VANISHING MEAN CURVATURE AND RELATED PROPERTIES

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1. Introduction

The project deals with surfaces that have vanishing mean curvature. A surface with vanishing mean curvature possesses interesting properties as well as an interesting relationship to the area of the surface. With the help of calculus, differential geometry, and the involvement of complex numbers, these relationships can be described accurately. Using these relationships, conditions can be found to generate surfaces with vanishing mean curvature.

2. Mean Curvature

The surfaces in the project possess vanishing mean curvature. In order to discuss the mean curvature of a surface, some restrictions need to be made on the surfaces being used.

Definition 2.1. [DoC76] *A subset $S \subset \mathbb{R}^3$ is a* ***regular surface*** *if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\boldsymbol{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

(1) *$\boldsymbol{x}$ is differentiable.*
(2) *$\boldsymbol{x}$ is continuous, one-to-one, onto, and has a continuous inverse.*
(3) *For each $q \in U$, the differential $d\boldsymbol{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one, where*

$$d\boldsymbol{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

The restriction to regular surfaces allows the derivatives at a point on the surface to be taken. Since the differential is one to one at all points, the tangent plane spanned by the tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ at a point p, exists at every point on the surface. The tangent plane at a point is also written $T_p(S)$.

Definition 2.2. [DoC76] *$I_p : T_p(S) \to \mathbb{R}$ defined by $I_p = \langle w, w\rangle = |w|^2 \geq 0$ is called the* ***first fundamental form*** *of the regular surface $S \in \mathbb{R}^3$, where w is in $T_p(S)$.*

The coefficients of the first fundamental form can be described by basis vectors in the tangent plane. Given a parameterization $x : U \to \mathbb{R}^3$ of a regular surface, with a basis $\{x_u, x_v\}$ of the tangent plane and a curve $\alpha(t) = x(u(t), v(t))$ such that $\alpha(0) = p$, a tangent vector at p can be represented by $\alpha'(0) = u'(0)x_u + v'(0)x_v$. Applying the first fundamental form to the tangent vector,

$$\begin{aligned}\langle u'x_u + v'x_v, u'x_u + v'x_v\rangle =& \langle x_u, x_u\rangle (u')^2 + \langle x_u, x_v\rangle (u'v') + \langle x_v, x_v\rangle (v')^2 \\ =& E(u')^2 + F(u'v') + G(v')^2\end{aligned}$$

where $E = \langle x_u, x_u\rangle$, $F = \langle x_u, x_v\rangle$, and $G = \langle x_v, x_v\rangle$ are defined as the coefficients of the first fundamental form. The coefficients are useful when simplifying equations. Another way to describe a surface is by the normal vectors at a point on a surface.

Definition 2.3. [DoC76] *Let C be a curve on a regular surface S passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N\rangle$, where n is the unit normal vector to C and N is the unit normal vector to S at p. The number $k_n = k\cos\theta$ is called the* ***normal curvature*** *of $C \subset S$ at p.*

Since the set of unit vectors at a point is closed and bounded, a maximum and a minimum normal curvature will be attained. The maximum and minimum normal curvatures and their corresponding directions are given a special label.

Definition 2.4. [DoC76] *The maximum normal curvature k_1 and the minimum normal curvature k_2 over the set of unit tangent vectors at p are called the* ***principal curvatures*** *at p; the corresponding directions of the unit vectors are called the* ***principal directions****.*

The definition of mean curvature can be given in terms of the principal curvatures.

Definition 2.5. [DoC76] *The* ***mean curvature*** *at a point is given as*

$$H = \frac{1}{2}(k_1 + k_2)$$

where k_1 and k_2 are the principal curvatures at the point.

Vanishing mean curvature was actually found by examining a condition developed by Lagrange.

3. Minimal Surface Equation

Definition 3.1. [DoC76] *If $f : U \to \mathbb{R}$ is a differentiable function in an open set $U \subset \mathbb{R}^2$, then the* ***graph*** *of f is given by the regular surface $F = (x, y, f(x, y))$ for $x, y \in U$.*

Definition 3.2. [BC76] *An open set $S \subset \mathbb{R}^2$ is* ***connected*** *if each pair of points a_1 and a_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S.*

Definition 3.3. [BC76] *A* ***domain*** *is a connected open set.*

In the 1700's Lagrange set out to find the surface with the smallest area bounded by a closed curve in space[Nit89]. Lagrange discovered the minimal surface equation, which must be satisfied by the function whose graph has the minimum area of all graphs bounded by the closed curve in space. Nitsche derives the minimal surface equation in *Lectures on Minimal Surfaces*. Let α be a closed curve in space which projects to a curve C in the x, y plane such that C encloses a domain D.

Definition 3.4. *A function with* ***compact support*** *is a function that vanishes outside a compact set.*

Theorem 3.5. *Given a continuous function $f(x)$ over an interval Q, if $\int_Q fg = 0$ for all integrable compactly supported $g(x)$ on Q, then $f(x) = 0$.*

Proof. Without loss of generality, assume $\int fg = 0$ and $f(x)$ is nonzero and positive at some point a in Q. Since $f(x)$ is continuous, there is a neighborhood (a_1, a_2) containing a such that $f(x)$ is also positive. Define

$$g(x) = \begin{cases} 1; \text{if } x \in [a_1, a_2] \\ 0; \text{otherwise,} \end{cases}$$

$g(x)$ is integrable, since all stair functions are integrable. Defining g in this way makes $\int f(x)g(x)dx$ nonzero and positive. Since g is zero outside (a_1, a_2), g is compactly supported on Q. So in order for $\int f(x)g(x)d = 0$ for all compactly supported functions, $f(x)$ must equal zero everywhere. □

From above, let α be a closed curve in space bounding the graph $(x, y, z(x, y))$ which has minimum area among all graphs bounded by α. Using $\epsilon \in \mathbb{R}$, and $\zeta(x, y)$, a differentiable function with compact support over D, Nitsche created a variation of z defined as $\overline{z}(x, y) = z(x, y) + \epsilon\zeta(x, y)$[Nit89]. The variation provides all graphs bounded by α with area given by $I(\epsilon) = \iint_D \sqrt{1 + \overline{z}_x^2 + \overline{z}_y^2}dxdy$. Lagrange knew that if $z(x, y)$ was the minimal solution, then $I(\epsilon)$ must have a minimum at $\epsilon = 0$, since $\overline{z}(x, y)|_{\epsilon=0} = z(x, y)$. In terms of the area function, $z(x, y)$ as a minimum requires that $I'(0) = 0$. The following theorem uses Lagrange's approach.

Theorem 3.6. *Let α be a closed curve in space bounding the graphs given by the variation $\overline{z}(x, y) = z(x, y) + \epsilon\zeta(x, y)$ and let C be the projection of α onto the x, y plane enclosing a domain D. If the area of the graph of $z(x, y)$ is the minimum area* *of all graphs, the function of area given by* $I(\epsilon) = \iint_D \sqrt{1+\overline{z}_x^2+\overline{z}_y^2}dxdy$ *must be at a minimum when* $\epsilon = 0$ *and* z *must satisfy the minimal surface equation*

$$(1+z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1+z_x^2)z_{yy} = 0. \tag{1}$$

