on the face and taking the dot product of some point in the face with the unit normal. From the graph in Figure 3.7, we see that there does exist an ϵ such that the volume of the constructed inflation is greater than the volume of the dodecahedron. We know that the construction is isometric to the octahedron. Hence we know that we have achieved our result of inflating the octahedron while preserving distance.

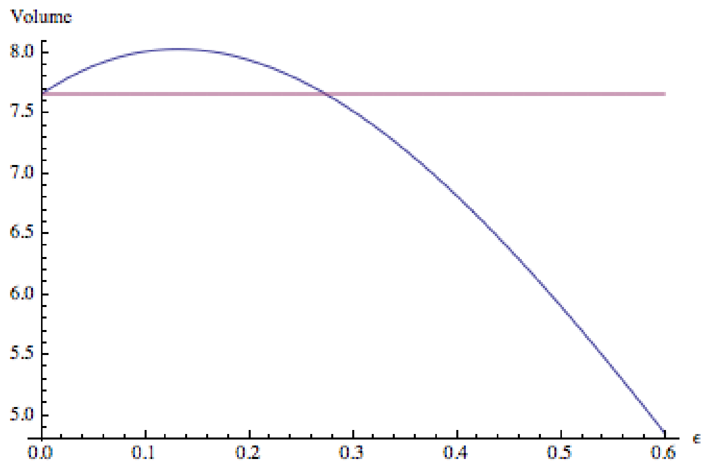
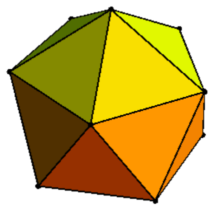
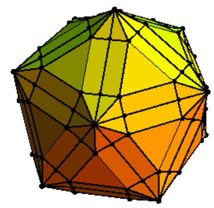
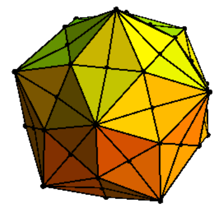


Figure 3.7: Volume of Inflated Dodecahedron vs. ϵ

3.5 Icosahedron

The inflated icosahedrons can be seen in Table 5. By using the star shaped properties of the icosahedron, we see that we have 20 equilateral triangular faces of area $\frac{\sqrt{3}}{4}$ and the height of the pyramids subdividing the volume of the icosahedron is $\frac{3+\sqrt{5}}{4\sqrt{3}}$. The volume of the inflated icosahedron is expressed graphically in Figure 3.8. Similarly to the dodecahedron, we see that a local extremum exist for some ϵ. The volume of the inflated icosahedron is greater than the volume of the icosahedron which is $\frac{5(3+\sqrt{5})}{12}$ for

Table 5: Inflations of the Icosahedron

		
$\epsilon = 0$	$\epsilon = 0.15$	$\epsilon = \frac{1}{2\sqrt{3}}$

some ϵ, and we know that our construction is an isometry. Hence, we have inflated the icosahedron while preserving distance. We also know that an upper bound on the volume of the inflations exist, and we conclude that an inflation exist that yields maximal volume which may or may not be different from our own construction.

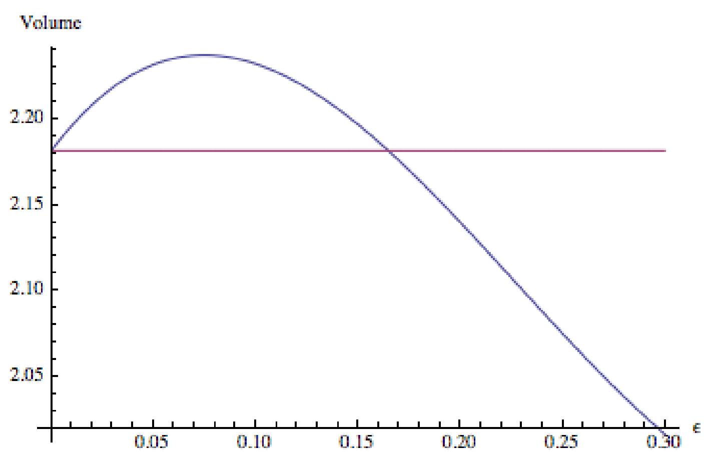


Figure 3.8: Volume of Inflated Icosahedron vs. ϵ

Table 6: Inflation Data

Solid	Volume	Max Inflation	% Inc.	Upper bound	% Inc. Needed
Tetrahedron	$\frac{1}{6\sqrt{2}}$	0.156032	32%	$\frac{1}{2\sqrt[4]{3}\sqrt{\pi}}$	82%
Cube	1	1.18205	18%	$\sqrt{\frac{6}{\pi}}$	38%
Octahedron	$\frac{\sqrt{2}}{3}$	0.518244	10%	$\frac{\sqrt{\frac{2}{\pi}}}{\sqrt[4]{3}}$	29%
Dodecahedron	$\frac{15+7\sqrt{5}}{4}$	≈ 8.00	$\approx 4\%$	$\frac{5\sqrt{3}(85+38\sqrt{5})^{\frac{1}{4}}}{2\sqrt{\pi}}$	15%
Icosahedron	$\frac{5(3+\sqrt{5})}{12}$	≈ 2.235	$\approx 2\%$	$\frac{5\sqrt{\frac{5}{\pi}}}{2\sqrt[4]{3}}$	9%

4 Conclusion

In this paper, we have shown that non-convex polyhedra isometric to the Platonic solids exist. More importantly, we have shown that these polyhedra have volumes larger than the Platonic solids for some ϵ parameter of the inflation. Also, by Osserman's Isoperimetric Inequality extension, we are able to bound the volumes of isometric polyhedra conceived from different inflation constructions. A summary of important data obtained from our constructions can be seen in Tabel 6. I have included percentages by which our inflations have increased the volume of the solids, and the percent increases necessary to meet the volume of the upper bounds. The construction shown within this paper is but one of many other potential inflation constructions. This leaves opportunities for others to research different inflation constructions that may be better or worse than our construction. Also, a more challenging question would be to find the best inflation construction because we have guaranteed that one exist. Lastly, there is also potential research in determining if our outlined construction applies to non-convex solids provided self-intersection does not occur.

5 Appendix: Octahedron Mathematica Construction

I have developed Mathematica code that inflates each one of the Platonic solids by our construction. I will provide the code for the octahedron inflation and the dodecahedron inflation, but for the sake of brevity, I will not include the Mathematica code for all five Platonic solids.

