Deformations of Triply Periodic Minimal Surfaces

A Family of Gyroids

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Outline

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Definition of a Minimal Surface

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A minimal surface is a 2-dimensional surface in $\mathbb{R}^3$ with mean curvature $H \equiv 0$.

Where does the name minimal come from?

Let $F : U \subset \mathbb{C} \to \mathbb{R}^3$ parameterize a minimal surface; let $d : U \to \mathbb{R}$ be smooth with compact support. Define a deformation of M by $F_\varepsilon : p \mapsto F(p) + \varepsilon d(p) N(p)$.

$$\frac{d}{d\varepsilon} Area(F_\varepsilon(U))\Big|_{\varepsilon=0} = 0 \iff H \equiv 0$$

Thus, “minimal surfaces” may really only be critical points for the area functional (but the name has stuck).

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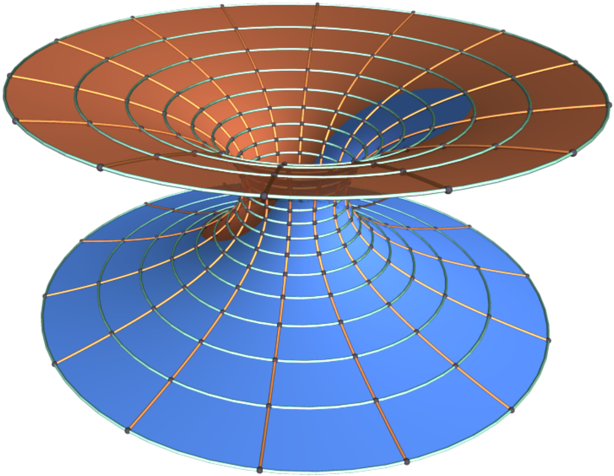
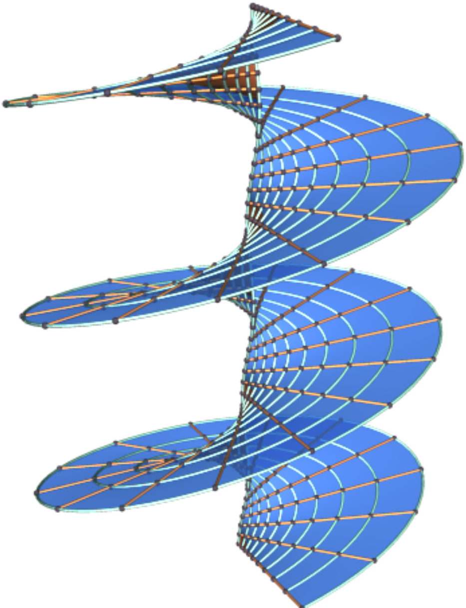
Examples - Catenoid



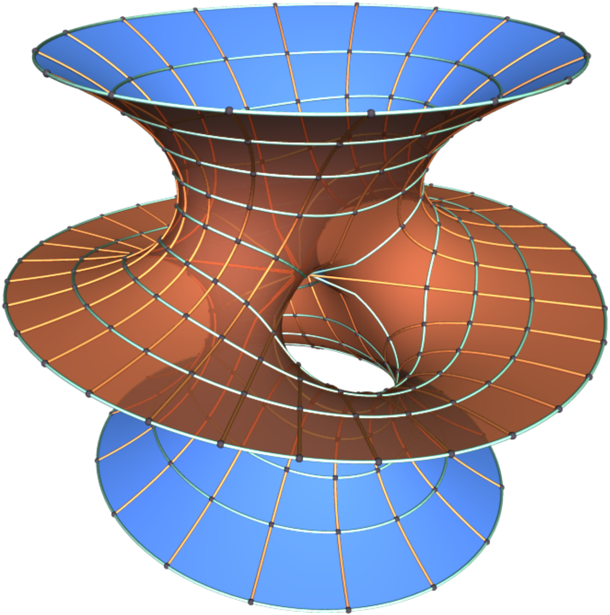
- Euler (1741), Meusnier (1776)
- Only minimal surface of revolution (except for the plane)