Proof. To start, assume $z(x,y)$ gives the graph with minimum area bounded of all graphs bounded by α, then $I(\epsilon)$ must be at a minimum when $\epsilon = 0$ since $\overline{z}(x,y)|_{\epsilon=0} = z(x,y)$. For this same reason, if $I(\epsilon)$ is at a minimum when $\epsilon = 0$, then the graph of $z(x,y)$ must give the minimal area of all the graphs bounded by α. Now, $z(x,y)$ must be shown to satisfy the minimal surface equation. By differentiating $I(\epsilon)$ with respect to ϵ we get,

$$\begin{aligned}
\tfrac{\partial}{\partial\epsilon}I(\epsilon) =& \tfrac{\partial}{\partial\epsilon}\iint_D \sqrt{1+\overline{z}_x^2+\overline{z}_y^2}dxdy \\
=& \tfrac{\partial}{\partial\epsilon}\iint_D \sqrt{1+(z_x+\epsilon\zeta_x)^2+(z_y+\epsilon\zeta_y)^2}dxdy \\
=& \tfrac{\partial}{\partial\epsilon}\iint_D \sqrt{1+(z_x^2+2z_x\epsilon\zeta_x+\epsilon^2\zeta_x^2)+\left(z_y^2+2z_y\epsilon\zeta_y+\epsilon^2\zeta_y^2\right)}dxdy,
\end{aligned}$$

since ϵ is independent of x and y, the differential can be taken inside the integral,

$$= \iint_D \frac{\left(2z_x\zeta_x+2\epsilon\zeta_x^2+2z_y\zeta_y+2\epsilon\zeta_y^2\right)}{2\sqrt{1+\overline{z}_x^2+\overline{z}_y^2}}dxdy,$$

factoring out ζ_x and ζ_y,

$$\begin{aligned}
&= \iint_D \frac{\left[\zeta_x\left(z_x+\epsilon\zeta_x\right)+\zeta_y\left(z_y+\epsilon\zeta_y\right)\right]}{\sqrt{1+\overline{z}_x^2+\overline{z}_y^2}}dxdy \\
&= \iint_D \frac{\overline{z}_x\zeta_x+\overline{z}_y\zeta_y}{\sqrt{1+\overline{z}_x^2+\overline{z}_y^2}}dxdy.
\end{aligned}$$

Since $I(e)$ is at a minimum when $\epsilon = 0$, $I'(\epsilon)$ must be zero when $\epsilon = 0$. Setting $\epsilon = 0$ makes $\overline{z}(x,y)|_{\epsilon=0} = z(x,y)$, which makes $\overline{z}_x|_{\epsilon=0} = z_x$ and $\overline{z}_y|_{\epsilon=0} = z_y$, giving

$$I'(0) = \iint_D \frac{\overline{z}_x\zeta_x+\overline{z}_y\zeta_y}{\sqrt{1+\overline{z}_x^2+\overline{z}_y^2}}dxdy\bigg|_{\epsilon=0} = \iint_D \frac{z_x\zeta_x+z_y\zeta_y}{\sqrt{1+z_x^2+z_y^2}}dxdy.$$

The integral can be written as a sum,

$$\iint_D \frac{z_x\zeta_x+z_y\zeta_y}{\sqrt{1+z_x^2+z_y^2}}dxdy = \iint_D \frac{z_x\zeta_x}{\sqrt{1+z_x^2+z_y^2}}dxdy + \iint_D \frac{z_y\zeta_y}{\sqrt{1+z_x^2+z_y^2}}dxdy. \tag{2}$$

In order to evaluate the double integrals as iterated integrals, integration limits need to be found for the integrals with respect to x and y. The shape of the domain could

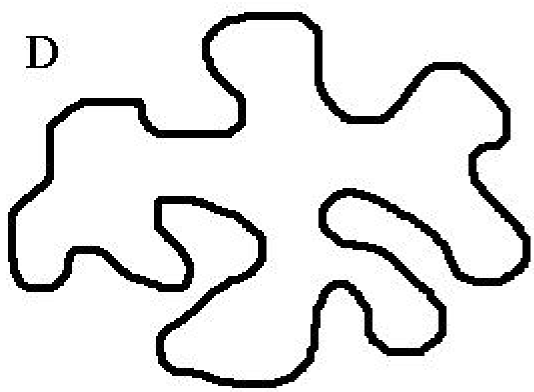

FIGURE 1. Possible Domain

make breaking the original double integral into iterated integrals difficult – consider a slice of figure 1 – so the integral needs to be redefined. By letting $g : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$g(x,y) = \begin{cases} \frac{z_x \zeta_x}{\sqrt{1+z_x^2+z_y^2}}; \text{if } (x,y) \in D \\ 0; \text{if } (x,y) \in \mathbb{R}^2 - D \end{cases}$$

and letting $A_y = \{(x,y) | \forall x \in \mathbb{R}\} \cap D$, for the first term of (2), evaluating the integral with respect to x gives

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) dx dy &= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}-D} 0 dx + \int_D \frac{z_x \zeta_x}{\sqrt{1+z_x^2+z_y^2}} dx \right) dy \\ &= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}-A_y} 0 dx + \int_{A_y} \frac{z_x \zeta_x}{\sqrt{1+z_x^2+z_y^2}} dx \right) dy. \end{aligned}$$

Since $g(x,y)$ is 0 everywhere but on A_y, integrating over $\mathbb{R} - A_y$ will be zero. Setting

$$\begin{aligned} u =& \frac{z_x}{(\sqrt{1+z_x^2+z_y^2})} & dv = \zeta_x dx \\ du =& \frac{\partial}{\partial x} \left(\frac{z_x}{(\sqrt{1+z_x^2+z_y^2})} \right) dx & v = \zeta(x,y), \end{aligned}$$

the integration can be done by parts. Evaluating the integral gives

$$\int_{-\infty}^{\infty}\left(uv-\int vdu\right)dy=\int_{-\infty}^{\infty}\left(\frac{z_x\zeta(x,y)}{\sqrt{1+z_x^2+z_y^2}}\bigg|_{\partial A_y}-\int_{A_y}\zeta(x,y)\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)dx\right)dy,$$

where ∂A_y is the boundary of A_y. Since A_y is just a slice of the domain, the boundary of A_y will be in C. So evaluating $\zeta(x,y)$ at any points in ∂A_y will be zero, which makes the first term zero. Thus rewriting the integral over D gives

$$\iint_D \frac{z_x\zeta_x}{\sqrt{1+z_x^2+z_y^2}}dxdy=-\iint_D\zeta(x,y)\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)dxdy. \tag{3}$$

Proceeding in the same fashion, the other double integral yields

$$\iint_D \frac{z_y\zeta_y}{\sqrt{1+z_x^2+z_y^2}}dxdy=-\iint_D\zeta(x,y)\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)dxdy. \tag{4}$$

Substituting (3) and (4) back into equation (2),

$$I'(0)=-\iint_D\left[\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)\right]\zeta(x,y)dxdy=0,$$

since $I'(0)$ must be at a minimum when $\epsilon=0$. Since $\zeta(x,y)$ can be any differentiable function with compact support over D, by theorem(3.5) the sum of the partial derivatives must be zero, which gives

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=0.\text{[Nit89]}$$

Simplifying the differentials, we find

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)&=z_{xx}(1+z_x^2+z_y^2)^{-1/2}+-\tfrac{1}{2}z_x\left[\left(1+z_x^2+z_y^2\right)^{-3/2}(2z_x(z_{xx})+2z_y(z_{yx})\right]\\&=(1+z_x^2+z_y^2)^{-3/2}[z_{xx}(1+z_x^2+z_y^2)-z_x^2z_{xx}-z_xz_yz_{yx}]\\&=(1+z_x^2+z_y^2)^{-3/2}\left[z_{xx}(1+z_y^2)-z_xz_yz_{yx}\right]\end{aligned}$$

likewise,

$$\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=(1+z_x^2+z_y^2)^{-3/2}\left[z_{yy}(1+z_{xx})-z_yz_xz_{xy}\right].$$

Substituting back into the original,

$$(1+z_x^2+z_y^2)^{-3/2}\left[(1+z_y^2)z_{xx}-2z_xz_yz_{xy}+(1+z_x^2)z_{yy}\right]=0,$$

which is true if and only if

$$(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy} = 0 \tag{5}$$

which is the minimal surface equation(1). □

4. Lagrange's Condition and Vanishing Mean Curvature

Shortly after Lagrange developed his minimal surface equation, Meusnier found that surfaces satisfying Lagrange's condition must also possess vanishing mean curvature. In order to show this relationship, a more suitable formula for mean curvature can be derived. Since the mean curvature everywhere on the surface will be examined, a normal curvature will be needed everywhere on the surface. This requirement can be met by examining only orientable surfaces.

