New Deformations of Classical Minimal Surfaces

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Outline

Definition of a Minimal Surface

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A minimal surface is a 2-dimensional surface in $\mathbb{R}^3$ with constant mean curvature $H \equiv 0$.

Where does the name minimal come from?

Let $F : U \subset \mathbb{C} \to \mathbb{R}^3$ parameterize a minimal surface; let $d : U \to \mathbb{R}$ be smooth with compact support. Define a deformation of M by $F_\varepsilon(p) = F(p) + \varepsilon d(p) N(p)$.

$$\frac{d}{d\varepsilon} Area(F_\varepsilon(U))\Big|_{\varepsilon=0} = 0 \iff H \equiv 0$$

The intersection of a minimal surface with sufficiently small balls is a surface patch which minimizes area with respect to the boundary.

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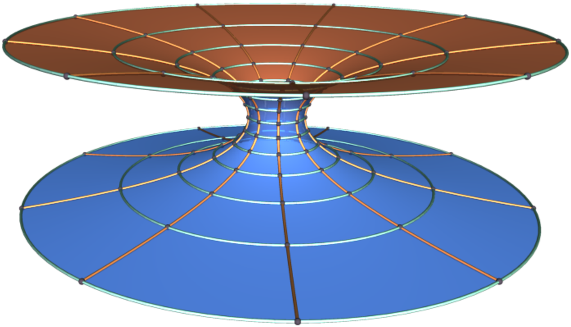
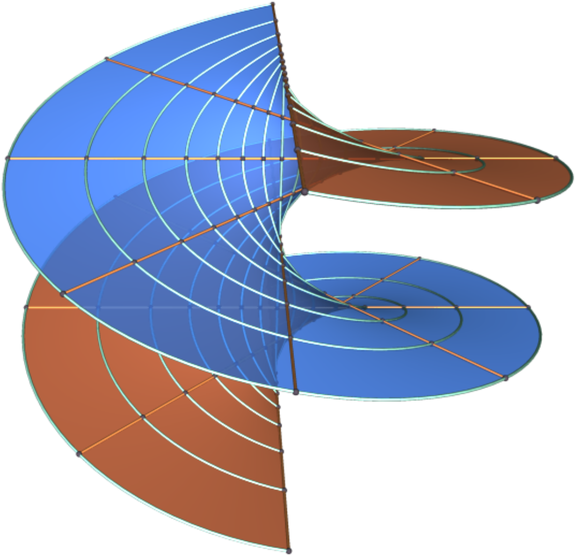
Examples - Catenoid



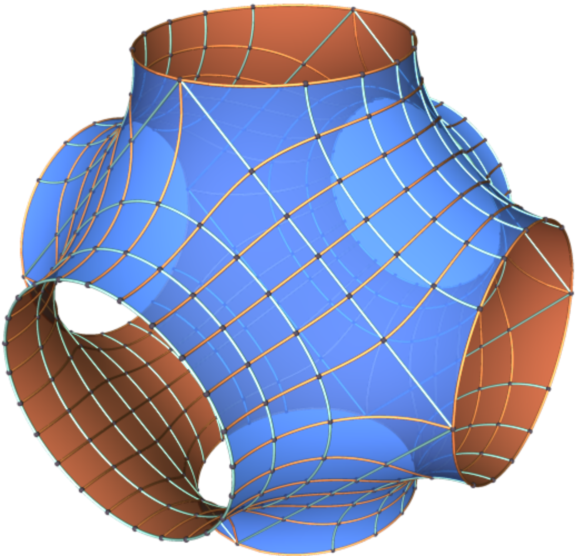
- Euler (1741), Meusnier (1776)
- Only minimal surface of revolution (except for the plane)