Examples - Helicoid



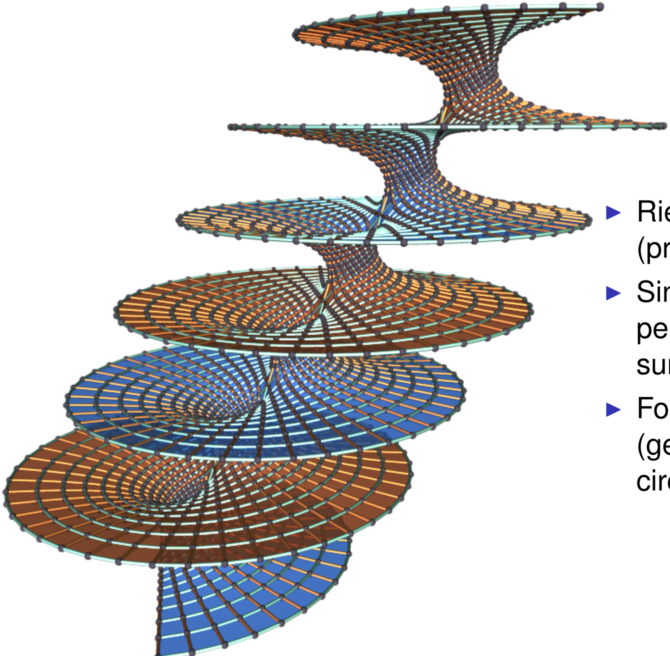
- Meusnier (1776)
- Only ruled minimal surface
- From 1776 - 1992, these three were the only examples that were complete, embedded, finite topology

Examples - Costa's Surface



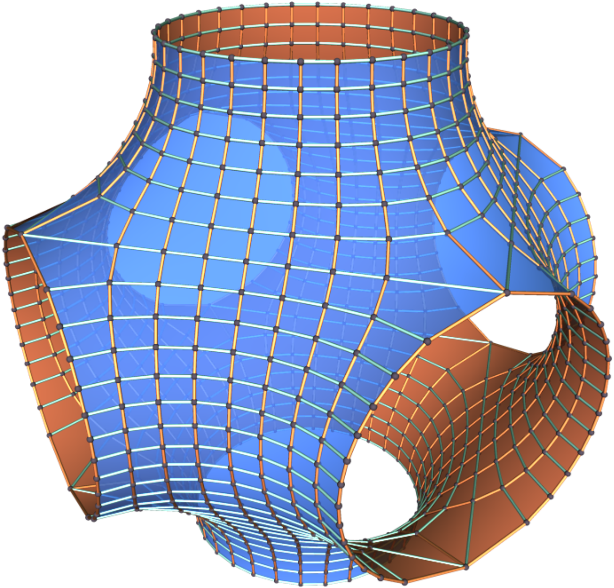
- Costa (1992) as a graduate student
- Complete, embedded, finite topology
- Rejuvenated the study of minimal surfaces

Examples - Riemann



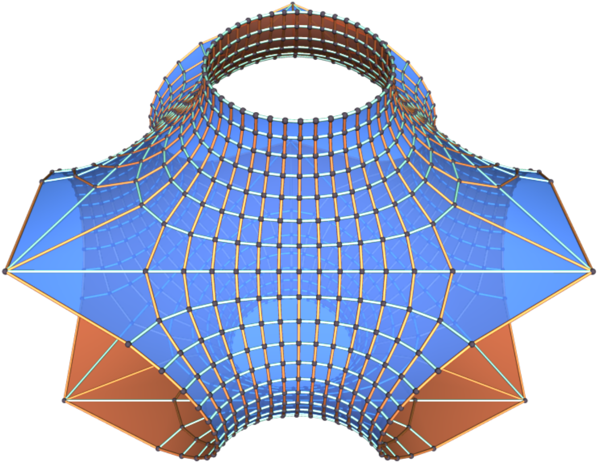
- Riemann (pre-1866)
- Singly periodic surface
- Foliated by (generalized) circles

Examples - P Surface



- Schwarz (1865)
- Triply periodic surface; cubical lattice
- “Square catenoid”

Examples - H Surface



- Schwarz (1865)
- Triply periodic surface; hexagonal lattice
- “Triangular catenoid”

Definition of Triply Periodic Minimal Surface

Definition

A triply periodic minimal surface M is a minimal surface in $\mathbb{R}^3$ that is invariant under the action of a lattice Λ. The quotient surface $M/\Lambda \subset \mathbb{R}^3/\Lambda$ is compact and minimal.