Definition 4.1. [DoC76] *A regular surface is **orientable** if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field is called an **orientation**.*

Definition 4.2. [DoC76] *Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere(S^2). The map $N : S \to S^2$ is called the **Gauss map** of S.*

By definition, orientation guarantees a differentiable field of unit normal vectors, thus the Gauss map is differentiable. The differential of the Gauss map is a map from the tangent space of a surface to the tangent space of the unit sphere. For a curve $\alpha(t) = x(u(t), v(t))$ on S such that $\alpha(0) = p$ for some $p \in S$, define $N(t) = N \circ \alpha(t)$, this keeps the normal vector N on α.([DoC76]) Taking the differential of $N(t)$ at p, will give the change of the unit normal vector to the surface in the direction of $\alpha'(t)$ at p. The differential of the Gauss map, dN, also gives the change of the surface normal at a point in the direction of a tangent vector. By evaluating $dN(\alpha'(t))$ over the same curve as $N'(t)$, the two functions describe the same thing. For this reason, $\frac{d}{dt}N(\alpha(t)) = dN_p(\alpha'(t))$. An orientable surface S with parameterization $x(u, v)$ has a basis $\{x_u, x_v\}$ for $T_p(S)$. Applying dN to $\alpha'(0)$, written in terms of basis vectors of the tangent space, can be written in terms of partial derivatives of N

$$\begin{aligned} dN_p(\alpha'(0)) =& dN_p(\alpha'(0)) \\ =& \frac{d}{dt}\left[N_p\left((u(t), (v(t))\right]\right|_{t=0} \\ =& N_u u'(0) + N_v v'(0). \end{aligned}$$

Applying dN to basis vectors of the tangent plane at p, we obtain

$$\begin{aligned} dN_p(x_u) &= dN\left((1)x_u + (0)x_v\right) \\ &= N_u \\ dN_p(x_v) &= dN\left((0)x_u + (1)x_v\right) \\ &= N_v. \end{aligned}$$

Definition 4.3. *Given two vector spaces U, V a function $A : U \to V$ is said to be* ***linear*** *if A preserves addition and scalar multiplication.*

Using the information above, dN can be shown to be operation preserving. Applying dN to the tangent vectors of two curves, $\alpha = x(u_1(t), v_1(t))$ and $\beta = x(u_2(t), v_2(t))$, we find

$$\begin{aligned} dN(\alpha' + \lambda\beta') &= dN\left[u_1' x_u + v_1' x_v + \lambda(u_2' x_u + v_2' x_v)\right] \\ &= dN\left[(u_1' + \lambda u_2')x_u + (v_1' + \lambda v_2')x_v)\right] \\ &= N_u\left(u_1' + \lambda u_2'\right) + N_v\left(v_1' + \lambda v_2'\right) \\ &= u_1' N_u + v_1' N_v + \lambda\left(u_2' N_u + v_2' N_v\right) \\ &= dN\left(u_1' + v_1'\right) + \lambda dN\left(u_2' + v_2'\right) \\ &= dN(\alpha') + \lambda dN(\beta'). \end{aligned}$$

Since dN is operation preserving from $T_p(S)$ to $T_{N(p)}(S^2)$, two vector spaces, dN is linear.

Definition 4.4. [DoC76] *Given a vector space V, a linear map $A : V \to V$ is* ***self-adjoint*** *if $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in V$.*

Next, dN can be shown to be self-adjoint. Using the fact $\langle N, x_u\rangle = 0$ and differentiating,

$$\frac{d}{dv}\langle N, x_u\rangle = \langle N_v, x_u\rangle + \langle N, x_{uv}\rangle = 0,$$

also $\langle N, x_v\rangle = 0$, which makes

$$\frac{d}{du}\langle N, x_v\rangle = \langle N_u, x_v\rangle + \langle N, x_{vu}\rangle = 0,$$

Since x is twice differentiable $x_{uv} = x_{vu}$, combined with the above equalities $\langle N_u, x_v\rangle = -\langle N, x_{uv}\rangle = \langle N_v, x_u\rangle$. Using this substitution,

$$\begin{aligned} \langle dN(x_u), x_v\rangle &= \langle N_u, x_v\rangle \\ &= \langle N_v, x_u\rangle \\ &= \langle dN(x_v), x_u\rangle \\ &= \langle x_u, dN(x_v)\rangle, \end{aligned}$$

because the above equalities hold for basis vectors and dN is linear, the above equalities will hold for all vectors in $T_p(s)$. This means $\langle dN(m), n\rangle = \langle m, dN(n)\rangle$ for all $n, m \in T_p(S)$, thus dN is self-adjoint. Since dN has been shown to be a self-adjoint linear map, the following result can be applied.

Theorem 4.5. [DoC76] *Let $A : V \to V$ be a self-adjoint, linear map. Then there exists an orthonormal basis e_1, e_2 of V such that $A(e_1) = \lambda e_1$, $A(e_2) = \lambda e_2$. In the basis e_1, e_2, the matrix of A is clearly diagonal and the elements λ_1, λ_2 on the diagonal are the maximum and the minimum, respectively of the quadratic form $Q(v) = \langle Av, v\rangle$ on the unit circle of V.*

Since dN has been shown to be a linear self-adjoint map, by theorem(4.5) there is an orthogonal basis e_1, e_2 in the principal directions. In addition, the eigenvalues of dN are the maximum and minimum of the quadratic form on the tangent plane. The quadratic form of dN referred to in the theorem is the second fundamental form:

Definition 4.6. [DoC76] *Given a $v \in T_p(S)$, the function $II_p(v) = -\langle dN_p(v), v\rangle$ is called the **second fundamental form**.*

The second fundamental form can be manipulated to give the normal curvature. Using the notation from above and the fact that the surface normal at a point is orthogonal to any tangent vector at the point, $\langle N(0), \alpha'(0)\rangle = 0$. Differentiating,

$$\langle N(0), \alpha'(0)\rangle' = 0 = \langle N'(0), \alpha'(0)\rangle + \langle N(0), \alpha''(0)\rangle,$$

which gives II_p applied to a tangent vector $\alpha'(0) = p$ as

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N'(0), \alpha'(0)\rangle \\ &= \langle N(0), \alpha''(0)\rangle \\ &= \langle N(0), kn\rangle \\ &= k\langle N(0), n\rangle \\ &= k_n(p), \end{aligned}$$

where n is the unit normal to α at p and k_n is the normal curvature at p. Of course, the mean curvature is half the sum of the principal curvatures, which are the maximum and minimum normal curvatures, which from theorem (4.5) are given by the eigenvalues of dN. Using the same notation as before, a tangent vector to a surface can be represented by

$$\alpha' = x_u u' + x_v v',$$

and applying dN yields

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'.$$

Since dN maps onto the tangent plane of S at p, N_u and N_v can be represented as a linear combination of x_u and x_v,

$$N_u = a_{11}x_u + a_{21}x_v \qquad N_v = a_{12}x_u + a_{22}x_v.$$

Writing $dN(\alpha')$ in the basis x_u, x_v gives

$$dN(\alpha') = (a_{11}x_u + a_{21}x_v)u' + (a_{12}x_u + a_{22}x_v)v',$$

which simplifies to

$$(a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v,$$

leading to

$$dN \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}.$$

Applying the second fundamental form to a tangent vector gives

$$\begin{aligned}
-\langle dN(\alpha'), \alpha' \rangle &= -\langle dN(u'x_u + v'x_v), u'x_u + v'x_v \rangle \\
&= -\langle N_u u' + N_v v', x_u u' + x_v v' \rangle \\
&= -\langle N_u u', x_u u' \rangle - \langle N_u u', x_v v' \rangle - \langle N_v v', x_u u' \rangle - \langle N_v v', x_v v' \rangle \\
&= -\langle N_u u', x_u u' \rangle - (u'v') \langle N_u, x_v \rangle - (u'v') \langle N_v, x_u \rangle - \langle N_v v', x_v v' \rangle \\
&= -\langle N_u u', x_u u' \rangle - (u'v') \langle N_u, x_v \rangle - (u'v') \langle dN(x_v), x_u \rangle - \langle N_v v', x_v v' \rangle .
\end{aligned}$$

