Students, Sudoku, permanents, and combinatorial proof: An upper bound for permanents of $(0, 1)$-matrices

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How this project came about

SIUE senior project

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Disadvantage:

- Huge time commitment for faculty (and students)

Beth's Project

Beth was interested in a senior project related to Sudoku. Beth was a senior mathematics major with a focus on secondary education. (She is now a high school mathematics teacher at a school in southern Illinois.)

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Around the same time, this article appeared in the AMS Notices:

[HM07b] Agnes Herzberg and M. Ram Murty, *Sudoku squares and chromatic polynomials*, Notices Amer. Math. Soc. **54** (2007), no. 6, 708–717. MR 2327972

The article:

- involved a nice blend of combinatorics, graph theory, and analysis
- left enough details to the reader without writing the article at a sky-high level

Combinatorial background

Let $A_1, A_2, \ldots A_n \subseteq \{1, 2, \ldots n\}$.

System of distinct representatives

A system of distinct representatives is an n-tuple $(x_1, x_2, \ldots, x_n)$ where $x_j \in A_j$ and $x_i = x_j$ if and only if $i = j$.

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Hall Matrix

The Hall matrix corresponding to sets $A_1, A_2, \dots A_n$ is the matrix $A = (a_{ij})$ where $a_{ij} = 1$ if $j \in A_i$ and $a_{ij} = 0$ otherwise.

Combinatorial background

One of the tools that the paper used was the permanent of a matrix (which neither Beth nor I had used before).

Definition of permanent

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

The permanent of the Hall matrix, Per(A), gives the number of different systems of distinct representatives.

Why?

$a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ is nonzero iff each $a_{i\sigma(i)}$ is nonzero.

so

$a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ is nonzero iff $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a SDR.

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Counting 9×9 Sudoku grids

There are 9! ways to complete the first row.

1	2	3	4	5	6	7	8	9

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There are 9! ways to complete the first row. There are 6 possible numbers that can go in each entry of the second row. The Hall matrix for the second row has row sums equal to 6.

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$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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We can continue this way and each time bound the permanent of the Hall matrix to obtain the following upper bound on the number of Sudoku squares.

$$\prod_{i=1}^{n}\left(\prod_{j=0}^{i-1}\left(\left(n^2-(i-1)n-j\right)!\right)^{\frac{n^2}{n^2-(i-1)n-j}}\prod_{k=i}^{n-1}\left(\left(n^2-kn\right)!\right)^{\frac{n^2}{n^2-kn}}\right)$$

What got me thinking about permanent bounds

Actual number of distinct 9×9 Sudoku squares

6,670,903,752,021,072,936,960

Upper bound of Murty and Herzberg

170,719,448,452,571,374,040,662,731

We wondered why these differed by a factor of 25,500. One place to look was at an upper bound for the permanent that Murty and Herzberg used.

Computational complexity

Naïvely one expects that the determinant and permanent are both computationally expensive since we must sum over S_n which requires $n!$ additions. However:

Determinant

Amazingly (to me) $\mathcal{O}(n^3)$ algorithms exist to compute the determinant of a matrix.

Permanent

Amazingly (to me) no such algorithms are known for the permanent. It is likely that no polynomial time algorithm exists.

Therefore, many efforts have been made to find upper and lower bounds for the permanent of a matrix.

Some upper bounds on the permanent

Theorem (Minc-Brégman)

Let A be a $n \times n$ nonnegative integer matrix. Then

$$\operatorname{Per}(A) \leq \prod_{i=1}^{n} (r_i)!^{1/r_i}$$

where r_i is the i^{th} row sum of A.

Theorem (Liang and Bai, [LB04])

Let A be a $n \times n$ $(0,1)$-matrix with rows sums r_i. Then

$$\operatorname{Per}(A) \leq \prod_{i=1}^{n} \sqrt{x_i(r_i - x_i + 1)}$$

where $x_i = \min(\lceil \frac{r_i+1}{2} \rceil, \lceil \frac{i}{2} \rceil)$.

Main result

Al-Kurdi proved the following theorem using algebraic means:

Theorem (from [AK07])

Let A be a $(0,1)$ $n \times n$ matrix. Let $A_i = \{j \mid a_{ij} = 1\}$ (the set of non-zero column indicies of row i). Assume that $|A_1| \leq |A_2| \leq \cdots \leq |A_n|$ and $|\cup_{j>i}(A_i \cap A_j)| \neq 0$ for $1 \leq i \leq n-1$. Then

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n-1} \left| \bigcup_{j>i} (A_i \cap A_j) \right| .$$

Main result

We prove the following refinement of Al-Kurdi's bound with a short combinatorial proof:

Theorem (W.)