5.1 Octahedron Inflation

```
In[21]:= octverts = PolyhedronData["Octahedron", "VertexCoordinates"]; (*Utilizes built-
          in data for the vertices of the unit octahedron centered at the origin*)

         midpoints = {octverts[[1]] + (1 / 2) (octverts[[2]] - octverts[[1]]),
            octverts[[1]] + (1 / 2) (octverts[[3]] - octverts[[1]]),
            octverts[[1]] + (1 / 2) (octverts[[4]] - octverts[[1]]),
            octverts[[1]] + (1 / 2) (octverts[[5]] - octverts[[1]]),
            octverts[[6]] + (1 / 2) (octverts[[2]] - octverts[[6]]),
            octverts[[6]] + (1 / 2) (octverts[[3]] - octverts[[6]]),
            octverts[[6]] + (1 / 2) (octverts[[4]] - octverts[[6]]),
            octverts[[6]] + (1 / 2) (octverts[[5]] - octverts[[6]]),
            octverts[[3]] + (1 / 2) (octverts[[2]] - octverts[[3]]),
            octverts[[4]] + (1 / 2) (octverts[[2]] - octverts[[4]]),
            octverts[[3]] + (1 / 2) (octverts[[5]] - octverts[[3]]),
            octverts[[4]] + (1 / 2) (octverts[[5]] - octverts[[4]])};
         verts = Join[octverts, midpoints];
         (*Calculates the midpoints of each edge of the octahedron. These
          midpoints are joined with the vertex coordinates in "verts"*)

In[24]:= Clear[ϵ];
         ϵ;
         everts = {verts[[1]] + 2 ϵ (verts[[15]] - verts[[1]]) / Norm[verts[[15]] - verts[[1]]],
            verts[[2]] + 2 ϵ (verts[[8]] - verts[[2]]) / Norm[verts[[8]] - verts[[2]]],
            verts[[3]] + 2 ϵ (verts[[7]] - verts[[3]]) / Norm[verts[[7]] - verts[[3]]],
            verts[[1]] + 2 ϵ (verts[[16]] - verts[[1]]) / Norm[verts[[16]] - verts[[1]]],
            verts[[4]] + 2 ϵ (verts[[7]] - verts[[4]]) / Norm[verts[[7]] - verts[[4]]],
            verts[[2]] + 2 ϵ (verts[[9]] - verts[[2]]) / Norm[verts[[9]] - verts[[2]]],
            verts[[1]] + 2 ϵ (verts[[18]] - verts[[1]]) / Norm[verts[[18]] - verts[[1]]],
            verts[[5]] + 2 ϵ (verts[[9]] - verts[[5]]) / Norm[verts[[9]] - verts[[5]]],
            verts[[4]] + 2 ϵ (verts[[10]] - verts[[4]]) / Norm[verts[[10]] - verts[[4]]],
            verts[[1]] + 2 ϵ (verts[[17]] - verts[[1]]) / Norm[verts[[17]] - verts[[1]]],
            verts[[3]] + 2 ϵ (verts[[10]] - verts[[3]]) / Norm[verts[[10]] - verts[[3]]],
            verts[[5]] + 2 ϵ (verts[[8]] - verts[[5]]) / Norm[verts[[8]] - verts[[5]]],
            verts[[6]] + 2 ϵ (verts[[17]] - verts[[6]]) / Norm[verts[[17]] - verts[[6]]],
            verts[[5]] + 2 ϵ (verts[[12]] - verts[[5]]) / Norm[verts[[12]] - verts[[5]]],
            verts[[3]] + 2 ϵ (verts[[14]] - verts[[3]]) / Norm[verts[[14]] - verts[[3]]],
            verts[[6]] + 2 ϵ (verts[[18]] - verts[[6]]) / Norm[verts[[18]] - verts[[6]]],
            verts[[4]] + 2 ϵ (verts[[14]] - verts[[4]]) / Norm[verts[[14]] - verts[[4]]],
            verts[[5]] + 2 ϵ (verts[[13]] - verts[[5]]) / Norm[verts[[13]] - verts[[5]]],
            verts[[6]] + 2 ϵ (verts[[16]] - verts[[6]]) / Norm[verts[[16]] - verts[[6]]],
            verts[[2]] + 2 ϵ (verts[[13]] - verts[[2]]) / Norm[verts[[13]] - verts[[2]]],
            verts[[4]] + 2 ϵ (verts[[11]] - verts[[4]]) / Norm[verts[[11]] - verts[[4]]],
            verts[[6]] + 2 ϵ (verts[[15]] - verts[[6]]) / Norm[verts[[15]] - verts[[6]]],
            verts[[3]] + 2 ϵ (verts[[11]] - verts[[3]]) / Norm[verts[[11]] - verts[[3]]],
            verts[[2]] + 2 ϵ (verts[[12]] - verts[[2]]) / Norm[verts[[12]] - verts[[2]]]};
         (*Calculates the internal polygonal vertices x_{v,F} as a function of ϵ*)
```