Since dN has been shown to be self-adjoint, $\langle dN(x_v), x_u \rangle = \langle x_v, dN(x_u) \rangle$, making

$$\begin{aligned}
-\langle dN(\alpha'), \alpha' \rangle &= -\langle N_u u', x_u u' \rangle - (u'v') \langle N_u, x_v \rangle - (u'v') \langle dN(x_u), x_v \rangle - \langle N_v v', x_v v' \rangle \\
&= -\langle N_u u', x_u u' \rangle - (u'v') \langle N_u, x_v \rangle - (u'v') \langle N_u, x_v \rangle - \langle N_v v', x_v v' \rangle \\
&= -\langle N_u, x_u \rangle (u')^2 - \langle N_u, x_v \rangle (2u'v') - \langle N_v, x_v \rangle (v')^2.
\end{aligned}$$

This equation defines the coefficients of the second fundamental form,

$$-\langle dN(\alpha'), \alpha' \rangle = e(u')^2 + f(2u'v') + g(v')^2$$

with $e = -\langle N_u, x_u \rangle$, $f = -\langle N_u, x_v \rangle$, and $g = -\langle N_v, x_v \rangle$. The coefficients can be further simplified into terms of the first fundamental form and members of dN.

$$\begin{aligned}
-e &= \langle N_u, x_u \rangle \\
&= \langle a_{11}x_u + a_{21}x_v, x_u \rangle \\
&= \langle a_{11}x_u, x_u \rangle + \langle a_{21}x_v, x_u \rangle \\
&= a_{11} \langle x_u, x_u \rangle + a_{21} \langle x_v, x_u \rangle \\
&= a_{11}E + a_{21}F
\end{aligned}$$

Similarly, f and g can also be shown as

$$\begin{aligned} -f =& a_{11}F + a_{21}G = a_{12}E + a_{22}F \\ -g =& a_{12}F + a_{22}G \end{aligned}$$

These relationships can be shown in matrix form

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

To solve for dN, multiplying the system by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix},$$

gives

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} &= - \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{fF-eG}{EG-F^2} & \frac{gF-fG}{EG-F^2} \\ \\ \frac{eF-fE}{EG-F^2} & \frac{fF-gE}{EG-F^2}. \end{bmatrix} \end{aligned}$$

With dN computed, mean curvature can be computed in the form of coefficients of the first and second fundamental forms:

$$H = \frac{1}{2}(k_1 + k_2) = -\tfrac{1}{2}trace[dN] = -(a_{11} + a_{22}) = \frac{1}{2}\left(\frac{eG - 2fF + gE}{EG - F^2}\right). \tag{6}$$

Using the above results, the proof of the first theorem is possible.

Theorem 4.7. [Web06] *Let U be a planar domain and $j = j(u, v)$ a twice differentiable function. The graph is then the parametrized surface J given by $J(u, v) = (u, v, j(u, v))$. The function $j(u, v)$ satisfies the minimal surface equation*

$$(1 + j_u^2)j_{vv} - 2j_u j_v j_{uv} + (1 + j_v^2)j_{uu} = 0$$

if and only if $J = (u, v, j(u, v))$ has vanishing mean curvature everywhere.

Proof. Given a graph $J(u, v) = (u, v, j(u, v))$, the mean curvature is given as

$$H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\left(\frac{eG - 2fF + gE}{EG - F^2}\right), \text{ where}$$

$$E = \langle j_u, j_u \rangle, \qquad F = \langle j_u, j_v \rangle, \text{ and} \qquad G = \langle j_v, j_v \rangle$$

are the coefficients of the first fundamental form. Moreover,

$$e = -\langle N_u, j_u \rangle \qquad f = -\langle N_u, j_v \rangle \qquad g = -\langle N_v, j_v \rangle$$

are the coefficients of the second fundamental form, and

$$N = \frac{j_u \times j_v}{|j_u \times j_v|}$$

is the unit normal to the surface. Using the fact that N is orthogonal to j_u and j_v, we obtain

$$0 = \frac{\partial}{\partial u}\langle N, j_u\rangle = \langle N_u, j_u\rangle + \langle N, j_{uu}\rangle$$

so that $e = -\langle N_u, j_u\rangle = \langle N, j_{uu}\rangle$. By the same reasoning

$$0 = \frac{\partial}{\partial v}\langle N, x_u\rangle = \langle N_v, x_u\rangle + \langle N, x_{uv}\rangle$$

which gives $f = -\langle N_u, j_v\rangle = \langle N, j_{uv}\rangle$. Also,

$$0 = \frac{\partial}{\partial v}\langle N, j_v\rangle = \langle N_v, j_v\rangle + \langle N, j_{vv}\rangle$$

which makes $g = -\langle N_v, j_v\rangle = \langle N, j_{vv}\rangle$. The following partial derivatives of $J(u, v)$ can be computed,

$$\begin{aligned} J_u &= (1, 0, j_u) & J_v &= (0, 1, j_v) \\ J_{uu} &= (0, 0, j_{uu}) & J_{vv} &= (0, 0, j_{vv}) \\ J_{uv} &= J_{vu} = (0, 0, j_{uv}) = (0, 0, j_{vu}). \end{aligned}$$

This gives N as

$$N = \frac{(1, 0, j_u) \times (0, 1, j_v)}{|(1, 0, j_u) \times (0, 1, j_v)|} = \frac{(-j_u, -j_v, 1)}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}}.$$

Substituting the derivatives into the fundamental forms,

$$\begin{aligned} E &= \langle(1, 0, j_u), (1, 0, j_u)\rangle = 1 + j_u^2 \\ F &= \langle(1, 0, j_u), (0, 1, j_v)\rangle = j_u j_v \\ G &= \langle(0, 1, j_v), (0, 1, j_v)\rangle = 1 + j_v^2 \end{aligned}$$

and

$$\begin{aligned} e &= \left\langle \frac{(-j_u, -j_v, 1)}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}}, (0, 0, j_{uu}) \right\rangle = \frac{j_{uu}}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}} \\ f &= \left\langle \frac{(-j_u, -j_v, 1)}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}}, (0, 0, j_{vu}) \right\rangle = \frac{j_{uv}}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}} \\ g &= \left\langle \frac{(-j_u, -j_v, 1)}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}}, (0, 0, j_{vv}) \right\rangle = \frac{j_{vv}}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}}. \end{aligned}$$

Substituting back into H gives

$$H = \frac{1}{(j_u^2 + j_v^2 + 1)^{\frac{1}{2}}} \left(\frac{(j_{uu})(1 + j_v^2) - 2(j_{uv})(j_u j_v) + (j_{uv})(1 + j_u^2)}{(1 + j_u^2)(1 + j_v^2) - (j_u j_v)^2} \right)$$

after factoring the denominator

$$H = \frac{(j_{uu})(1 + j_v^2) - 2(j_{uv})(j_u j_v) + (j_{uv})(1 + j_u^2)}{(j_u^2 + j_v^2 + 1)^{\frac{3}{2}}}$$

which gives the numerator as the minimal surface equation. Therefore, the graph satisfying the minimal surface equation is a necessary and sufficient condition for the mean curvature to vanish everywhere on the surface. □

5. Normal Variations

The next theorem takes a Lagrange-like approach to finding qualities of a surface with vanishing mean curvature. Instead of a variation of a graph, a variation of an orientable surface will be taken. If a surface is orientable, then the surface possesses a differentiable field of unit normals vectors over the entire surface. With this in mind, the normal variation can now be introduced. For an orientable surface, the surface can be deformed by the surface normal.