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$$\mathrm{Per}(A) \leq \prod_{i=1}^{n-1} \min \left\{ \max \left\{ \left| \bigcup_{j>i} (A_i \cap A_j) \right|, 1 \right\}, n - i + 1 \right\}.$$

One example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Minc-Brégman	1189.74
Liang-Bai	509.12
Our bound	192
Per(A)	48

Percentage of time that our bound improves on others

$p \backslash n$	10	15	20	25	30	35	40	45	50
.06	0.05	0.04	0.04	0.20	0.53	1.00	1.12	0.48	0.07
.08	0.36	0.69	1.62	2.97	3.29	0.81	0.05	0	0
.10	1.32	3.1	6.02	4.46	0.73	0.01	0	-	-
.12	3.56	8.74	9.12	1.17	0	0	0	-	-
.14	7.98	13.97	5.02	0.11	0	0	0	-	-
.16	13.73	15.26	1.0	0	0	0	0	-	-
.18	20.5	10.7	0.1	0	0	0	0	-	-
.20	25.29	5.42	0	0	0	0	0	-	-
.25	25.25	0.13	0	0	-	-	-	-	-
.30	11.60	0	0	0	-	-	-	-	-
.35	2.26	0	0	0	-	-	-	-	-
.40	0.37	0	0	0	-	-	-	-	-

p: probability of an entry being 1

n: size of matrix

10,000 trials

Proof of our bound

We want to show that

$$\mathrm{Per}(A) \le \prod_{i=1}^{n-1} \min\left\{ \max\left\{ \left| \bigcup_{j>i} (A_i \cap A_j) \right|, 1 \right\}, n-i+1 \right\}$$

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Since A is a $(0,1)$-matrix, we interpret it as the Hall matrix for sets A_i $(1 \leq i \leq n)$.

- We choose a representative for each set A_i as i increases.

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  - Therefore, either k has already been chosen as a representative or k is the representative for A_i.
- Therefore, there is either 1 choice for the representative of A_i or we can choose a k that satisfies $k \in H_i$ and $k \in \bigcup_{j>i}(A_i \cap A_j)$.

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- Therefore, there is either 1 choice for the representative of A_i or we can choose a k that satisfies $k \in H_i$ and $k \in \bigcup_{j>i}(A_i \cap A_j)$.
- Therefore, there are at most $\left|\bigcup_{j>i}(A_i \cap A_j)\right|$ choices for the A_i.
- On the other hand, when choosing the representative for A_i we have already chosen $i-1$ representatives, so at most $n-(i-1)$ possible choices remain.

Open questions / undergraduate research

Caveat: I'm not an expert, but it seems...

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Students can easily grasp the definition of the permanent and an undergraduate can understand the combinatorial interpretation of these.

- Are there combinatorial interpretations of these other bounds?
- Are there combinatorial interpretations for integer matrices?
- Under what circumstances can a certain bound be guaranteed to beat another?
- Can one improve upon Minc-Brégman (or any of the others) for all $(0, 1)$-matrices?

More information and details

[Wey09] Adam G. Weyhaupt, *A note on some upper bounds for permanents of $(0, 1)$-matrices*, J. Interdiscip. Math. **12** (2009), no. 1, 123–128. MR 2501724

[HM07a] Agnes M. Herzberg and M. Ram Murty, *Sudoku squares and chromatic polynomials*, Notices Amer. Math. Soc. **54** (2007), no. 6, 708–717. MR 2327972

[LB04] Heng Liang and Fengshan Bai, *An upper bound for the permanent of $(0, 1)$-matrices*, Linear Algebra Appl. **377** (2004), 291–295. MR 2022177

[AK07] Ahmad H. Al-Kurdi, *Some upper bounds for permanents of* $(0, 1)$*-matrices*, J. Interdiscip. Math. **10** (2007), no. 2, 169–175. MR 2317527

Information about SIUE's senior project:

`http://www.siue.edu/assessment/seniorassignment/`

Some past senior projects in mathematics:

`http://www.siue.edu/~aweyhau/`