Examples - Helicoid



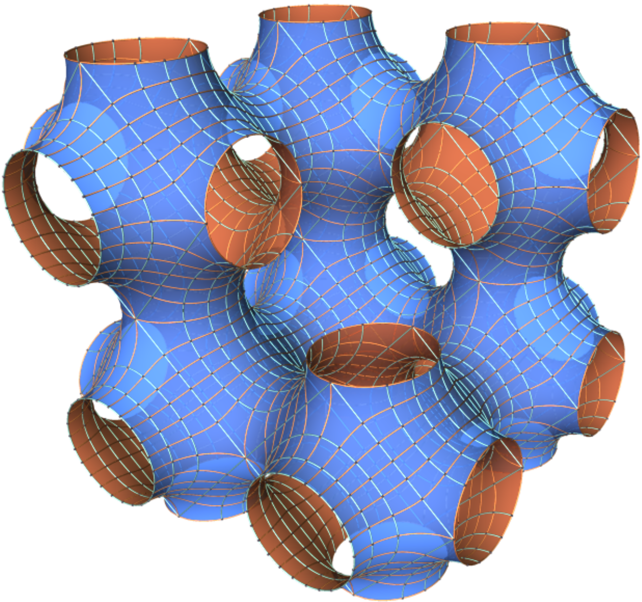
- Meusnier (1776)
- Only ruled minimal surface
- Plane and helicoid are only complete, embedded, simply connected minimal surfaces in $\mathbb{R}^3$ (ColMin-MeeRos).

Examples - P Surface

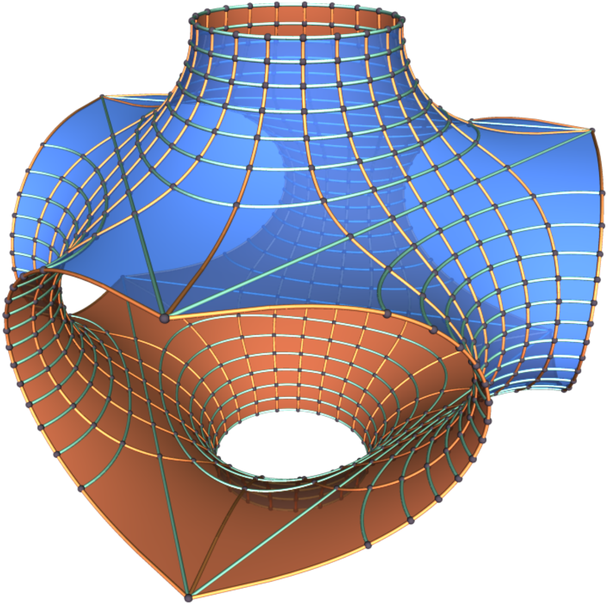


- Schwarz (1865)
- Triply periodic surface; cubical lattice
- Tiled by right angled hexagons

Examples - P Surface

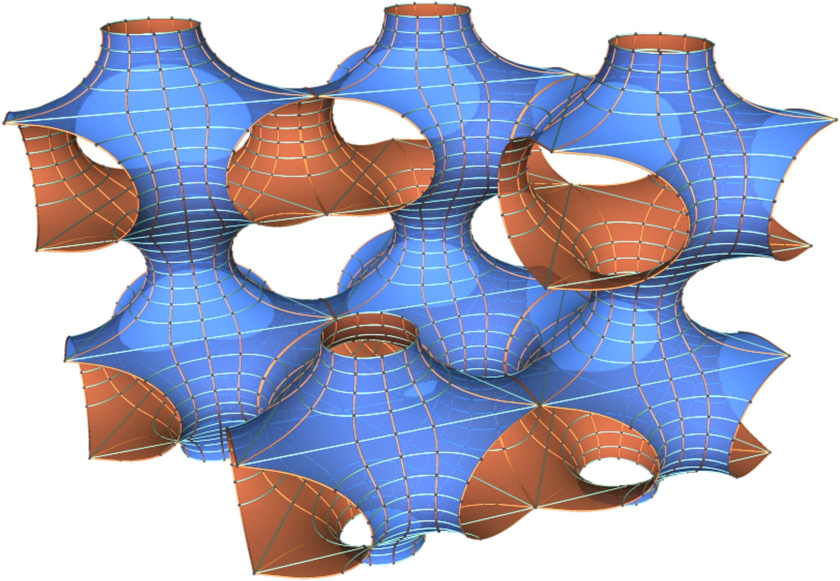


Examples - H Surface



- Schwarz (1865)
- Triply periodic surface; hexagonal lattice
- Lots of straight lines, planar symmetries

Examples - H Surface



Definition of Triply Periodic Minimal Surface

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A triply periodic minimal surface M is a minimal surface in $\mathbb{R}^3$ that is invariant under the action of a lattice Λ. The quotient surface $M/\Lambda \subset \mathbb{R}^3/\Lambda$ is compact and minimal.