Definition 5.1. [BC76] *A* ***simply connected domain*** $D \subset \mathbb{R}^2$ *is a domain such that every simple closed contour within in it encloses only points in* D*.*

More precisely, given a simply connected domain $U \subset \mathbb{R}^2$, and an orientable surface $x : U \to \mathbb{R}^3$, $x^t : U \to \mathbb{R}^3$ is defined such that

$$x^t(p) = x(p) + th(p)N(p), \tag{7}$$

where $t \in \mathbb{R}$, N is the surface normal to x at a point p on the surface, and $h : U \to \mathbb{R}$ is a differentiable function with compact support in U. The function h deforms the surface at the point by some factor in the direction of the surface normal. Since h is differentiable, the variation x^t will be differentiable.

Definition 5.2. [DoC76] *Let* $\boldsymbol{x} : U \to S$ *be a coordinate system in a regular surface* S *and let* $R = x(Q)$ *be a bounded region of* S *contained in* $x(U)$*. Then* R *has an area given by*

$$A(R) = \iint_Q |\boldsymbol{x}_u \times \boldsymbol{x}_v| dudv = \iint_Q \sqrt{EG - F^2} dudv \tag{8}$$

where E,F,G are the coefficients of the first fundamental form.

Combining the variation with the area form allows for the introduction of the second theorem.

Theorem 5.3. *Let $x : U \to \mathbb{R}^3$ be an orientable surface and $x^t : U \to \mathbb{R}^3$ such that $x^t(p) = x(p) + th(p)N(p)$. A surface x has vanishing mean curvature over the entire surface iff $A'(0) = 0$, where $A(t)$ is the area of x^t.*

Proof. Given an orientable surface $x : U \to \mathbb{R}^3$, a variation $x^t : U \to \mathbb{R}^3$ can be created such that $x^t(p) = x(p) + th(p)N(p)$, where $t \in \mathbb{R}$, N is the unit normal to the surface, and h is a differentiable function with compact support in U. A basis for the tangent space of the variation can be found:

$$x^t_u(p) = x_u(p)+th(p)N_u(p)+th_u(p)N(p) \qquad x^t_v(p) = x_v(p)+th(p)N_v(p)+th_v(p)N(p).$$

Using this basis and letting all terms be evaluated at p, the coefficients of the first fundamental form of the variation can be computed giving

$$\begin{aligned} E^t &= \left\langle x^t_u, x^t_u \right\rangle \\ &= \langle x_u + thN_u + th_uN, x_u + thN_u + th_uN\rangle \\ &= \langle x_u, x_u\rangle + \langle x_u, thN_u\rangle + \langle x_u, th_uN\rangle + \langle thN_u, x_u\rangle + \langle thN_u, thN_u\rangle + \\ &\quad \langle thN_u, th_uN\rangle + \langle th_uN, x_u\rangle + \langle th_uN, thN_u\rangle + \langle th_uN, th_uN\rangle . \end{aligned}$$

Collecting like terms and using the fact $\langle N, N_u\rangle = 0$ and $\langle N, x_u\rangle = 0$ gives

$$E^t = \langle x_u, x_u\rangle + th\,\langle x_u, N_u\rangle + th\,\langle N_u, x_u\rangle + t^2h^2\,\langle N_u, N_u\rangle + t^2h_u^2\,\langle N, N\rangle .$$

N is a unit vector, thus $\langle N, N\rangle = |N|^2 = 1$, which makes

$$E^t = E + 2th\,\langle x_u, N_u\rangle + t^2h^2\,\langle N_u, N_u\rangle + t^2h_uh_u;$$

substituting in the coefficients of the second fundamental form of x gives

$$E^t = E + 2th(-e) + t^2h^2\,\langle N_u, N_u\rangle + t^2h_uh_u.$$

Similar computation gives

$$\begin{aligned} F^t &= \left\langle x^t_u, x^t_v \right\rangle \\ &= \langle x_u + thN_u + th_uN, x_v + thN_v + th_vN\rangle \\ &= F + th\left(\langle x_u, N_v\rangle + \langle x_v, N_u\rangle\right) + \langle N_u, N_v\rangle + t^2h_uh_v \\ &= F + th(-2f) + t^2h^2\,\langle N_u, N_v\rangle + t^2h_uh_v, \end{aligned}$$

and,

$$\begin{aligned} G^t &= \left\langle x^t_v, x^t_v \right\rangle \\ &= \langle x_v + thN_v + th_vN, x_v + thN_v + th_vN\rangle \\ &= G + 2th\,\langle x_v, N_v\rangle + t^2h^2\,\langle N_v, N_v\rangle + t^2h_vh_v \\ &= G + 2th(-g) + t^2h^2\,\langle N_v, N_v\rangle + t^2h_vh_v. \end{aligned}$$

Since t will be set to zero after differentiating with respect to t, all of the terms with a t with order greater than 2 will be given by $O(t^2)$, this gives

$$\begin{aligned}
E^tG^t-(F^t)^2 = &\left(E+2th(-e)+O(t^2)\right)\left(G+2th(-g)+O(t^2)\right)-\left(F+th(-2f)+O(t^2)\right)^2\\
=&EG+E(2th(-g))+G(2th(-e))+(2th(-e))(2th(-g))\\
&-\left(F^2+2F(th(-2f))+(th(-2f))(th(-2f))\right)\\
=&EG-2th(Eg)-2th(Ge)-F^2+4(Ff)th+O(t^2)\\
=&(EG-F^2)-2th(Eg+-2Ff+Ge)+O(t^2)\\
=&(EG-F^2)\left[1-2th\left(\frac{Eg+-2Ff+Ge}{EG-F^2}+O(t^2)\right)\right]\\
=&(EG-F^2)\left[1-4th(\tfrac{1}{2})\left(\frac{Eg+-2Ff+Ge}{EG-F^2}+O(t^2)\right)\right]\\
=&(EG-F^2)\left[1-4th\left(H+O(t^2)\right)\right]
\end{aligned}$$

Substituting into the area form,

$$\begin{aligned}
A(t)&=\iint_D\sqrt{(E^t)(G^t)-(F^t)^2}dudv\\
&=\iint_D\sqrt{(EG-F^2)\left(1-4thH+O(t^2)\right)}dudv.
\end{aligned}$$

Differentiating $A(t)$ with respect to t and setting $t=0$ makes $O(t)=0$ and gives

$$\begin{aligned}
A'(0)=&\frac{\partial}{\partial t}\iint_D\sqrt{(EG-F^2)\left(1-4thH+O(t^2)\right)}dudv\bigg|_{t=0}\\
=&\iint_D\sqrt{EG-F^2}\left(1-4thH+O(t^2)\right)^{-1/2}\frac{1}{2}\left(4Hh+O(t)\right)dudv\bigg|_{t=0}\\
=&\iint_D\sqrt{EG-F^2}\left(2Hh\right)dudv
\end{aligned}$$

From this equation, if $H=0$, $A'(0)$ will also be zero. The converse gives $A'(0)=0$ which is true when $Hh=0$. Since h is any compactly supported function over D, by theorem(3.5), H must be zero. $\square$

6. Harmonic Component Functions

The final theorem relates vanishing mean curvature with a conformal, harmonic map.