In[27]:=
```
Clear[δ];
δ;
push[list_] := Module[{pushvec = {0, 0, 0}, newlist = {}},
  For[i = 1, i <= Length[list], i = i + 3,
   pushvec = δ (Cross[list[[i + 1]] - list[[i]], list[[i + 2]] - list[[i]]]) / Simplify[
       Norm[Cross[list[[i + 1]] - list[[i]], list[[i + 2]] - list[[i]]]], ϵ ∈ Reals];
   AppendTo[newlist, list[[i]] + pushvec];
   AppendTo[newlist, list[[i + 1]] + pushvec];
   AppendTo[newlist, list[[i + 2]] + pushvec];
  ];
  newlist
 ]
(*This is the pushing parameter δ determined
 by the cross product to two edge vectors of X_F*)

intpoly = Join[octverts, push[everts]]; (*This is the compilation
 of the pushed faces of X_F and the vertices of the octahedron*)

flapverts = {intpoly[[7]] + .5 (intpoly[[16]] - intpoly[[7]]),
   intpoly[[9]] + .5 (intpoly[[17]] - intpoly[[9]]),
   intpoly[[16]] + .5 (intpoly[[13]] - intpoly[[16]]),
   intpoly[[18]] + .5 (intpoly[[14]] - intpoly[[18]]),
   intpoly[[13]] + .5 (intpoly[[10]] - intpoly[[13]]),
   intpoly[[15]] + .5 (intpoly[[11]] - intpoly[[15]]),
   intpoly[[7]] + .5 (intpoly[[10]] - intpoly[[7]]),
   intpoly[[8]] + .5 (intpoly[[12]] - intpoly[[8]]),
   intpoly[[8]] + .5 (intpoly[[30]] - intpoly[[8]]),
   intpoly[[9]] + .5 (intpoly[[29]] - intpoly[[9]]),
   intpoly[[17]] + .5 (intpoly[[21]] - intpoly[[17]]),
   intpoly[[18]] + .5 (intpoly[[20]] - intpoly[[18]]),
   intpoly[[14]] + .5 (intpoly[[24]] - intpoly[[14]]),
   intpoly[[15]] + .5 (intpoly[[23]] - intpoly[[15]]),
   intpoly[[11]] + .5 (intpoly[[27]] - intpoly[[11]]),
   intpoly[[12]] + .5 (intpoly[[26]] - intpoly[[12]]),
   intpoly[[26]] + .5 (intpoly[[30]] - intpoly[[26]]),
   intpoly[[25]] + .5 (intpoly[[28]] - intpoly[[25]]),
   intpoly[[29]] + .5 (intpoly[[21]] - intpoly[[29]]),
   intpoly[[28]] + .5 (intpoly[[19]] - intpoly[[28]]),
   intpoly[[20]] + .5 (intpoly[[24]] - intpoly[[20]]),
   intpoly[[19]] + .5 (intpoly[[22]] - intpoly[[19]]),
   intpoly[[23]] + .5 (intpoly[[27]] - intpoly[[23]]),
   intpoly[[22]] + .5 (intpoly[[25]] - intpoly[[22]])};
(*These points serve only aestheic purposes. They are the midpoints
 of line segment connecting adjacent pushed x_{v,F}'s *)
```

```
finverts = Join[intpoly, flapverts];
(*Compiles all important vertices into one variable*)
ratio = (Norm[(finverts[[22]] + finverts[[25]] + finverts[[28]] + finverts[[19]]) / 4] +
      Sqrt[2] ϵ) / Norm[finverts[[6]]];
(*This scalar quantity will adjust the initial vertices to
 maintain an isometry with the initial solid*)
gfinverts = Join[Join[ratio * octverts, push[everts]], flapverts];
(*Defines the vertex set used for the remainder of the notebook*)
ivfin = {{7, 8, 9}, {10, 11, 12}, {13, 14, 15}, {16, 17, 18}, {19, 20, 21}, {22, 23, 24},
    {25, 26, 27}, {28, 29, 30}, {7, 31, 32, 9}, {9, 40, 39, 8}, {8, 38, 37, 7}, {37, 7, 1},
    {7, 31, 1}, {9, 32, 3}, {9, 40, 3}, {8, 39, 2}, {8, 38, 2}, {10, 37, 38, 12},
    {12, 46, 45, 11}, {11, 36, 35, 10}, {10, 35, 1}, {10, 1, 37}, {12, 38, 2}, {12, 46, 2},
    {11, 36, 4}, {11, 45, 4}, {13, 35, 36, 15}, {15, 44, 43, 14}, {14, 34, 33, 13},
    {13, 35, 1}, {13, 33, 1}, {15, 4, 36}, {15, 4, 44}, {14, 5, 34}, {14, 5, 43},
    {18, 42, 41, 17}, {16, 33, 34, 18}, {17, 32, 31, 16}, {16, 31, 1}, {16, 1, 33},
    {18, 34, 5}, {18, 5, 42}, {17, 3, 32}, {17, 3, 41}, {28, 48, 47, 30}, {30, 39, 40, 29},
    {29, 49, 50, 28}, {28, 50, 6}, {28, 6, 48}, {30, 47, 2}, {30, 39, 2}, {29, 49, 3},
    {29, 40, 3}, {19, 50, 49, 21}, {21, 41, 42, 20}, {20, 51, 52, 19}, {19, 52, 6},
    {19, 6, 50}, {21, 49, 3}, {21, 41, 3}, {20, 42, 5}, {20, 51, 5}, {22, 52, 51, 24},
    {24, 43, 44, 23}, {23, 53, 54, 22}, {22, 54, 6}, {22, 52, 6}, {24, 51, 5}, {24, 5, 43},
    {23, 44, 4}, {23, 4, 53}, {25, 54, 53, 27}, {27, 45, 46, 26}, {26, 47, 48, 25},
    {25, 48, 6}, {25, 6, 54}, {27, 53, 4}, {27, 45, 4}, {26, 46, 2}, {26, 47, 2}};
 (*Describes the relationships between vertices to develop
 the faces of the inflated solid*)

Show[Graphics3D[{Opacity[1], Yellow, GraphicsComplex[gfinverts, Polygon[ivfin]]}],
    Graphics3D[{Thick, GraphicsComplex[gfinverts, Line[ivfin]]}],
    Graphics3D[{PointSize[0.02], Tooltip[Point[gfinverts[[#]]], #]}] & /@
     Range[Length[gfinverts]], Boxed → False] /.
   Solve[Norm[gfinverts[[13]] - gfinverts[[10]]] == 2 ϵ, δ][[2]] /.
 ϵ → .999 / (2 Sqrt[3]); (*Graphic that Shows the Inflated Octahedron*)

a1 = Cross[(gfinverts[[12]] - gfinverts[[10]]), (gfinverts[[11]] - gfinverts[[10]])] /
    Norm[Cross[(gfinverts[[12]] - gfinverts[[10]]),
      (gfinverts[[11]] - gfinverts[[10]])]];
(*Determines the height of the pryamid with the pushed face of X_F as the base*)
v1 = (1 / 3) * (Sqrt[3] / 4) * (1 - 2 Sqrt[3] ϵ) ^2 * Abs[(a1.gfinverts[[10]])];
(*Find the Volume of the Pyramid*)

a2 = Cross[(gfinverts[[10]] - gfinverts[[13]]),
    (gfinverts[[15]] - gfinverts[[13]])] / Norm[
    Cross[(gfinverts[[10]] - gfinverts[[13]]), (gfinverts[[15]] - gfinverts[[13]])]];
(*Determines the height of the pryamid with the trapezoidal face as the base*)
v2 = (1 / 3) * (2 ϵ) (1 - 2 Sqrt[3] ϵ) * Abs[a2.gfinverts[[13]]];
(*Finds the Volume of the Pyramid*)

a3 = Cross[(gfinverts[[10]] - gfinverts[[1]]),
     (gfinverts[[13]] - gfinverts[[1]])] / Norm[
     Cross[(gfinverts[[10]] - gfinverts[[1]]), (gfinverts[[13]] - gfinverts[[1]])]];
(*Determines the height of the pyramid with one of the triangular
 faces defining the vertex pyramid as its base.*)
v3 = (1 / 3) * (Sqrt[3] / 4) (2 ϵ) ^2 * Abs[a3.gfinverts[[1]]];
(*Finds the Volume of the Pyramid*)
```

In[43]:=
```
v[e_] := Module[{d},
  d = δ /. (Solve[Norm[gfinverts[[13]] - gfinverts[[10]]] == 2 ϵ /. ϵ → e, δ][[2]]);
  If[e >= 1/(2 √3) - .001, .47, (8 * v1 + 12 * v2 + 24 * v3) /. {ϵ → e, δ → d} ]
 ] (*Numerical definition of the volume function*)

Plot[{v[e], Sqrt[2] / 3}, {e, 0.0001, 1/(2 √3) - .01}, AxesLabel → {ϵ, Volume}];
(*Graphs the Volume function*)
Maximize[v[e], e ]; (*Numerically determines the maximum of the Volume Function*)
```