Physical scientists are interested in these surfaces:

- Interface in polymers
- Physical assembly during chemical reactions
- Microcellular membrane structures

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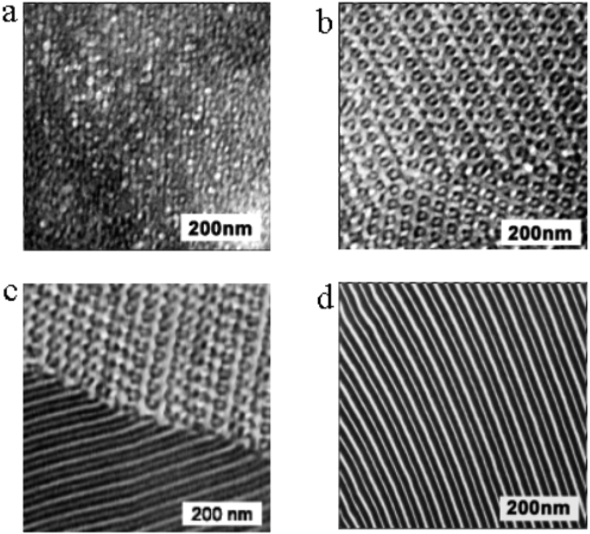
TEM of Polymers Showing Periodic Structure



Novel Morphologies of Block Copolymer Blends via Hydrogen Bonding. Jiang, S., Gopfert, A., and Abetz, V. Macromolecules, 36, 16, 6171 - 6177, 2003, 10.1021/ma0342933

Classification of TPMS

Rough classification by the genus of M/Λ:

Theorem

(Meeks, 1975) Let M be a triply periodic minimal surface of genus g. The Gauss map of M/Λ is a conformal branched covering map of the sphere of degree $g-1$.

Corollary

The smallest possible genus of M/Λ is 3.

Theorem

(Traizet, 2006) There exists an embedded triply periodic minimal surface of every genus at least 3.

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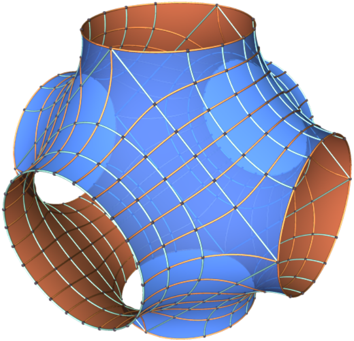

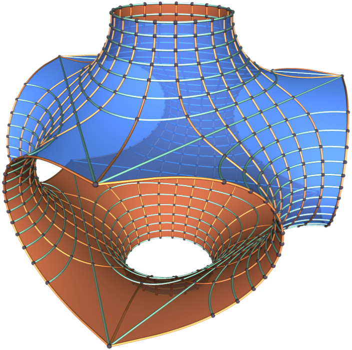

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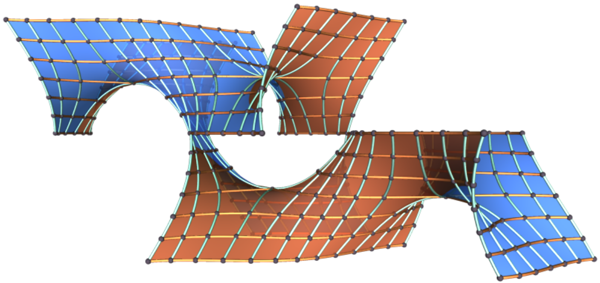
What's in the Meeks' family?

All proven examples of genus 3 triply periodic surfaces are in the Meeks' family, with two exceptions, the gyroid and the Lidinoid. Do these surfaces admit no deformations?

The Gyroid



- Schoen, 1970 (while trying to find strong / light structures for NASA)
- Triply periodic surface
- Contains no straight lines or planar symmetry curves

Results

Theorem

(W., 2005) There is a continuous, one-parameter family of embedded triply periodic minimal surfaces of genus 3 that contains the gyroid. Each surface admits an order 2 rotational symmetry.

A slight extension gives:

- Another 1-parameter gyroid family (preserves order 3 rotation)
- Analogous results for the Lidinoid

As a consequence, all known examples of triply periodic minimal surfaces of genus 3 are deformable.