Definition 6.1. [DoC76] *Given two surfaces S and $\overline{S}$, $\varphi : S \to \overline{S}$ is called **conformal** if there is $\lambda : S \to \mathbb{R}^+$ such that for all $p \in S$ and all $v_1, v_2 \in T_p(S)$*

$$\langle d\varphi_p(v_1), d\varphi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle .$$

Definition 6.2. [BC76] *A twice differentiable function $F : U \to \mathbb{R}$ is said to be **harmonic** iff $F_{uu} + F_{vv} = 0$.*

A conformal map preserves angles of tangent vectors on their respective surfaces. Combining the two definitions will result in vanishing mean curvature.

Theorem 6.3. *Let $U \subset \mathbb{R}^2$ be a simply connected domain and define $x : U \to \mathbb{R}^3$ to be a conformal map. x is harmonic if and only if mean curvature vanishes on x.*

Proof. Since x is conformal, given a basis of U,$\{(0,1),(1,0)\}$, applying the definition of conformal to the coefficients of the first fundamental form gives

$$\begin{aligned} E &= \langle x_u, x_u\rangle = \langle dx(1,0), dx(1,0)\rangle = \lambda^2\langle (1,0),(1,0)\rangle = \lambda^2 \\ F &= \langle x_u, x_v\rangle = \langle dx(1,0), dx(0,1)\rangle = \lambda^2\langle (1,0),(0,1)\rangle = 0 \\ G &= \langle x_v, x_v\rangle = \langle dx(0,1), dx(0,1)\rangle = \lambda^2\langle (0,1),(0,1)\rangle = \lambda^2. \end{aligned}$$

For a conformal map, the mean curvature (6) can be rewritten as

$$\begin{aligned} H = \frac{1}{2}\frac{Eg - Ff + Ge}{EG - F^2} &= \frac{1}{2}\frac{\lambda^2(g) - (0)(f) + (\lambda^2)(e)}{\lambda^4 - (0)^2} \\ &= \frac{1}{2}\frac{g+e}{\lambda^2} \\ &= \frac{1}{2}\frac{-\langle N_v, x_v\rangle - \langle N_u, x_u\rangle}{\lambda^2} \\ &= \frac{1}{2}\frac{\langle N, x_{vv}\rangle + \langle N, x_{uu}\rangle}{\lambda^2} \\ &= \frac{1}{2}\frac{\langle N, x_{vv} + x_{uu}\rangle}{\lambda^2}. \end{aligned}$$

From this form, if x is harmonic then H is zero. If mean curvature vanishes on the surface $\langle N, x_{vv} + x_{uu}\rangle$ must equal zero. Since x is conformal, $E = G = \lambda^2$ and $F = 0$. Taking the differentials of E and G, we find

$$\begin{aligned} \tfrac{\partial}{\partial u}E &= \tfrac{\partial}{\partial u}G \\ \tfrac{\partial}{\partial u}\langle x_u, x_u\rangle &= \tfrac{\partial}{\partial u}\langle x_v, x_v\rangle \\ 2\langle x_u, x_{uu}\rangle &= 2\langle x_v, x_{vu}\rangle \\ \langle x_u, x_{uu}\rangle &= \langle x_v, x_{vu}\rangle , \end{aligned}$$

similar computation shows $\langle x_u, x_{uv} \rangle = \langle x_v, x_{vv} \rangle$.
Using the fact that

$$\frac{\partial}{\partial u} F = 0 = \frac{\partial}{\partial u} \langle x_u, x_v \rangle = \langle x_{uu}, x_v \rangle + \langle x_u, x_{vu} \rangle$$

and

$$\frac{\partial}{\partial v} F = 0 = \frac{\partial}{\partial v} \langle x_u, x_v \rangle = \langle x_{uv}, x_v \rangle + \langle x_u, x_{vv} \rangle$$

which makes $\langle x_{uv}, x_v \rangle = - \langle x_u, x_{vv} \rangle$ and $\langle x_{uu}, x_v \rangle = - \langle x_u, x_{vu} \rangle$. Substituting into the first equations,

$$\begin{aligned} \langle x_u, x_{uu} \rangle &= \langle x_v, x_{vu} \rangle \\ \langle x_u, x_{uu} \rangle &= - \langle x_u, x_{vv} \rangle \end{aligned}$$

which makes,

$$\begin{aligned} 0 &= \langle x_u, x_{uu} \rangle + \langle x_u, x_{vv} \rangle \\ &= \langle x_u, x_{uu} + x_{vv} \rangle \end{aligned}$$

similar computation also shows $0 = \langle x_v, x_{vv} + x_{uu} \rangle$. The vectors x_u,x_v and N form a basis for $\mathbb{R}^3$. Since $x_{uu} + x_{vv}$ is in $\mathbb{R}^3$, it can be written as a linear combination as $a_1 x_u + a_2 x_v + a_3 N$ where $a_i \in \mathbb{R}$. From above, $x_{uu} + x_{vv}$ is orthogonal to x_u and x_v, which means

$$\begin{aligned} \langle x_u, x_{uu} + x_{vv} \rangle &= \langle x_u, a_1 x_u + a_2 x_v + a_3 N \rangle \\ &= \langle x_u, a_1 x_u \rangle + \langle x_u, a_2 x_v \rangle + \langle x_u, a_3 N \rangle \\ &= a_1 \langle x_u, x_u \rangle + a_2 \langle x_u, x_v \rangle + a_3 \langle x_u, N \rangle . \end{aligned}$$

which is zero only if a_1 and a_2 are zero. This makes $x_u u + x_v v$ a multiple of N as well as parallel to N. By definition N is a unit vector and is non-zero, this means $\langle N, x_{uu} + x_{vv} \rangle$ equals zero if and only if $x_{uu} + x_{vv} = 0$ which makes x harmonic. $\square$

7. Complex Analytic Functions and Vanishing Mean Curvature

A complex function $f : \mathbb{C} \to \mathbb{C}$ can be represented by two real valued functions, $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy \in \mathbb{C}$ and $x, y \in \mathbb{R}$. Just like real valued functions, the derivatives of complex functions can also be taken. Since a point in the complex plane can be approached along two axes, additional conditions can be made about the derivative of a complex function.

Theorem 7.1. [BC76] *Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations*

$$u_x = v_y \qquad u_y = -v_x.$$

The Cauchy-Riemann equations are the result of taking the partial derivatives of the function and setting the resulting real and imaginary parts equal to each other. The equality guarantees the limit of the function at all points will exist. Since the derivative will be needed at more than a point, a stronger condition will be needed.

Definition 7.2. [BC76] *A function $f(z)$ is **analytic** in an open set if it has a derivative at each point in that set.*

Using line integrals, harmonic functions can be shown to be a part of a corresponding analytic function.

Definition 7.3. [Lar04] *Let M and N have continuous partial derivatives in a simply connected domain. The vector field given by $V(x,y) = M\hat{\imath} + N\hat{\jmath}$ is **conservative** if and only if $N_x = M_y$.*

Theorem 7.4. [Lar04] *$V(x,y) = M\hat{\imath} + N\hat{\jmath}$ is conservative in a simply connected domain if and only if the line integral*

$$\int_C Mdx + Ndy$$

is independent of path.