5.2 Dodecahedron Inflation

$$\text{dodecV} = \Big\{\Big\{-\sqrt{1+\frac{2}{\sqrt{5}}},\ 0,\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big]\Big\},$$

$$\Big\{\sqrt{1+\frac{2}{\sqrt{5}}},\ 0,\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 2\big]\Big\},$$

$$\Big\{\text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 1\big],\ \frac{1}{4}\left(-3-\sqrt{5}\right),\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big]\Big\},$$

$$\Big\{\text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 1\big],\ \frac{1}{4}\left(3+\sqrt{5}\right),\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big]\Big\},$$

$$\Big\{\sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}},\ \frac{1}{4}\left(-1-\sqrt{5}\right),\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big]\Big\},$$

$$\Big\{\sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}},\ \frac{1}{4}\left(1+\sqrt{5}\right),\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big]\Big\},$$

$$\Big\{\text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 2\big],\ \frac{1}{4}\left(-1-\sqrt{5}\right),\ \sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}}\Big\},$$

$$\Big\{\text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 2\big],\ \frac{1}{4}\left(1+\sqrt{5}\right),\ \sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}}\Big\},$$

$$\Big\{-\frac{1}{2}\sqrt{1+\frac{2}{\sqrt{5}}},\ -\frac{1}{2},\ \text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big]\Big\},$$

$$\Big\{-\frac{1}{2}\sqrt{1+\frac{2}{\sqrt{5}}},\ \frac{1}{2},\ \text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big]\Big\},\ \Big\{\sqrt{\frac{1}{4}+\frac{1}{2\sqrt{5}}},\ -\frac{1}{2},\ \sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}}\Big\},$$

$$\Big\{\sqrt{\frac{1}{4}+\frac{1}{2\sqrt{5}}},\ \frac{1}{2},\ \sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}}\Big\},\ \Big\{\sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)},\ 0,\ \text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big]\Big\},$$

$$\Big\{\text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big],\ \frac{1}{4}\left(-1-\sqrt{5}\right),\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 2\big]\Big\},$$

$$\Big\{\text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big],\ \frac{1}{4}\left(1+\sqrt{5}\right),\ \text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 2\big]\Big\},$$

$$\Big\{\text{Root}\big[1-5\,\#1^2+5\,\#1^4\,\&,\ 1\big],\ 0,\ \sqrt{\frac{5}{8}+\frac{11}{8\sqrt{5}}}\Big\},$$

$$\Big\{\text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big],\ \frac{1}{4}\left(-1-\sqrt{5}\right),\ \text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big]\Big\},$$

$$\Big\{\text{Root}\big[1-20\,\#1^2+80\,\#1^4\,\&,\ 3\big],\ \frac{1}{4}\left(1+\sqrt{5}\right),\ \text{Root}\big[1-100\,\#1^2+80\,\#1^4\,\&,\ 1\big]\Big\},$$

$$\left\{\sqrt{\frac{1}{8}+\frac{1}{8\sqrt{5}}},\ \frac{1}{4}\left(-3-\sqrt{5}\right),\ \text{Root}\left[1-20\,\#1^2+80\,\#1^4\,\&,\,2\right]\right\},$$

$$\left\{\sqrt{\frac{1}{8}+\frac{1}{8\sqrt{5}}},\ \frac{1}{4}\left(3+\sqrt{5}\right),\ \text{Root}\left[1-20\,\#1^2+80\,\#1^4\,\&,\,2\right]\right\}\};$$

```
(*Utilizes built in Mathematica Data for the vertices
 of the Dodecahedron of unit
 length centered at the origin*)

(*Determines the midpoints of each
 edge used to find vertices of internal polygon*)
midpoint = {dodecV[[11]] +
     (1 / 2) * (dodecV[[5]] - dodecV[[11]]) / Norm[dodecV[[5]] - dodecV[[11]]],
   dodecV[[2]] + (1 / 2) * (dodecV[[5]] - dodecV[[2]]) / Norm[dodecV[[5]] - dodecV[[2]]],
   dodecV[[6]] + (1 / 2) * (dodecV[[2]] - dodecV[[6]]) / Norm[dodecV[[2]] - dodecV[[6]]],
   dodecV[[12]] + (1 / 2) *
     (dodecV[[6]] - dodecV[[12]]) / Norm[dodecV[[6]] - dodecV[[12]]], dodecV[[11]] +
    (1 / 2) * (dodecV[[12]] - dodecV[[11]]) / Norm[dodecV[[12]] - dodecV[[11]]],
   dodecV[[13]] + (1 / 2) * (dodecV[[2]] - dodecV[[13]]) /
     Norm[dodecV[[2]] - dodecV[[13]]], dodecV[[18]] +
    (1 / 2) * (dodecV[[13]] - dodecV[[18]]) / Norm[dodecV[[13]] - dodecV[[18]]],
   dodecV[[20]] + (1 / 2) * (dodecV[[18]] - dodecV[[20]]) /
     Norm[dodecV[[18]] - dodecV[[20]]], dodecV[[6]] + (1 / 2) *