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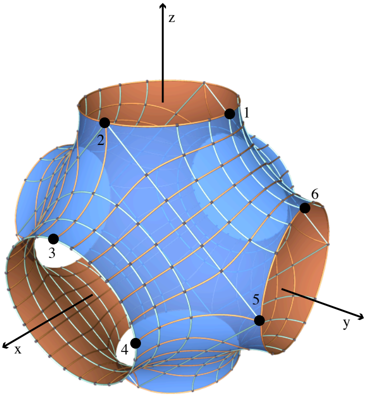
Overview of the Construction

To construct the gyroid deformation, we'll:

- Describe gyroid with certain (complex analytic) data
- Modify this data to construct new surfaces that are still minimal and still embedded
- Ensuring that the surfaces are embedded is the tricky part. To obtain embeddedness we use flat structures, a technique that transfers a difficult analysis problem to one involving Euclidean polygons.

As a toy problem and to get some intuition, we'll construct the P surface and study its "periods" using flat structures.

The P Surface

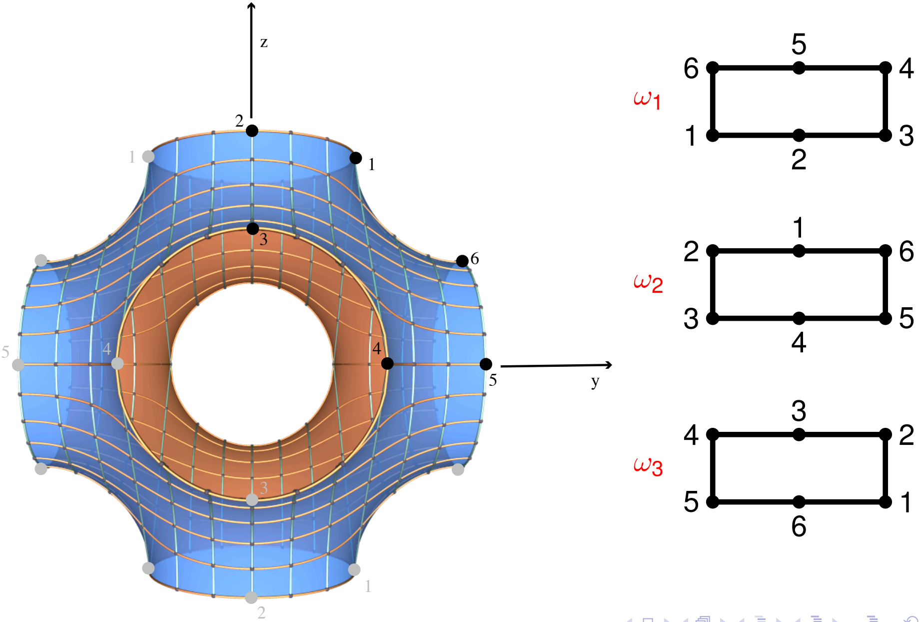


- Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by $F(z) = (f_1(z), f_2(z), f_3(z))$ be a conformal parameterization of one of the P surface hexagons.
- The f_i are harmonic.
- In a simply connected domain, can write

$$f_i(w) = \text{Real} \int^{w} \omega_i$$

- We can explicitly write down ω_i using Schwarz-Christoffel maps from the upper half plane.

Analytic Continution



What is the Gyroid?

The Associate Family

Modify coordinate functions by:

$$\text{Real}\int \omega_i \longrightarrow \text{Real}\int e^{i\theta}\omega_i$$

This new surface patch is locally isometric to the original.

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The Gyroid

In general, this transformation does not yield patches that fit together to give an embedded (or even immersed) surface.

For exactly one value of θ ($\theta \approx 51.9852^\circ$), this transformation applied to the P surface gives an embedded, triply periodic minimal surface, called the gyroid. Flat structures make this curious value of θ less mysterious.

Sketch of the Proof (of the gyroid deformation)

To construct P surface:

Parameterize a surface patch by mapping $F : U \subset \mathbb{C} \to \mathbb{R}^3/\Lambda$ with U simply connected.

To construct the gyroid deformation:

We need more of than just a surface patch, because

- it's not clear how to assemble the pieces in more complicated settings
- embeddedness would be very difficult to prove

To get more than a patch, consider as domain a Riemann surface X:

$$F : X \to \mathbb{R}^3/\Lambda \quad \text{by} \quad z \in X \mapsto \mathrm{Real} \int^z (\omega_1, \omega_2, \omega_3)$$

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Sketch of the Proof (Period Problem)

To construct P surface:

The integral is well-defined because integrals are path independent in U (simply connected).