Given a curve C in a simply connected domain D, and differentiable functions M and N such that $M_y = N_x$, then the integral

$$\int_C Mdx + Ndy$$

is independent of path. Since the integral is independent of path, integrating over any path in D with the same endpoints as C will yield the same result. Let C have endpoints (x_0, y_0) and (x, y), then C can be split into two pieces. Define C_1 as the section of C from (x_0, y_0) to (x_1, y) and let C_2 be the section of C curve from (x_1, y) to (x, y) such that $x_1 \neq x$. By fixing (x_0, y_0), the integral over C becomes a function of x and y, given by

$$\begin{aligned} F(x,y) &= \int_C Mdx + Ndy \\ &= \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy. \end{aligned}$$

The first term does not vary with x, differentiating with respect to x makes it zero. In the second integral, parameterizing $C_2 = (x(t), y(t))$ for $a < t < b$ where $a, b, t \in D$,

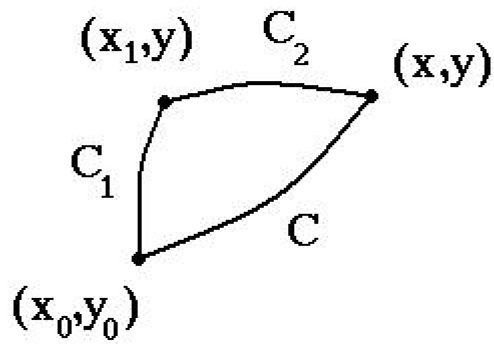

FIGURE 2. Curves from (x_0, y_0) to (x, y)

the second term is written

$$\int_{C_2} N dy = \int_a^b N y' dt.$$

Since C_2 is from (x_1, y) to (x, y), the y value does not change when parameterized by t. For this reason, y' is zero over C_2. Using the two pieces of information,

$$\begin{aligned}
F_x(x,y) =&\tfrac{\partial}{\partial x}\left(\int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy\right)\\
=&0 + \tfrac{\partial}{\partial x}\int_{C_2} Mdx + Ndy\\
=&\tfrac{\partial}{\partial x}\int_{C_2} Mdx + Ndy\\
=&\tfrac{\partial}{\partial x}\int_{C_2} Mdx + 0\\
=&\tfrac{\partial}{\partial x}\int_{C_2} Mdx\\
=&M(x,y),
\end{aligned}$$

similar computation shows that $F_y = N$.

Theorem 7.5. *If $f : D \to \mathbb{R}^3$ is harmonic, and D is simply connected, then there exists an analytic function $\varphi : D \to \mathbb{C}$ such that $Re(\varphi) = f$.*

Proof. Given a harmonic function $f(x, y)$, and a curve C in a simply connected domain D, define

$$g(x,y) = \int_C -f_y dx + f_x dy.$$

The integral is independent of path since D is simply connected and

$$\begin{aligned} \tfrac{\partial}{\partial y}(-f_y) =& - f_{yy}, \\ =& f_{xx} (f \text{ is harmonic}) \\ =& \tfrac{\partial}{\partial x}(f_x). \end{aligned}$$

Since the integral is independent of path, from before, we know $g_x = -f_y$ and $g_y = f_x$. For a complex function, $\varphi(x, y) = f(x, y) + ig(x, y)$, the Cauchy-Riemann equations are met and φ is analytic. ☐

The last theorem, combined with conformality can be related to vanishing mean curvature.

Theorem 7.6. *Assume $J(x, y) = (a(x, y), b(x, y), c(x, y))$ is conformal. Then $\varphi_1 = a_x - ia_y, \varphi_2 = b_x - ib_y, \varphi_3 = c_x - ic_y$ are analytic if and only if J has vanishing mean curvature everywhere.*

Since J is assumed to be conformal, showing J is also harmonic when φ_i are analytic will prove the theorem.

Proof. If φ_i are analytic, then the Cauchy-Riemann equations hold for each φ_i. The Cauchy-Riemann equations give

$$\begin{aligned} a_{xx} =& - a_{yy}, & a_{xy} =& a_{yx}, \\ b_{xx} =& - b_{yy}, & b_{xy} =& b_{yx}, \\ c_{xx} =& - c_{yy}, & c_{xy} =& c_{yx}. \end{aligned}$$

From the left side, the component functions a,b, and c must all be harmonic. From theorem(6.3) a conformal map is harmonic if and only if the surface has vanishing mean curvature everywhere. Since the φ_i's are analytic if and only if a,b, and c are harmonic, the φ_i's are analytic if and only if J has vanishing mean curvature. ☐

The fact that if φ_i's are analytic then the corresponding component functions must be harmonic allows another observation to be made. Something interesting happens when the squares of the analytic functions of J are added. The sum of the squares of the φ_i's give

$$\begin{aligned}
&\varphi_1^2+\varphi_2^2+\varphi_3^2=(a_x-ia_y)^2+(b_x-ib_y)^2+(c_x-ic_y)^2\\
=&(a_x)^2-i2(a_xa_y)-(a_y)^2+(b_x)^2-i2(b_xb_y)-(b_y)^2+(c_x)^2-i2(c_xc_y)-(c_y)^2\\
=&(a_x^2+b_x^2+c_x^2)+i2(-a_xa_y-b_xb_y-c_xc_y)-(a_y^2+b_y^2+c_y^2)\\
=&\left\langle(a_x,b_x,c_x),(a_x,b_x,c_x)\right\rangle-i2\left\langle(a_x,b_x,c_x),(a_y,b_y,c_y)\right\rangle-\left\langle(a_y,b_y,c_y),(a_y,b_y,c_y)\right\rangle\\
=&E-G-2iF.
\end{aligned}$$

If we instead add the absolute value of the squares we get

$$\begin{aligned}
&|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2=|\,(a_x-ia_y)^2\,|+|\,(b_x-ib_y)^2\,|+|\,(c_x-ic_y)^2\,|\\
=&|(a_x)^2-i2(a_xa_y)+(a_y)^2|+|(b_x)^2-i2(b_xb_y)+(b_y)^2|+|(c_x)^2-i2(c_xc_y)+(c_y)^2|\\
=&(a_x)^2+|-i2(a_xa_y)|+(a_y)^2+(b_x)^2|-i2(b_xb_y)|+(b_y)^2+(c_x)^2+|-i2(c_xc_y)|+(c_y)^2,\\
&|i|=1,\text{ so we have}\\
=&(a_x^2+b_x^2+c_x^2)+2(a_xa_y+b_xb_y+c_xc_y)+(a_y^2+b_y^2+c_y^2)\\
=&\left\langle(a_x,b_x,c_x),(a_x,b_x,c_x)\right\rangle+2\left\langle(a_x,b_x,c_x),(a_y,b_y,c_y)\right\rangle+\left\langle(a_y,b_y,c_y),(a_y,b_y,c_y)\right\rangle\\
=&E+G+2F
\end{aligned}$$

If we assume $\varphi_1^2+\varphi_2^2+\varphi_3^2=0$ and $|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2\neq 0$, we force the parameterization to be conformal. Combining this piece of information with harmonic component functions gives another theorem.

Theorem 7.7. [Web06] *Given three harmonic functions f_1, f_2, and f_3, all real parts to analytic functions F_1, F_2, and F_3, and the parameterization $x=Re(F_1,F_2,F_3)$; if $F_1'^2+F_2'^2+F_3'^2=0$ and $|F_1'|^2+|F_2'|^2+|F_3'|^2\neq 0$, then x has vanishing mean curvature everywhere.*

The proof is similar to theorem(6.3) since the functions are harmonic and the restrictions give conformal. The converse of this theorem is also true. However, unlike theorem(6.3), the map is not assumed to be conformal. The lack of this property makes the proof very difficult and is outside the scope of this project. Using these results, we can now generate zero mean curvature surfaces with analytic functions.