     (dodecV[[20]] - dodecV[[6]]) / Norm[dodecV[[20]] - dodecV[[6]]], dodecV[[4]] +
    (1 / 2) * (dodecV[[20]] - dodecV[[4]]) / Norm[dodecV[[20]] - dodecV[[4]]],
   dodecV[[4]] + (1 / 2) * (dodecV[[8]] - dodecV[[4]]) / Norm[dodecV[[8]] - dodecV[[4]]],
   dodecV[[12]] + (1 / 2) *
     (dodecV[[8]] - dodecV[[12]]) / Norm[dodecV[[8]] - dodecV[[12]]], dodecV[[18]] +
    (1 / 2) * (dodecV[[10]] - dodecV[[18]]) / Norm[dodecV[[10]] - dodecV[[18]]],
   dodecV[[10]] + (1 / 2) * (dodecV[[15]] - dodecV[[10]]) /
     Norm[dodecV[[15]] - dodecV[[10]]], dodecV[[15]] +
    (1 / 2) * (dodecV[[4]] - dodecV[[15]]) / Norm[dodecV[[4]] - dodecV[[15]]],
   dodecV[[13]] + (1 / 2) * (dodecV[[17]] - dodecV[[13]]) /
     Norm[dodecV[[17]] - dodecV[[13]]], dodecV[[9]] +
    (1 / 2) * (dodecV[[17]] - dodecV[[9]]) / Norm[dodecV[[17]] - dodecV[[9]]],
   dodecV[[9]] + (1 / 2) * (dodecV[[10]] - dodecV[[9]]) /
     Norm[dodecV[[10]] - dodecV[[9]]], dodecV[[9]] +
    (1 / 2) * (dodecV[[14]] - dodecV[[9]]) / Norm[dodecV[[14]] - dodecV[[9]]],
   dodecV[[1]] + (1 / 2) * (dodecV[[14]] - dodecV[[1]]) /
     Norm[dodecV[[14]] - dodecV[[1]]], dodecV[[1]] +
    (1 / 2) * (dodecV[[15]] - dodecV[[1]]) / Norm[dodecV[[15]] - dodecV[[1]]],
   dodecV[[19]] + (1 / 2) * (dodecV[[17]] - dodecV[[19]]) /
     Norm[dodecV[[17]] - dodecV[[19]]], dodecV[[19]] +
    (1 / 2) * (dodecV[[3]] - dodecV[[19]]) / Norm[dodecV[[3]] - dodecV[[19]]],
   dodecV[[3]] + (1 / 2) * (dodecV[[14]] - dodecV[[3]]) /
     Norm[dodecV[[14]] - dodecV[[3]]], dodecV[[5]] +
    (1 / 2) * (dodecV[[19]] - dodecV[[5]]) / Norm[dodecV[[19]] - dodecV[[5]]],
   dodecV[[3]] + (1 / 2) * (dodecV[[7]] - dodecV[[3]]) / Norm[dodecV[[7]] - dodecV[[3]]],
   dodecV[[7]] + (1 / 2) *
     (dodecV[[11]] - dodecV[[7]]) / Norm[dodecV[[11]] - dodecV[[7]]], dodecV[[7]] +
    (1 / 2) * (dodecV[[16]] - dodecV[[7]]) / Norm[dodecV[[16]] - dodecV[[7]]],
   dodecV[[8]] + (1 / 2) * (dodecV[[16]] - dodecV[[8]]) /
```

```
      Norm[dodecV[[16]] - dodecV[[8]]], dodecV[[1]] +
    (1 / 2) * (dodecV[[16]] - dodecV[[1]]) / Norm[dodecV[[16]] - dodecV[[1]]]};
(*Finds the midpoints of each edge
 of
 the
 dodecahedron*)
extvert = Join[dodecV, midpoint];
(*Joins the midpoints and vertices into a single variable*)
Show[ Graphics3D[{Opacity[1], Yellow, GraphicsComplex[extvert, Polygon[ivb]]}],
  Graphics3D[{Thick, GraphicsComplex[extvert, Line[ivb]]}],
  Graphics3D[{PointSize[0.03], Tooltip[Point[extvert[[#]]], #]}] & /@ Range[50],
  Boxed → False];


inverts = {extvert[[1]] + (ϵ / (1/4 (1 + √5))) *
      (extvert[[38]] - extvert[[1]]) / Norm[(extvert[[38]] - extvert[[1]])],
    extvert[[14]] + (ϵ / (1/4 (1 + √5))) * (extvert[[34]] - extvert[[14]]) /
      Norm[(extvert[[34]] - extvert[[14]])], extvert[[9]] + (ϵ / (1/4 (1 + √5))) *
     (extvert[[41]] - extvert[[9]]) / Norm[(extvert[[41]] - extvert[[9]])],
    extvert[[10]] + (ϵ / (1/4 (1 + √5))) * (extvert[[40]] - extvert[[10]]) /
      Norm[(extvert[[40]] - extvert[[10]])], extvert[[15]] + (ϵ / (1/4 (1 + √5))) *
     (extvert[[39]] - extvert[[15]]) / Norm[(extvert[[39]] - extvert[[15]])],
    extvert[[15]] + (ϵ / (1/4 (1 + √5))) * (extvert[[49]] - extvert[[15]]) /
      Norm[(extvert[[49]] - extvert[[15]])], extvert[[4]] + (ϵ / (1/4 (1 + √5))) *
     (extvert[[50]] - extvert[[4]]) / Norm[(extvert[[50]] - extvert[[4]])],
    extvert[[8]] + (ϵ / (1/4 (1 + √5))) * (extvert[[41]] - extvert[[8]]) /
      Norm[(extvert[[41]] - extvert[[8]])], extvert[[16]] + (ϵ / (1/4 (1 + √5))) *
     (extvert[[35]] - extvert[[16]]) / Norm[(extvert[[35]] - extvert[[16]])],
    extvert[[1]] + (ϵ / (1/4 (1 + √5))) * (extvert[[31]] - extvert[[1]]) /
      Norm[(extvert[[31]] - extvert[[1]])], extvert[[1]] + (ϵ / (1/4 (1 + √5))) *
     (extvert[[46]] - extvert[[1]]) / Norm[(extvert[[46]] - extvert[[1]])],
    extvert[[16]] + (ϵ / (1/4 (1 + √5))) * (extvert[[44]] - extvert[[16]]) /
      Norm[(extvert[[44]] - extvert[[16]])], extvert[[7]] + (ϵ / (1/4 (1 + √5))) *
     (extvert[[40]] - extvert[[7]]) / Norm[(extvert[[40]] - extvert[[7]])],
    extvert[[3]] + (ϵ / (1/4 (1 + √5))) * (extvert[[50]] - extvert[[3]]) /
```