Physical scientists are interested in these surfaces:

- Interface in polymers
- Physical assembly during chemical reactions
- Microcelluar membrane structures

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Classification of TPMS

Rough classification by the genus of M/Λ:

Theorem

(Meeks, 1975) Let M be a triply periodic minimal surface of genus g. The Gauss map of M/Λ is a conformal branched covering map of the sphere of degree $g-1$.

Proof.

Since M is minimal, G is holomorphic (Weierstraß). Then M/Λ is a conformal branched cover of S^2. By Gauss-Bonnet:

$$-\mathit{degree}(G)4\pi = -\int |K|dA = \int KdA = 2\pi\chi(M) = 4\pi(1-g)$$

$\square$

Corollary

The smallest possible genus of M/Λ is 3.

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Other classifications?

Many triply periodic surfaces are known to come in a continuous family (or deformation).

Theorem

(Meeks, 1975) There is a five-dimensional continuous family of embedded triply periodic minimal surfaces of genus 3.

Picture

All proven examples of genus 3 triply periodic surfaces are in the Meeks' family, with two exceptions, the gyroid and the lidinoid.

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▸ Picture

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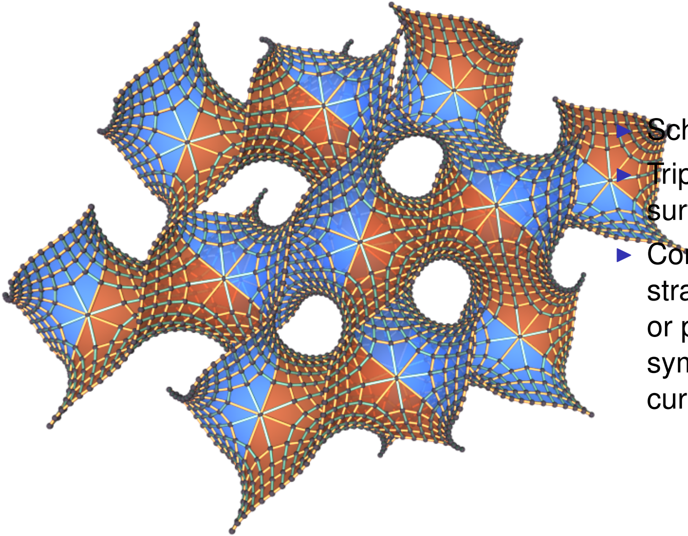
The Gyroid



- Schoen, 1970
- Triply periodic surface
- Contains no straight lines or planar symmetry curves

Results

Theorem

(W., 2005) There is a continuous, one-parameter family of embedded triply periodic minimal surfaces of genus 3 that contains the gyroid. Each surface admits an order 2 rotational symmetry.

A slight extension gives:

- Another 1-parameter gyroid family (preserves order 3 rotation)
- Analogous results for the lidinoid

These show that all known examples of surfaces are not isolated.

Our goal today is to sketch the main ideas in the proof of the theorem.

Outline

The Weierstraß Representation

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain, with $F : U \to \mathbb{R}^3$ a minimal immersion. Then we can write F as:

$$F(z) = \mathrm{Real} \begin{cases} \frac{1}{2}\int_{\cdot}^{z}(\frac{1}{G} - G)dh \\ \frac{i}{2}\int_{\cdot}^{z}(\frac{1}{G} + G)dh \\ \int_{\cdot}^{z} dh \end{cases}$$

Here $G(z)$ is a holomorphic function called the Gauss map (stereographic projection of the normal), and dh is a holomorphic 1-form. Where G has a zero of order k, dh must have a zero of order k.

Conversely, under mild compatibility conditions, any G and dh will construct a surface.

The Period Problem

To construct a larger piece of a surface, we could use as domain a Riemann Surface X and write

$$F(z) = \mathrm{Real} \begin{cases} \frac{1}{2} \int_{\cdot}^{z} (\frac{1}{G} - G) dh \\ \frac{i}{2} \int_{\cdot}^{z} (\frac{1}{G} + G) dh \\ \int_{\cdot}^{z} dh \end{cases}$$

where now $z \in X$ and G is a meromorphic function on X and dh is a holomorphic 1-form on X.

Unfortunately, this is not well-defined. If $\gamma \in H_1(X, \mathbb{Z})$, then in general

$$\mathrm{Real} \frac{1}{2} \int_{\gamma} \left(\frac{1}{G} - G \right) dh \neq 0.$$

The Period Problem, Continued...