8. Examples

After talking about surfaces with vanishing mean curvature, here are some examples. First, the graph of Scherk's doubly periodic surface will be shown to satisfy the minimal surface equation(1). The graph of Scherk's doubly periodic surface, discovered in the 1830's, is given as

$$F(x,y) = \left(x, y, \ln\frac{\cos y}{\cos x}\right)$$

where $x, y \in (0, 2\pi)$ such that $x, y \neq n\frac{\pi}{2}$ for $n \in \mathbb{Z}$ and the sign of $\cos x$ is the same as the sign of $\cos y$.[Web06] In order to use the minimal surface equation, the following partial derivatives of $f(x,y)$ are needed

$$\begin{aligned} f_x &= \tan x & f_{xx} &= \sec^2 x \\ f_y &= -\tan y & f_{yy} &= -\sec^2 y \\ f_{xy} &= 0. \end{aligned}$$

Substituting into the minimal surface equation

$$\begin{aligned} (1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} \\ &= (1+-\tan y^2)\sec^2 x - 2\tan x(-\tan y)(0) + (1+\tan x^2)(-\sec^2 y) \\ &= \sec^2 y \sec^2 x - \sec^2 x \sec^2 y \\ &= 0. \end{aligned}$$

By theorem(4.7), Scherk's doubly periodic surface has vanishing mean curvature everywhere. The catenoid(pictured below), shown to have vanishing mean curvature by Meusnier in 1776, can be given by the parameterization

$$x(u,v) = (a\cosh v \cos u, a\cosh v \sin u, av)$$
$$\text{for } a \in \mathbb{R},\ 0 < u < 2\pi, \text{ and } -\infty < v < \infty.\text{[DoC76]}$$

The catenoid can be shown to have vanishing mean curvature by solving for mean curvature or, from theorem(6.3), by showing x is conformal and harmonic.

Finding the coefficients of the first fundamental form gives

$$\begin{aligned} E = \langle x_u, x_u\rangle &= a^2\left\langle(-\cosh v \sin u, \cosh v \cos u, 0), (-\cosh v \sin u, \cosh v \cos u, 0)\right\rangle \\ &= a^2\left(\cosh^2 v \sin^2 u + \cosh^2 v \cos^2 u + 0\right) \\ &= a^2 \cosh^2 v\left(\sin^2 u + \cos^2 u\right) \\ &= a^2\cosh^2 v. \end{aligned}$$

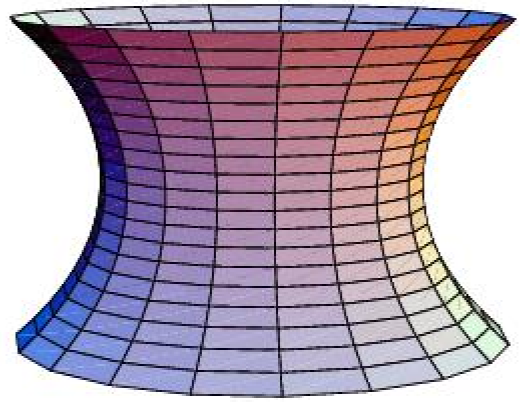

FIGURE 3. Catenoid

Similar computations give $F = 0$ and $G = a^2 \cosh^2 v$, which makes x conformal. Next, x needs to shown to be harmonic(6.2). To do this we need to we find

$$\begin{aligned} x_{uu} &= \frac{\partial}{\partial u}\frac{\partial}{\partial u}\left(a\cosh v\cos u, a\cosh v\sin u, av\right) \\ &= \frac{\partial}{\partial u}\left(-a\cosh v\sin u, a\cosh v\cos u, 0\right) \\ &= \left(-a\cosh v\cos u, -a\cosh v\sin u, 0\right) \text{ and} \\ x_{vv} &= \frac{\partial}{\partial v}\frac{\partial}{\partial v}\left(a\cosh v\cos u, a\cosh v\sin u, av\right) \\ &= \frac{\partial}{\partial v}\left(a\sinh v\cos u, a\sinh v\sin u, a\right) \\ &= \left(a\cosh v\cos u, a\cosh v\sin u, 0\right). \end{aligned}$$

Adding the two together we have

$$(-a\cosh v\cos u, -a\cosh v\sin u, 0) + (a\cosh v\cos u, a\cosh v\sin u, 0) = (0, 0, 0)$$

which gives x as harmonic. Since x, the catenoid, is both conformal and harmonic, the catenoid has vanishing mean curvature. Since each component function of the catenoid is harmonic, by theorem (7.5) each component function of the catenoid can be given as the real part of an analytic function. The creation of these analytic functions is very interesting because of the resulting imaginary part of the newly created functions. The imaginary parts of the functions give the real component functions of the helicoid, another surface also discovered by Meusnier to have vanishing mean curvature. This is a direct relationship to theorem(7.7), since the sum of the squares of the derivatives of the component analytic functions equals zero, by multiplying by a constant the sum will still be zero. Since the sum is still zero, the new parameterization will still be conformal. The relationship between the catenoid and the

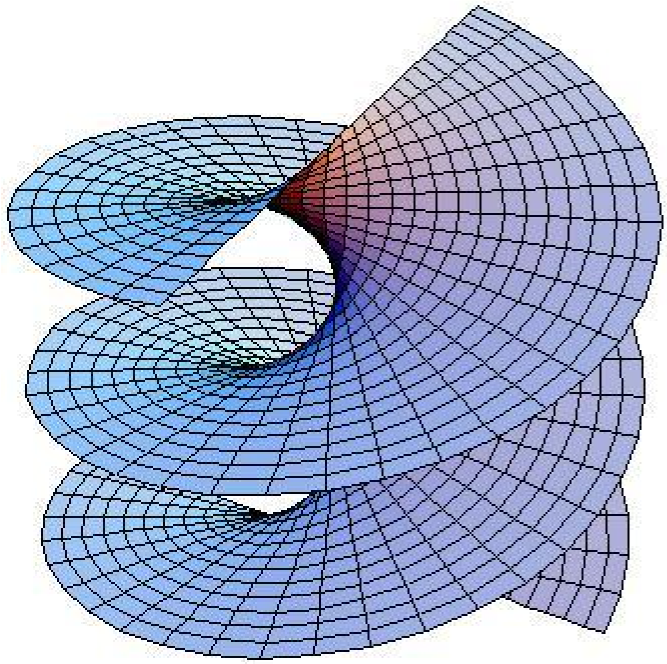

FIGURE 4. Helicoid

helicoid is displayed by

$$Re\int\left(\frac{1}{2}\left(\frac{1}{z}-z\right),\frac{i}{z}\left(\frac{1}{2}\left(\frac{1}{z}-z\right)\right),1\right)dz$$ a parameterization for the catenoid, and

$$Re\int e^{i\frac{\pi}{2}}\left(\frac{1}{2}\left(\frac{1}{z}-z\right),\frac{i}{z}\left(\frac{1}{2}\left(\frac{1}{z}-z\right)\right),1\right)dz$$ a parameterization for the helicoid.

In the complex plane, multiplication by $e^{i\theta}$ is a rotation by θ. This is incredible because not only do the catenoid and helicoid have vanishing mean curvature, any rotation of the catenoid in $\mathbb{C}^3$ will give a surface with vanishing mean curvature.

9. Conclusion

Surfaces with vanishing mean curvature are more commonly referred to as minimal surfaces. The term minimal surface was originally used to describe surfaces that satisfied Lagrange's minimal surface equation. After Meusnier discovered surfaces satisfying Lagrange's equation possessed vanishing mean curvature, the term minimal surface began being used to describe surfaces with just vanishing mean curvature. From theorem (5.3), the area of surfaces with vanishing mean curvature is not necessarily a minimum but rather a critical point of the area function. For this reason, in the paper the word minimal was avoided when possible. Aside from a misleading naming convention, vanishing mean curvature has an interesting relationship to harmonic, conformal maps in particular the connection with complex analysis. A short while ago the list of surfaces with vanishing mean curvature was fairly short and now they can be constructed with analytic functions. In any subject, this sort of breakthrough is always interesting, especially in topics where visualizing is not hard. To end, I give a quote from Fra Luca Bartolomeo de Pacioli, an Italian mathematician and collaborator with Leonardo da Vinci, "Without mathematics there is no art."[Phi08]. Minimal surfaces, among other things, certainly justify this statement.

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