$$
\begin{aligned}
&\text{Norm}[(\text{extvert}[[50]]-\text{extvert}[[3]])],\ \text{extvert}[[14]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[48]]-\text{extvert}[[14]])/\text{Norm}[(\text{extvert}[[48]]-\text{extvert}[[14]])],\\
&\text{extvert}[[14]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[42]]-\text{extvert}[[14]])/\\
&\text{Norm}[(\text{extvert}[[42]]-\text{extvert}[[14]])],\ \text{extvert}[[3]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[37]]-\text{extvert}[[3]])/\text{Norm}[(\text{extvert}[[37]]-\text{extvert}[[3]])],\\
&\text{extvert}[[19]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[39]]-\text{extvert}[[19]])/\\
&\text{Norm}[(\text{extvert}[[39]]-\text{extvert}[[19]])],\ \text{extvert}[[17]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[44]]-\text{extvert}[[17]])/\text{Norm}[(\text{extvert}[[44]]-\text{extvert}[[17]])],\\
&\text{extvert}[[9]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[43]]-\text{extvert}[[9]])/\\
&\text{Norm}[(\text{extvert}[[43]]-\text{extvert}[[9]])],\ \text{extvert}[[16]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[25]]-\text{extvert}[[16]])/\text{Norm}[(\text{extvert}[[25]]-\text{extvert}[[16]])],\\
&\text{extvert}[[8]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[47]]-\text{extvert}[[8]])/\\
&\text{Norm}[(\text{extvert}[[47]]-\text{extvert}[[8]])],\ \text{extvert}[[12]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[48]]-\text{extvert}[[12]])/\text{Norm}[(\text{extvert}[[48]]-\text{extvert}[[12]])],\\
&\text{extvert}[[11]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[49]]-\text{extvert}[[11]])/\\
&\text{Norm}[(\text{extvert}[[49]]-\text{extvert}[[11]])],\ \text{extvert}[[7]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[32]]-\text{extvert}[[7]])/\text{Norm}[(\text{extvert}[[32]]-\text{extvert}[[7]])],\\
&\text{extvert}[[5]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[46]]-\text{extvert}[[5]])/\\
&\text{Norm}[(\text{extvert}[[46]]-\text{extvert}[[5]])],\ \text{extvert}[[19]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[47]]-\text{extvert}[[19]])/\text{Norm}[(\text{extvert}[[47]]-\text{extvert}[[19]])],\\
&\text{extvert}[[3]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[21]]-\text{extvert}[[3]])/\\
&\text{Norm}[(\text{extvert}[[21]]-\text{extvert}[[3]])],\ \text{extvert}[[7]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[45]]-\text{extvert}[[7]])/\text{Norm}[(\text{extvert}[[45]]-\text{extvert}[[7]])],\\
&\text{extvert}[[11]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[43]]-\text{extvert}[[11]])/\\
&\text{Norm}[(\text{extvert}[[43]]-\text{extvert}[[11]])],\ \text{extvert}[[11]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*\\
&(\text{extvert}[[23]]-\text{extvert}[[11]])/\text{Norm}[(\text{extvert}[[23]]-\text{extvert}[[11]])],\\
&\text{extvert}[[12]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*(\text{extvert}[[22]]-\text{extvert}[[12]])/\\
&\text{Norm}[(\text{extvert}[[22]]-\text{extvert}[[12]])],\ \text{extvert}[[6]]+\left(\epsilon\Big/\left(\tfrac{1}{4}\left(1+\sqrt{5}\right)\right)\right)*
\end{aligned}
$$

```
(extvert[[21]] - extvert[[6]]) / Norm[(extvert[[21]] - extvert[[6]])],
extvert[[2]] + (ϵ / (1/4 (1 + √5))) * (extvert[[25]] - extvert[[2]]) /
  Norm[(extvert[[25]] - extvert[[2]])], extvert[[5]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[24]] - extvert[[5]]) / Norm[(extvert[[24]] - extvert[[5]])],
extvert[[5]] + (ϵ / (1/4 (1 + √5))) * (extvert[[36]] - extvert[[5]]) /
  Norm[(extvert[[36]] - extvert[[5]])], extvert[[2]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[42]] - extvert[[2]]) / Norm[(extvert[[42]] - extvert[[2]])],
extvert[[13]] + (ϵ / (1/4 (1 + √5))) * (extvert[[45]] - extvert[[13]]) /
  Norm[(extvert[[45]] - extvert[[13]])], extvert[[17]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[22]] - extvert[[17]]) / Norm[(extvert[[22]] - extvert[[17]])],
extvert[[19]] + (ϵ / (1/4 (1 + √5))) * (extvert[[26]] - extvert[[19]]) /
  Norm[(extvert[[26]] - extvert[[19]])], extvert[[2]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[28]] - extvert[[2]]) / Norm[(extvert[[28]] - extvert[[2]])],
extvert[[6]] + (ϵ / (1/4 (1 + √5))) * (extvert[[27]] - extvert[[6]]) /
  Norm[(extvert[[27]] - extvert[[6]])], extvert[[20]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[26]] - extvert[[20]]) / Norm[(extvert[[26]] - extvert[[20]])],
extvert[[18]] + (ϵ / (1/4 (1 + √5))) * (extvert[[23]] - extvert[[18]]) /
  Norm[(extvert[[23]] - extvert[[18]])], extvert[[13]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[29]] - extvert[[13]]) / Norm[(extvert[[29]] - extvert[[13]])],
extvert[[10]] + (ϵ / (1/4 (1 + √5))) * (extvert[[36]] - extvert[[10]]) /
  Norm[(extvert[[36]] - extvert[[10]])], extvert[[9]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[27]] - extvert[[9]]) / Norm[(extvert[[27]] - extvert[[9]])],
extvert[[17]] + (ϵ / (1/4 (1 + √5))) * (extvert[[33]] - extvert[[17]]) /
  Norm[(extvert[[33]] - extvert[[17]])], extvert[[13]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[38]] - extvert[[13]]) / Norm[(extvert[[38]] - extvert[[13]])],
extvert[[18]] + (ϵ / (1/4 (1 + √5))) * (extvert[[37]] - extvert[[18]]) /
  Norm[(extvert[[37]] - extvert[[18]])], extvert[[4]] + (ϵ / (1/4 (1 + √5))) *
  (extvert[[33]] - extvert[[4]]) / Norm[(extvert[[33]] - extvert[[4]])],
```

```
extvert[[15]] + (ϵ/ ((1/4) (1 + √5))) * (extvert[[28]] - extvert[[15]]) /
    Norm[(extvert[[28]] - extvert[[15]])], extvert[[10]] + (ϵ/ ((1/4) (1 + √5))) *
   (extvert[[30]] - extvert[[10]]) / Norm[(extvert[[30]] - extvert[[10]])],
extvert[[18]] + (ϵ/ ((1/4) (1 + √5))) * (extvert[[35]] - extvert[[18]]) /
    Norm[(extvert[[35]] - extvert[[18]])], extvert[[20]] + (ϵ/ ((1/4) (1 + √5))) *