To make this well defined, we'll require that G and dh satisfy

$$\int_\gamma G dh + \overline{\int \frac{1}{G} dh} \in \Lambda$$

and

$$\text{Real} \int_\gamma dh \in \Lambda.$$

Cooking up G and dh so that these equations hold is called the period problem.

Flat Structures and Cone Points

Any (holomorphic) 1-form ω on a Riemann surface induces a flat (translation) structure (away from the zeros) by integration

$$g_\alpha : V_\alpha \to \mathbb{C} \qquad \text{by} \quad q \in V_\alpha \mapsto \int_{p_\alpha}^{q} \omega$$

— note that the change of coordinates are Euclidean translations:

$$g_{\alpha\beta}(z) = z + \int_{p_\beta}^{p_\alpha} \omega$$

Near the zeros of ω, we get a cone point: a local chart looks like $z^{\frac{1}{k}}$.

Outline

Philosophy of the Problem

From $H \equiv 0$ to Complex Analysis

Using Weierstraß Representation construct surfaces by finding a Riemann surface X, a meromorphic function G on X, and a holomorphic 1-form dh on the X so that:

- The period problem is solved
- Certain mild compatibility conditions are satisfied

From Complex Analysis to Euclidean Polygons

The period problem is typically hard. Using flat structures, transfer the period problem to one involving Euclidean polygons and compute explicitly (algebraically!) the periods. To achieve this we:

- Assume (fix) some symmetries of the surface to reduce the number of parameters (and the number of conditions)
- Find a suitable class of polygons to study

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Description of the P Surface

Picture

- Since P surface is invariant under order 2 rotation, we take as a conformal model of the Riemann surface a 2-fold branched cover of a torus (rectangular).
- G^2 is holomorphic on torus, can write down using theta functions:

$$\frac{\theta(z,\tau)\theta(z-\frac{\tau}{2},\tau)}{\theta(z-\frac{1}{2},\tau)\theta(z-\frac{1}{2}-\frac{\tau}{2})}$$

- Exact value of τ is determined by elliptic function (for most symmetric P surface).
- Only one holomorphic 1-form on torus, so $dh = dz$
- Flat structures of Gdh and dh

The Angle of Association

Recall that any minimal surface can be expressed as:

$$F(z) = \mathrm{Real}\begin{cases} \frac{1}{2}\int_{\cdot}^{z}(\frac{1}{G} - G)dh \\ \frac{i}{2}\int_{\cdot}^{z}(\frac{1}{G} + G)dh \\ \int_{\cdot}^{z} dh \end{cases}$$

in a simply connected domain U. We can obtain a one-parameter family of surface patches by multiplying dh by $e^{i\theta}$. The compatibility conditions are still satisfied.

The Angle of Association

If we try to do this on a general Riemann Surface, this will typically destroy the period condition that

$$\int_\gamma G\,dh + \overline{\int \frac{1}{G}\,dh} \in \Lambda$$

and

$$\mathrm{Real} \int_\gamma dh \in \Lambda.$$

We can ask: are there any values of θ for which this transformation yields an embedded minimal surface? Answer: Yes

The Gyroid

Set G and dh to the P surface data. If we modify dh to be $e^{i\theta}dh$ with

$$\theta\frac{180}{\pi} \approx 51.9852^o$$

the result is an embedded TPMS called the gyroid. How can we see what makes this strange number special?

The vertical period problem is one clue.

We'd like to construct a family of surfaces — to do that, we need to enlarge the family of flat structures we want to consider.

Sketch of the Proof of the Family of Gyroids

Step 1: Restrict to a reasonable moduli space of flat structures

- We will require all surface to have order 2 rotation, so conformal model is same: branched torus
- Tori will not need to be rectangular (too strict)
- Will consider *Gdh* flat structures as shown

Step 2: Solve the vertical and horizontal period problems simultaneously with compatible data

- Restrict to slices of the moduli space of tori
- Use an intermediate value type argument to show that for each slice there is a torus which solves vertical period problem and corresponding *Gdh* solves horizontal problem
- Can explicitly compute only isolated points; extrapolate from rectangular case using Dehn twist information

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- The use of intermediate value theorem means that this is not immediate.
- Essentially just a topological argument.