To construct the gyroid deformation:

Since X is not simply connected, the integral

$$F : X \to \mathbb{R}^3/\Lambda \quad \text{by} \quad z \in X \mapsto \mathrm{Real} \int^{z} (\omega_1, \omega_2, \omega_3)$$

is not path independent. We need to ensure that

$$\int_{\gamma \in H_1(X,\mathbb{Z})} \omega_i \in \Lambda.$$

This is the so-called period problem.

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Sketch of the Proof (G and dh)

To construct P surface:

The 1-forms ω_i satisfied $\omega_1^2 + \omega_2^2 + \omega_3^2 \equiv 0$. To construct surfaces this way, one must modify the data in such a way that this condition is satisfied.

To construct the gyroid deformation:

To remove this over-determination, we write

$$\omega_1 = \frac{1}{2}\left(G + \frac{1}{G}\right)dh \qquad \omega_2 = \frac{i}{2}\left(G - \frac{1}{G}\right)dh \qquad \omega_3 = dh$$

where G is a meromorphic function (the Gauss map!). Then (almost) any G, dh will satisfy $\omega_1^2 + \omega_2^2 + \omega_3^2 \equiv 0$.

Note that this emphasizes the importance of three 1-forms:

$$Gdh \qquad \frac{1}{G}dh \qquad dh$$

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Sketch of the Proof (Period Problem (reprise))

To construct P surface:

We checked that after analytic continuation (using Schwarz Reflection), the patches "fit together" by understanding the periods on the developed flat structures of ω_i.

To construct the gyroid deformation:

To construct surfaces, we modify G and dh (and the Riemann surface X) and ensure that the period problem is solved. We can express the period problem in terms of Gdh, $\frac{1}{G}dh$, and dh.

Each of these 1-forms puts a flat structure on the Riemann surface X, which we can develop into the plane to get Euclidean polygons. We can understand the period problem in terms of these polygons.

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Homework Problems to Work On

Describe a 5-parameter family of gyroids

- The gyroid is v.p.-stable; this suggests that a 5-parameter family (like Meeks' should exist).
- Meeks' methods do not seem immediately adaptable
- Neither do these methods (not enough symmetries)
- Maybe exploit a "hidden symmetry"

Homework Problems to Work On

Understand limits of this family

- Traizet works “backward” from proposed limit surfaces to construct minimal surface examples
- An understanding of the limits of this and other families may help to discover more examples
- Beautiful stuff!

Attack genus four surfaces

- Precious little is known about genus 4 surfaces
- Method generalizes nicely, even though these surfaces are no longer hyperelliptic
- Would be nice to find examples, deformations, start of a classification?

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For More Information and Pictures

The details I omitted, and more (in progress):

A. Weyhaupt. *Deformations of triply periodic minimal surfaces*. Preprint.

The definitive introduction to applying flat structures to minimal surfaces:

M. Weber and M. Wolf. *Teichmüller theory and handle addition for minimal surfaces*. Ann. of Math. (2), 156(3):713-795, 2002.

The Virtual Museum of Minimal Surfaces:

`http://www.indiana.edu/~minimal/`

(click on "Archive")

The Periods of Any Admissible Flat Structure

We can develop a basis for the homology into the plane using the flat structure to calculate the horizontal periods:

$$
\begin{aligned}
\int_{A_1}(Gdh + \overline{\tfrac{1}{G}dh}) &= (1+i)(\widehat{\xi_1} - \xi_2 + \overline{(\xi_2 + i\xi_1 - \widehat{\xi_1} + \xi_1)}) \\
\int_{B_1}(Gdh + \overline{\tfrac{1}{G}dh}) &= (1+i)(\widehat{\xi_1} - \xi_1 + \overline{\xi_2}) \\
\int_{A_2}(Gdh + \overline{\tfrac{1}{G}dh}) &= (i-1)\xi_1 + (1+i)\overline{\xi_1} \\
\int_{B_2}(Gdh + \overline{\tfrac{1}{G}dh}) &= (1-i)(\xi_2 + \overline{(\widehat{\xi_1} - \xi_1)}) \\
\int_{A_3}(Gdh + \overline{\tfrac{1}{G}dh}) &= (-1-i)\xi_1 + (-1+i)\overline{\xi_1} \\
\int_{B_3}(Gdh + \overline{\tfrac{1}{G}dh}) &= 2(\widehat{\xi_1} - \xi_1 + \overline{\xi_2})
\end{aligned}
$$