   (extvert[[34]] - extvert[[20]]) / Norm[(extvert[[34]] - extvert[[20]])],
extvert[[20]] + (ϵ/ ((1/4) (1 + √5))) * (extvert[[32]] - extvert[[20]]) /
    Norm[(extvert[[32]] - extvert[[20]])], extvert[[6]] + (ϵ/ ((1/4) (1 + √5))) *
   (extvert[[31]] - extvert[[6]]) / Norm[(extvert[[31]] - extvert[[6]])],
extvert[[12]] + (ϵ/ ((1/4) (1 + √5))) * (extvert[[30]] - extvert[[12]]) /
    Norm[(extvert[[30]] - extvert[[12]])], extvert[[8]] + (ϵ/ ((1/4) (1 + √5))) *
   (extvert[[29]] - extvert[[8]]) / Norm[(extvert[[29]] - extvert[[8]])],
extvert[[4]] + (ϵ/ ((1/4) (1 + √5))) * (extvert[[24]] - extvert[[4]]) /
    Norm[(extvert[[24]] - extvert[[4]])]};(*Finds the vertices of X_F*)

Clear[δ];
δ;
push[list_] := Module[{pushvec = {0, 0, 0}, newlist = {}},
  For[i = 1, i <= Length[list], i = i + 5,
   pushvec = -δ (Cross[list[[i + 1]] - list[[i]], list[[i + 2]] - list[[i]]]) /
      Norm[Cross[list[[i + 1]] - list[[i]], list[[i + 2]] - list[[i]]]];
   AppendTo[newlist, list[[i]] + pushvec];
   AppendTo[newlist, list[[i + 1]] + pushvec];
   AppendTo[newlist, list[[i + 2]] + pushvec];
   AppendTo[newlist, list[[i + 3]] + pushvec];
   AppendTo[newlist, list[[i + 4]] + pushvec];
  ];
  newlist
 ](*This function is responsible for
  determining the δ parameter for pushing each face of X_F*)

In[13]:= penvert = push[N[inverts]]; (*Set of push vertices appoximated numerically*)

flapverts = {penvert[[1]] + (1 / 2) (penvert[[11]] - penvert[[1]]),
   penvert[[1]] + (1 / 2) (penvert[[10]] - penvert[[1]]),
   penvert[[2]] + (1 / 2) (penvert[[15]] - penvert[[2]]),
   penvert[[5]] + (1 / 2) (penvert[[6]] - penvert[[5]]),
   penvert[[2]] + (1 / 2) (penvert[[16]] - penvert[[2]]),
   penvert[[3]] + (1 / 2) (penvert[[20]] - penvert[[3]]),
   penvert[[3]] + (1 / 2) (penvert[[47]] - penvert[[3]]),
   penvert[[4]] + (1 / 2) (penvert[[46]] - penvert[[4]]),
   penvert[[4]] + (1 / 2) (penvert[[53]] - penvert[[4]]),
```

```
   penvert[[5]] + (1 / 2) (penvert[[52]] - penvert[[5]]),
   penvert[[6]] + (1 / 2) (penvert[[52]] - penvert[[6]]),
   penvert[[7]] + (1 / 2) (penvert[[51]] - penvert[[7]]),
   penvert[[7]] + (1 / 2) (penvert[[60]] - penvert[[7]]),
   penvert[[8]] + (1 / 2) (penvert[[59]] - penvert[[8]]),
   penvert[[9]] + (1 / 2) (penvert[[21]] - penvert[[9]]),
   penvert[[8]] + (1 / 2) (penvert[[22]] - penvert[[8]]),
   penvert[[9]] + (1 / 2) (penvert[[12]] - penvert[[9]]),
   penvert[[10]] + (1 / 2) (penvert[[11]] - penvert[[10]]),
   penvert[[12]] + (1 / 2) (penvert[[21]] - penvert[[12]]),
   penvert[[13]] + (1 / 2) (penvert[[25]] - penvert[[13]]),
   penvert[[13]] + (1 / 2) (penvert[[29]] - penvert[[13]]),
   penvert[[14]] + (1 / 2) (penvert[[28]] - penvert[[14]]),
   penvert[[14]] + (1 / 2) (penvert[[17]] - penvert[[14]]),
   penvert[[15]] + (1 / 2) (penvert[[16]] - penvert[[15]]),
   penvert[[17]] + (1 / 2) (penvert[[28]] - penvert[[17]]),
   penvert[[18]] + (1 / 2) (penvert[[27]] - penvert[[18]]),
   penvert[[18]] + (1 / 2) (penvert[[40]] - penvert[[18]]),
   penvert[[19]] + (1 / 2) (penvert[[39]] - penvert[[19]]),
   penvert[[19]] + (1 / 2) (penvert[[48]] - penvert[[19]]),
   penvert[[20]] + (1 / 2) (penvert[[47]] - penvert[[20]]),
   penvert[[22]] + (1 / 2) (penvert[[59]] - penvert[[22]]),
   penvert[[23]] + (1 / 2) (penvert[[58]] - penvert[[23]]),
   penvert[[23]] + (1 / 2) (penvert[[32]] - penvert[[23]]),
   penvert[[24]] + (1 / 2) (penvert[[31]] - penvert[[24]]),
   penvert[[24]] + (1 / 2) (penvert[[30]] - penvert[[24]]),
   penvert[[25]] + (1 / 2) (penvert[[29]] - penvert[[25]]),
   penvert[[26]] + (1 / 2) (penvert[[36]] - penvert[[26]]),
   penvert[[27]] + (1 / 2) (penvert[[40]] - penvert[[27]]),
   penvert[[26]] + (1 / 2) (penvert[[35]] - penvert[[26]]),
   penvert[[30]] + (1 / 2) (penvert[[31]] - penvert[[30]]),
   penvert[[32]] + (1 / 2) (penvert[[58]] - penvert[[32]]),
   penvert[[33]] + (1 / 2) (penvert[[57]] - penvert[[33]]),
   penvert[[33]] + (1 / 2) (penvert[[42]] - penvert[[33]]),
   penvert[[34]] + (1 / 2) (penvert[[41]] - penvert[[34]]),
   penvert[[34]] + (1 / 2) (penvert[[37]] - penvert[[34]]),
   penvert[[35]] + (1 / 2) (penvert[[36]] - penvert[[35]]),
   penvert[[37]] + (1 / 2) (penvert[[41]] - penvert[[37]]),
   penvert[[38]] + (1 / 2) (penvert[[45]] - penvert[[38]]),
   penvert[[38]] + (1 / 2) (penvert[[49]] - penvert[[38]]),
   penvert[[39]] + (1 / 2) (penvert[[48]] - penvert[[39]]),
   penvert[[50]] + (1 / 2) (penvert[[54]] - penvert[[50]]),
   penvert[[46]] + (1 / 2) (penvert[[53]] - penvert[[46]]),
   penvert[[55]] + (1 / 2) (penvert[[56]] - penvert[[55]]),
   penvert[[51]] + (1 / 2) (penvert[[60]] - penvert[[51]]),
   penvert[[45]] + (1 / 2) (penvert[[49]] - penvert[[45]]),
   penvert[[44]] + (1 / 2) (penvert[[50]] - penvert[[44]]),
   penvert[[44]] + (1 / 2) (penvert[[54]] - penvert[[44]]),
   penvert[[43]] + (1 / 2) (penvert[[55]] - penvert[[43]]),
   penvert[[42]] + (1 / 2) (penvert[[57]] - penvert[[42]]),
   penvert[[43]] + (1 / 2) (penvert[[56]] - penvert[[43]])};
(*Finds the midpoints of the rectangular faces*)

finverts = Join[Join[penvert, dodecV], flapverts];
(*Compiles all vertices relevant to inflated dodecahedron*)
```

```
p1 = Sec[36 * π / 180] ∊;
p2 = Tan[36 * π / 180] ∊;
p3 = Sqrt[3] ∊;
(*p1,p2,and p3 determine the side lengths for the triangle that divides the
 volume of the cap vertex pyramid along an edge leading towards the apex,
the median of a triangular face, and the base of the pyramid*)
s = (p1 + p2 + p3) / 2;(*Determines the semiperimeter*)
area = FullSimplify[Sqrt[s * (s - p1) * (s - p2) * (s - p3)], ∊ ∈ Reals];
(*Heron's Formula to determine area*)
Solve[(1 / 2) Sqrt[3] ∊ * H == area, H];(*Determines the height to
 adjust the vertex of the cap pyramid to maintain an isometry*)
height = - ((-3 + √5) ∊) / √3;(*Declaration of height variable*)