Step 4: Show that the family is embedded

- Construction only guarantees immersed
- This comes from the continuity of the family and the fact that minimal surfaces are quite rigid
- Uses a “maximal principle” for minimal surfaces that is similar to that for harmonic functions

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Mild Modifications of the Arguments Yield

Another gyroid family

The family above preserved the order 2 rotation. Another family exists which preserves an order 3 rotation of the gyroid (and of the P surface).

Two families of “lidinoids”

The lidinoid is the surface the corresponds to the gyroid when we start with the H surface instead of the P surface. It will also contain two distinct 1-parameter families: one that preserves an order 2 symmetry and one that preserves an order 3 symmetry.

Outline

Homework Problems to Work On

Understand limits of this family

- Traizet works “backward” from proposed limit surfaces to construct minimal surface examples
- An understanding of the limits of this and other families may help to discover more examples
- Beautiful stuff!

Attack genus four surfaces

- Precious little is known about genus 4 surfaces
- Method generalizes nicely, even though these surfaces are no longer hyperelliptic
- Would be nice to find examples, deformations, start of a classification?

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Homework Problems to Work On

Describe a 5-parameter family of gyroids

- The gyroid is v.p.-stable; this suggests that a 5-parameter family (like Meeks' should exist).
- Meeks' methods do not seem immediately adaptable
- Neither do these methods (not enough symmetries)
- Maybe exploit a "hidden symmetry"

Outline

For More Information and Pictures

The details I omitted, and more:

A. Weyhaupt. *Deformations of Triply Periodic Minimal Surfaces*. Preprint, `http://aweyhaup.math.indiana.edu/`

The definitive introduction to applying flat structures to minimal surfaces:

M. Weber and M. Wolf. *Teichmüller theory and handle addition for minimal surfaces*. Ann. of Math. (2), 156(3):713-795, 2002.

The Virtual Museum of Minimal Surfaces:

`http://www.indiana.edu/~minimal/`
(click on “Archive”)

The Periods of Any Admissible Flat Structure

We can develop a basis for the homology into the plane using the flat structure to calculate the horizontal periods:

$$
\begin{aligned}
\int_{A_1}(Gdh+\overline{\tfrac{1}{G}dh}) &= (1+i)(\widehat{\xi_1}-\xi_2+\overline{(\xi_2+i\xi_1-\widehat{\xi_1}+\xi_1)})\\
\int_{B_1}(Gdh+\overline{\tfrac{1}{G}dh}) &= (1+i)(\widehat{\xi_1}-\xi_1+\overline{\xi_2})\\
\int_{A_2}(Gdh+\overline{\tfrac{1}{G}dh}) &= (i-1)\xi_1+(1+i)\overline{\xi_1}\\
\int_{B_2}(Gdh+\overline{\tfrac{1}{G}dh}) &= (1-i)(\xi_2+\overline{(\widehat{\xi_1}-\xi_1)})\\
\int_{A_3}(Gdh+\overline{\tfrac{1}{G}dh}) &= (-1-i)\xi_1+(-1+i)\overline{\xi_1}\\
\int_{B_3}(Gdh+\overline{\tfrac{1}{G}dh}) &= 2(\widehat{\xi_1}-\xi_1+\overline{\xi_2})
\end{aligned}
$$

The Periods of Any Admissible Flat Structure

This is nicer if we substitute $a = 2(\mathrm{Real}\xi_1 + \mathrm{Imag}\xi_2)$ and $b = 2(\mathrm{Imag}\xi_1 - \mathrm{Imag}\xi_2)$:

$$\begin{aligned}
\int_{A_1}(Gdh + \overline{\tfrac{1}{G}dh}) &= (a+b) + 0i \\
\int_{B_1}(Gdh + \overline{\tfrac{1}{G}dh}) &= a + bi \\
\int_{A_2}(Gdh + \overline{\tfrac{1}{G}dh}) &= (a+b) + 0i \\
\int_{B_2}(Gdh + \overline{\tfrac{1}{G}dh}) &= -a + bi \\
\int_{A_3}(Gdh + \overline{\tfrac{1}{G}dh}) &= 0 + (a+b)i \\
\int_{B_3}(Gdh + \overline{\tfrac{1}{G}dh}) &= (a+b) + (b-a)i
\end{aligned}$$

We can easily compute that for the standard P surface $b = -a$, and that for the gyroid $b = 0$.