The Periods of Any Admissible Flat Structure

This is nicer if we substitute $a = 2(\mathrm{Real}\xi_1 + \mathrm{Imag}\xi_2)$ and $b = 2(\mathrm{Imag}\xi_1 - \mathrm{Imag}\xi_2)$:

$$\int_{A_1}(Gdh + \overline{\tfrac{1}{G}dh}) = (a+b) + 0i$$
$$\int_{B_1}(Gdh + \overline{\tfrac{1}{G}dh}) = a + bi$$
$$\int_{A_2}(Gdh + \overline{\tfrac{1}{G}dh}) = (a+b) + 0i$$
$$\int_{B_2}(Gdh + \overline{\tfrac{1}{G}dh}) = -a + bi$$
$$\int_{A_3}(Gdh + \overline{\tfrac{1}{G}dh}) = 0 + (a+b)i$$
$$\int_{B_3}(Gdh + \overline{\tfrac{1}{G}dh}) = (a+b) + (b-a)i$$

We can easily compute that for the standard P surface $b = -a$, and that for the gyroid $b = 0$.

Description of the P Surface

▸ Picture

- Since P surface is invariant under order 2 rotation, we take as a conformal model of the Riemann surface a 2-fold branched cover of a torus (rectangular).
- G^2 is holomorphic on torus, can write down using theta functions:
$$\frac{\theta(z,\tau)\theta(z-\frac{\tau}{2},\tau)}{\theta(z-\frac{1}{2},\tau)\theta(z-\frac{1}{2}-\frac{\tau}{2})}$$
- Exact value of τ is determined by elliptic function (for most symmetric P surface).
- Only one holomorphic 1-form on torus, so $dh = dz$
- Flat structures of Gdh and dh

The Angle of Association

Recall that any minimal surface can be expressed as:

$$F(z) = \mathrm{Real} \begin{cases} \frac{1}{2}\int_{\cdot}^{z} (\frac{1}{G} - G)dh \\ \frac{i}{2}\int_{\cdot}^{z} (\frac{1}{G} + G)dh \\ \int_{\cdot}^{z} dh \end{cases}$$

in a simply connected domain U. We can obtain a one-parameter family of surface patches by multiplying dh by $e^{i\theta}$. The compatibility conditions are still satisfied.

The Angle of Association

If we try to do this on a general Riemann Surface, this will typically destroy the period condition that

$$\int_\gamma G dh + \overline{\int \frac{1}{G} dh} \in \Lambda$$

and

$$\mathrm{Real} \int_\gamma dh \in \Lambda.$$

We can ask: are there any values of θ for which this transformation yields an embedded minimal surface? Answer: Yes

The Gyroid

Set G and dh to the P surface data. If we modify dh to be $e^{i\theta} dh$ with

$$\theta \frac{180}{\pi} \approx 51.9852^o$$

the result is an embedded TPMS called the gyroid. How can we see what makes this strange number special?

The vertical period problem is one clue.

We'd like to construct a family of surfaces — to do that, we need to enlarge the family of flat structures we want to consider.

Sketch of the Proof of the Family of Gyroids

Step 1: Restrict to a reasonable moduli space of flat structures

- We will require all surface to have order 2 rotation, so conformal model is same: branched torus
- Tori will not need to be rectangular (too strict)
- Will consider *Gdh* flat structures as shown

Step 2: Solve the vertical and horizontal period problems simultaneously with compatible data

- Restrict to slices of the moduli space of tori
- Use an intermediate value type argument to show that for each slice there is a torus which solves vertical period problem and corresponding *Gdh* solves horizontal problem
- Can explicitly compute only isolated points; extrapolate from rectangular case using Dehn twist information

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Step 3: Show continuity of the family

- The use of intermediate value theorem means that this is not immediate.
- Essentially just a topological argument.

Step 4: Show that the family is embedded

- Construction only guarantees immersed
- This comes from the continuity of the family and the fact that minimal surfaces are quite rigid
- Uses a “maximal principle” for minimal surfaces that is similar to that for harmonic functions

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The Weierstraß Representation

Theorem

Let X be a Riemann surface, G a meromorphic function on X, and dh a holomorphic 1-form. Define $F : X \to \mathbb{R}^3/\Lambda$ by

$$F(z) = \mathrm{Real}\frac{1}{2}\int_{\cdot}^{z}\left(\frac{1}{G} - G, i\left(G + \frac{1}{G}\right), 2\right)dh$$

If $F(\gamma) = 0 \in \mathbb{R}^3/\Lambda$ for all curves $\gamma \in H_1(X, \mathbb{Z})$ (and other mild condition is met), then $F(X)$ is an immersed triply periodic minimal surface. Furthermore, $G(z)$ is the Gauss map (stereographic projection of the normal) of $F(X)$.