ratio = (Norm[(finverts[[25]] + finverts[[29]] + finverts[[13]]) / 3] + height) /
   Norm[finverts[[67]]];(*The scalar
 quantity that adjusts the cap vertices of the pyramid*)

gfinverts = Join[Join[penvert, ratio * dodecV], flapverts]; (*Compliation
 of all the vertices necessary to define the inflated dodecahedron*)
ivd = {{1, 2, 3, 4, 5}, {6, 7, 8, 9, 10}, {11, 12, 13, 14, 15}, {16, 17, 18, 19, 20},
  {21, 22, 23, 24, 25}, {26, 27, 28, 29, 30}, {31, 32, 33, 34, 35}, {36, 37, 38, 39, 40},
  {41, 42, 43, 44, 45}, {46, 47, 48, 49, 50}, {51, 52, 53, 54, 55}, {56, 57, 58, 59, 60},
  {1, 2, 83, 81}, {2, 3, 86, 85}, {3, 87, 88, 4}, {4, 89, 90, 5}, {5, 84, 82, 1},
  {1, 61, 81}, {1, 61, 82}, {2, 83, 74}, {2, 74, 85}, {3, 86, 69}, {3, 87, 69},
  {4, 88, 70}, {4, 70, 89}, {5, 75, 90}, {5, 75, 84}, {6, 10, 82, 84}, {6, 91, 92, 7},
  {7, 93, 94, 8}, {8, 96, 95, 9}, {9, 97, 98, 10}, {6, 84, 75}, {6, 91, 75},
  {7, 64, 92}, {7, 64, 93}, {8, 94, 68}, {8, 68, 96}, {9, 95, 76}, {9, 76, 97},
  {10, 61, 82}, {10, 98, 61}, {11, 98, 97, 12}, {12, 99, 100, 13}, {13, 101, 102, 14},
  {14, 103, 104, 15}, {15, 83, 81, 11}, {11, 81, 61}, {11, 98, 61}, {12, 97, 76},
  {12, 99, 76}, {13, 100, 67}, {13, 67, 101}, {14, 102, 63}, {14, 63, 103}, {15, 74, 83},
  {15, 104, 74}, {16, 104, 103, 17}, {17, 105, 106, 18}, {18, 107, 108, 19},
  {19, 109, 110, 20}, {20, 86, 85, 16}, {16, 85, 74}, {16, 74, 104}, {17, 63, 103},
  {17, 63, 105}, {18, 79, 106}, {18, 79, 107}, {19, 77, 108}, {19, 77, 109},
  {20, 69, 110}, {20, 69, 86}, {21, 95, 96, 22}, {22, 111, 112, 23}, {23, 113, 114, 24},
  {24, 115, 116, 25}, {25, 100, 99, 21}, {21, 76, 99}, {21, 76, 95}, {22, 68, 96},
  {22, 68, 111}, {23, 72, 112}, {23, 72, 113}, {24, 71, 114}, {24, 71, 115},
  {25, 67, 116}, {25, 67, 100}, {26, 117, 118, 27}, {27, 106, 105, 28},
  {28, 102, 101, 29}, {29, 116, 115, 30}, {30, 120, 119, 26}, {30, 71, 115},
  {30, 71, 120}, {26, 65, 119}, {26, 65, 117}, {27, 79, 118}, {27, 79, 106},
  {28, 63, 105}, {28, 63, 102}, {29, 67, 101}, {29, 67, 116}, {41, 124, 123, 42},
  {42, 139, 140, 43}, {43, 138, 137, 44}, {44, 136, 135, 45}, {45, 128, 127, 41},
  {41, 62, 124}, {41, 62, 127}, {42, 66, 123}, {42, 66, 139}, {43, 80, 140},
  {43, 80, 138}, {44, 78, 137}, {44, 78, 136}, {45, 73, 135}, {45, 73, 128},
  {51, 92, 91, 52}, {52, 90, 89, 53}, {53, 132, 131, 54}, {54, 137, 138, 55},
  {55, 133, 134, 51}, {51, 64, 92}, {51, 64, 134}, {52, 75, 91}, {52, 75, 90},
  {53, 70, 89}, {132, 53, 70}, {54, 78, 131}, {54, 78, 137}, {55, 80, 138}, {55, 80, 133},
  {50, 131, 132, 46}, {46, 88, 87, 47}, {47, 110, 109, 48}, {48, 130, 129, 49},
  {49, 135, 136, 50}, {46, 70, 132}, {46, 70, 88}, {47, 69, 87}, {47, 69, 110},
  {48, 77, 109}, {48, 77, 130}, {49, 73, 135}, {49, 73, 129}, {50, 78, 136},
  {50, 78, 131}, {36, 126, 125, 37}, {37, 127, 128, 38}, {38, 129, 130, 39},
  {39, 108, 107, 40}, {40, 118, 117, 36}, {36, 65, 117}, {36, 65, 126}, {37, 62, 125},
  {37, 62, 127}, {38, 73, 128}, {38, 73, 129}, {39, 77, 130}, {39, 77, 108},
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  {40, 79, 107}, {40, 79, 118}, {31, 114, 113, 32}, {32, 121, 122, 33},
  {33, 123, 124, 34}, {34, 125, 126, 35}, {35, 119, 120, 31}, {31, 71, 120},
  {31, 71, 114}, {32, 72, 113}, {32, 72, 121}, {33, 66, 122}, {33, 66, 123},
  {34, 62, 124}, {34, 62, 125}, {35, 65, 126}, {35, 65, 119}, {56, 140, 139, 57},
  {57, 122, 121, 58}, {58, 112, 111, 59}, {59, 94, 93, 60}, {60, 134, 133, 56},
  {56, 80, 133}, {56, 80, 140}, {57, 66, 139}, {57, 66, 122}, {58, 72, 121},
  {58, 72, 112}, {59, 68, 111}, {59, 68, 94}, {60, 64, 93}, {60, 64, 134}};
(*Information about how the vertices related to each other
 to develop faces for the inflated dodecahedron*)
Show[Graphics3D[{Opacity[1], Yellow, GraphicsComplex[gfinverts, Polygon[ivd]]}],
   Graphics3D[{Thick, GraphicsComplex[gfinverts, Line[ivd]]}],
   Graphics3D[{PointSize[0.02], Tooltip[Point[gfinverts[[#]]], #]}] & /@
    Range[Length[gfinverts]], Boxed → False] /. ϵ → 0.69 /.
 Solve[Norm[gfinverts[[30]] - gfinverts[[24]]] == 2 ϵ /. ϵ → 0.69, δ][[2]];
(*Displays the inflated dodecahedron for what ever ϵ value*)

a1 = Cross[(gfinverts[[6]] - gfinverts[[10]]),
     (gfinverts[[9]] - gfinverts[[10]])] / Norm[
     Cross[(gfinverts[[6]] - gfinverts[[10]]), (gfinverts[[9]] - gfinverts[[10]])]];
 (*Determines the height of the pyramid with the pushed
 face of X_F as a face*)
side = 1 - 2 ϵ Cot[54 * π / 180];(*Used to determine the area of the pushed face*)
v1 = (1 / 3) ((side^2 * Sqrt[25 + 10 Sqrt[5]]) / 4) * Abs[a1.gfinverts[[6]]];
(*Determines the volume of the pyramid*)

a2 = Cross[(gfinverts[[10]] - gfinverts[[11]]),
     (gfinverts[[12]] - gfinverts[[11]])] / Norm[
     Cross[(gfinverts[[10]] - gfinverts[[11]]), (gfinverts[[12]] - gfinverts[[11]])]];
 (*Determines the height of the pyramid with a rectangular face*)
v2 = (1 / 3) * 2 ϵ * side * Abs[gfinverts[[11]].a2];
(*Determines the volume of the pyramid*)

a3 = Cross[(gfinverts[[10]] - gfinverts[[61]]),
     (gfinverts[[11]] - gfinverts[[61]])] / Norm[
     Cross[(gfinverts[[10]] - gfinverts[[61]]), (gfinverts[[11]] - gfinverts[[61]])]];
 (*Determines the height of the pyramid with one of the triangles
 defining the cap pyramid as a face*)
v3 = (1 / 3) * area * Abs[gfinverts[[61]].a3];
(*Determines the volume of the pyramid*)

v[e_] := Module[{d},
  d = δ /. (Solve[Norm[gfinverts[[29]] - gfinverts[[25]]] == 2 ϵ /. ϵ → e, δ][[2]]);
  If[e > .7, .4, ((12 * v1 + 30 * v2 + 60 * v3)) /. {ϵ → e, δ → d}]
 ](*Numberically defines the volume as a function of ϵ*)

Plot[{v[e], (1 / 4) (15 + 7 Sqrt[5])}, {e, 0.00001, .6}, AxesLabel → {ϵ, Volume}]
(*Plots the volume function*)
```

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