The Period Problem

To construct minimal surfaces, we can take any Riemann surface X, meromorphic function G, and holomorphic 1-form dh (subject to a mild compatibility condition). We "only" need to ensure that closed curves on X map to closed curves in $\mathbb{R}^3/\Lambda$, i.e.,

$$\int_\gamma G dh + \overline{\int_\gamma \frac{1}{G} dh} \in \Lambda$$

and

$$\mathrm{Real} \int_\gamma dh \in \Lambda.$$

for all $\gamma \in H_1(X, \mathbb{Z})$. Cooking up G and dh that satisfy these is called the period problem.

Flat structures

To understand

$$\int_\gamma G dh + \overline{\int_\gamma \frac{1}{G} dh} \quad \text{and} \quad \text{Real} \int_\gamma dh$$

we need to understand the three 1-forms Gdh, $\frac{1}{G}dh$, and dh.

- Each is a holomorphic 1-form on the Riemann surface.
- Any holomorphic 1-form on a Riemann surface induces a flat structure with cone points (on the Riemann surface).
- Cut the (flat) Riemann surface apart to get a Euclidean polygon.

Flat Structures and Cone Points

Any (holomorphic) 1-form ω on a Riemann surface induces a flat (translation) structure (away from the zeros) by integration

$$g_\alpha : V_\alpha \to \mathbb{C} \qquad \text{by} \quad q \in V_\alpha \mapsto \int_{p_\alpha}^{q} \omega$$

— note that the change of coordinates are Euclidean translations:

$$g_{\alpha\beta}(z) = z + \int_{p_\beta}^{p_\alpha} \omega$$

Near the zeros of ω, we get a cone point: a local chart looks like $z^{\frac{1}{k}}$.

Description of the Gyroid

We need a Riemann surface X, a Gauss map G, and a 1-form dh.

- Gyroid is invariant under order 2 rotation. Quotient by this rotation is a branched torus.
- Riemann surface will be 2-fold branched cover of a torus.
- G can be written using theta functions.
- dh is holomorphic on torus, so $dh = re^{i\theta}dz$.

Philosophy of the Problem

From $H \equiv 0$ to Complex Analysis

Using Weierstraß Representation construct surfaces by finding a Riemann surface X, a meromorphic function G on X, and a holomorphic 1-form dh on the X so that:

- The period problem is solved
- Certain mild compatibility conditions are satisfied

Philosophy of the Problem

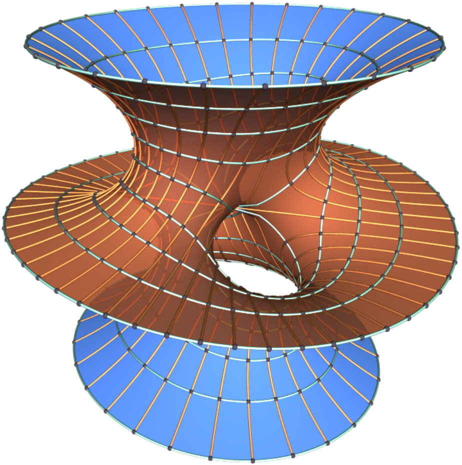
From $H \equiv 0$ to Complex Analysis

From Complex Analysis to Euclidean Polygons

The period problem is typically hard. Using flat structures, transfer the period problem to one involving Euclidean polygons and compute explicitly (algebraically!) the periods. To achieve this we:

- Assume (fix) some symmetries of the surface to reduce the number of parameters (and the number of conditions)
- Parameterize the moduli space of (admissible) polygons and compute the periods in terms of these parameters.
- Show that there is a 1-parameter family of polygons that solves the period problem

Examples - Costa's Surface



- Costa (1992) as a graduate student
- Complete, embedded, finite topology
- Rejuvenated the study of minimal surfaces

Examples - Riemann

- Riemann (pre-1866)
- Singly periodic surface
- Foliated by (generalized) circles